Mathematics Interdisciplinary Research **10** (3) (2025) 315 – 335

Original Scientific Paper

A Carleman-Knopp Type Inequality for Pseudo-Integral

Mohammadreza Yasamian and Mohsen Jaddi ⋆

Abstract

Pseudo-analysis has applications in several fields, including game theory and optimization problems. Pseudo-analysis is a generalized form of ordinary classical analysis that has two main operations. In fact, these two operations, which are called pseudo-multiplication $\otimes$ and pseudo-addition $\oplus$, are the basis of the formation of a semi-ring on the interval $[c, d]$ of $[-\infty, \infty]$. The pseudo-operations $\otimes$ and $\oplus$ on $[c, d]$ produce three types of semi-ring. First, the semi-ring $([c, d], \sup, \otimes)$ or $([c, d], \inf, \otimes)$ in which $\otimes$ is generated, the second, a semi-ring where $\otimes$ and $\oplus$ are defined by the continuous and strictly monotone function ψ, the third, a semi-ring in which both pseudo-operations $\otimes$ and $\oplus$ are idempotent. In this article, we intend to state and prove some of the most recent generalizations of Carleman-Knopp's type inequalities via pseudo-integrals.

Keywords: Pseudo-operation, Pseudo-integral, Pseudo-logarithm.

2020 Mathematics Subject Classification: 26D15; 28B15; 28A25.

How to cite this article
M. Yasamian and M. Jaddi, A Carleman-Knopp type inequality for pseudo -integral, *Math. Interdisc. Res.* **10** (3) (2025) 315-335.

1. Introduction

In fact, pseudo-analysis with two operations of pseudo-multiplication and pseudo-addition on the real interval $[c, d]$ of $[-\infty, \infty]$ is a generalized form of classical

⋆Corresponding author (E-mail: mohsenjaddi@gmail.com)
Academic Editor: Abbas Saadatmandi
Received 17 July 2025, Accepted 24 August 2025
DOI: 10.22052/MIR.2025.257212.1527

analysis [1–3]. According to this structure, concepts were created such as pseudo-division, pseudo-additive measure ($\oplus - measure$), pseudo-integral, pseudo-scalar product, pseudo-convolution, pseudo-analytic exponential, pseudo-logarithm, etc. A wide variety of applications of pseudo-analysis can be seen in applied sciences, such as fuzzy sets and systems [4], game theory and decision making [5], Laplace transform [6], optimization problems [7], etc.

Sugeno and Murofushi [8] introduced the concept of pseudo-integral and $\oplus - measure$ based on the definition of pseudo-addition. $\oplus - measure$ is a type of monotone measure. In the definition of a pseudo-integral, which is a generalization of the Lebesgue integral, a type of multiplication corresponding to a pseudo-addition is presented. The integral inequalities are a very useful tool in mathematics. Important and different integral inequalities, including Barnes-Godunova-Levin, Chebyshev, Carleman-Knopp, Jensen, Cauchy–Schwarz, are increasingly used in various mathematical fields such as probability theory, differential equations, system theory, optimization, control theory and difference equations.

So far, many inequalities have been proven in the field of pseudo-integrals. Pap and Štrboja [9] generalized the Jensen integral inequality. Abbaszadeh et al. [10, 11] proved Hölder's type integral inequality and Hadamard inequality. Agahi et al. [12, 13] proved Chebyshev type inequalities and generalized the integral inequalities of Hölder and Minkowski type.

We know that the well-known classical inequality of Carleman [14] is as follows:

$$\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du \leq e\int_0^{+\infty} f(u)du, \tag{1}$$

where $f : [0, +\infty) \to [0, +\infty)$ is a Riemann integrable function which $\int_0^{+\infty} f(u)du < \infty$. We also know that

$$\int_{[0,+\infty)} f d\mu = \int_0^{+\infty} f(u)du, \tag{2}$$

where $\int_0^{+\infty} f(u)du < \infty$ and $f \geq 0$ on $[0, +\infty)$ [15]. Equation (2) also holds [16] when$f$ is a nonnegative continuous function on $[0, +\infty)$.

The inequality (1) is known as Knopp's inequality [17]. But it is important to note that Hardy himself claimed that G. Pólya had previously pointed out this inequality [14]. Also, inequality (1) has been used in several mathematics and physics fields [18, 19]. Some applications and generalizations of this inequality can be found in [14, 18, 20–22].

In the following, a generalization of inequality (1) by Ma and Guo in the field of fuzzy logic is provided.

Theorem 1.1. *([23]). Let $h, H : [0, +\infty) \to [0, +\infty)$ be strictly increasing functions and $\int_0^{+\infty} h(u)du < \infty$. Then we have*

$$⨍_0^{+\infty} H\left(\frac{1}{u}\int_0^u H^{-1}(h(t))dt\right)d\mu \leq ⨍_0^{+\infty} h(u)d\mu.$$

Also, in 2020, RomPán-Flores et al. [24] proved the following extension of inequality (1) in the field of fuzzy logic.

Theorem 1.2. *Let $h : [0, +\infty) \to [1, +\infty)$ be a Sugeno-integrable function with respect to the Lebesgue measure. Then*

$$\fint_0^{+\infty} \exp\left(\frac{1}{u}\fint_0^u \ln(h(t))dt\right) du \leq e \fint_0^{+\infty} h(u)du.$$

It is important to note that in Theorem 1.1, the inner integral is the Lebesgue integral and the outer integrals are the Sugeno integral, while in Theorem 1.2 all integrals are Sugeno integrals.

2. Pseudo-integral

Let $[c, d]$ be a closed (in some cases semi-closed) internal of $[-\infty, \infty]$ and $\preceq$ be a total ordering on $[c, d]$.

Definition 2.1. ([6, 25]). A binary operation $\oplus$ on the interval $[c, d]$ is called pseudo-addition if, for all $u_1, u_2, w \in [c, d]$,

1. $u_1 \oplus u_2 = u_2 \oplus u_1$,
2. $(u_1 \oplus u_2) \oplus w = u_1 \oplus (u_2 \oplus w)$,
3. If $u_1 \preceq u_2$, then $u_1 \oplus w \preceq u_2 \oplus w$,
4. $\mathbf{0}_\oplus \oplus u_1 = u_1$, where $\mathbf{0}_\oplus \in [c, d]$ is a neutral element.

Now, let we define $[c, d]_+ = \{u_1 : u_1 \in [c, d], \mathbf{0}_\oplus \preceq u_1\}$.

Definition 2.2. ([6, 25]). Let $\oplus$ be a given pseudo-addition on $[c, d]$. A binary operation $\otimes$ defined on $[c, d]$ is pseudo-multiplication if for all $u_1, u_2, w \in [c, d]$ and $t \in [c, d]_+$,

1. $u_1 \otimes u_2 = u_2 \otimes u_1$,
2. $(u_1 \otimes u_2) \otimes w = u_1 \otimes (u_2 \otimes w)$,
3. If $u_1 \preceq u_2$, then $u_1 \otimes t \preceq u_2 \otimes t$,
4. $(u_1 \oplus u_2) \otimes w = (u_1 \otimes w) \oplus (u_2 \otimes w)$,
5. $\mathbf{1}_\otimes \otimes u_1 = u_1$, where $\mathbf{1}_\otimes \in [c, d]$ is a neutral element.

The pseudo-operation $* : [c, d]^2 \to [c, d]$ is idempotent if for any $u_1 \in [c, d]$, $u_1 * u_1 = u_1$ holds. Clearly, the structure $([c, d], \oplus, \otimes)$ is a semicircle, see [26].

We consider special semirings with continuous operations according to the following process:

Case I The pseudo-multiplication $\otimes$ is not idempotent, and the pseudo-addition $\oplus$ is an idempotent operation.

a)

$$(i) \quad u_1 \oplus u_2 := \sup\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := u_1 + u_2,$$

on the interval $[-\infty, +\infty[$. We have $\mathbf{0}_\oplus = -\infty$ and $\mathbf{1}_\otimes = 0$.

$$(ii) \quad u_1 \oplus u_2 := \inf\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := u_1 + u_2,$$

on the interval $]-\infty, +\infty]$. We have $\mathbf{1}_\otimes = 0$ and $\mathbf{0}_\oplus = +\infty$.

b)

$$(i) \quad u_1 \oplus u_2 := \sup\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := u_1.u_2,$$

on the interval $[0, +\infty[$. We have $\mathbf{0}_\oplus = 0$ and $\mathbf{1}_\otimes = 1$.

$$(ii) \quad u_1 \oplus u_2 := \inf\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := u_1.u_2,$$

on the interval $]0, +\infty]$. We have $\mathbf{1}_\otimes = 1$ and $\mathbf{0}_\oplus = +\infty$.

Case II Both pseudo-operations $\oplus$ and $\otimes$ are not idempotent. The pseudo-operations are generated by a continuous and strictly monotone function ψ [27]. In this case, we will focus exclusively on the strict pseudo-addition $\oplus$.

By Aczel's representation theorem [28] for each strict pseudo-addition $\oplus$ there exists a continuous and strictly monotone surjective function ψ (generator for $\oplus$), $\psi : [c, d] \to [0, +\infty]$ such that $\psi(\mathbf{0}_\oplus) = 0$ and

$$u_1 \oplus u_2 := \psi^{-1}(\psi(u_1) + \psi(u_2)).$$

Using a generator ψ of a strict pseudo-addition $\oplus$ we can define a pseudo-multiplication $\otimes$ by

$$u_1 \otimes u_2 := \psi^{-1}(\psi(u_1)\psi(u_2)),$$

with the convention $0 \times (+\infty) := 0$.

Case III Both pseudo-operations $\oplus$ and $\otimes$ are idempotent.

$$(i) \quad u_1 \oplus u_2 := \sup\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := \inf\{u_1, u_2\},$$

on the interval $[-\infty, +\infty]$. We have $\mathbf{0}_\oplus = -\infty$ and $\mathbf{1}_\otimes = +\infty$.

$$(ii) \quad u_1 \oplus u_2 := \inf\{u_1, u_2\}\ ,\ u_1 \otimes u_2 := \sup\{u_1, u_2\},$$

on the interval $[-\infty, +\infty]$. We have $\mathbf{0}_\oplus = +\infty$ and $\mathbf{1}_\otimes = -\infty$.

Definition 2.3. ([1, 29]). Let U be a non-empty set and $\mathcal{F}$ be a σ-algebra of the subsets of U. The set function $m : \mathcal{F} \to [c, d]_+$ is a σ-$\oplus$-measure if

1. $m(\emptyset) = \mathbf{0}_{\oplus}$,
2. For any sequence $(E_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{E}$,

$$m(\overset{+\infty}{\underset{i=1}{\cup}} E_i) = \overset{+\infty}{\underset{i=1}{\oplus}} m(E_i) := \lim_{n\to+\infty} \overset{n}{\underset{i=1}{\oplus}} m(E_i).$$

Definition 2.4. ([1, 29]). Let that U be a non-empty set, $\mathcal{F}$ is a σ-algebra of the subsets of U and $m : \mathcal{F} \to [c,d]_+$ is a σ-$\oplus$-measure. The pseudo-integral of a bounded measurable function $f : U \to [c,d]$, where the pseudo-operations are defined by a continuous and monotone function $\psi : [c,d] \to [0,\infty]$, is defined by

$$\int_U^{\oplus} f(u) \otimes \mathrm{d}m := \psi^{-1}\left(\int_U (\psi\circ f)\ \mathrm{d}(\psi\circ m)\right).$$

If $U \subseteq [-\infty,+\infty]$ is a closed (semiclosed) interval, $\mathcal{F} = \mathcal{B}_U$ is σ-algebra of Borel subsets of U and $m = \psi^{-1}\circ\mu$ where μ represents the standard Lebesgue measure on U, then the pseudo-integral for the function f takes the following form:

$$\int_U^{\oplus} f(u) \otimes \mathrm{d}m = \psi^{-1}\left(\int_U \psi\big(f(u)\big)\mathrm{d}\mu\right). \tag{3}$$

If we consider the semiring $([c,d], \sup, \otimes)$, where $\otimes$ is a pseudo-multiplication defined by means of a generator $\psi : [c,d] \to [0,+\infty]$ and ψ is increasing bijection, the pseudo-integral of a function $f : U \to [c,d]$ has the following form:

$$\int_U^{\oplus} f \otimes \mathrm{d}m := \sup_{u\in U}\big(f(u)\otimes\phi(u)\big),$$

where $\phi : U \to [c,d]$ is a density function given by $\phi(u) = m(\{u\})$. In this case, we prefer to use the notation $\int_U^{\sup} f \otimes \mathrm{d}m$ instead of $\int_U^{\oplus} f \otimes \mathrm{d}m$.

Theorem 2.5. *([29]). Let $([0,\infty], \sup, \otimes)$ be a semiring, when $\otimes$ is generated by the increasing and continuous function ψ. Let m be sup-measure on $\big([0,\infty], \mathcal{B}_{[0,+\infty]}\big)$, where $\mathcal{B}_{[0,+\infty]}$ is σ-algebra of Borel subsets of the interval $[0,\infty]$, $m(A) = \sup\big\{c \mid \mu(\{u \mid u\in A,\ u > c\}) > 0\big\}$ and $\psi : [0,\infty] \to [0,\infty]$ be a continuous density. Then, there exists a family m_λ of $\oplus_\lambda$-measure where $\oplus_\lambda$ is generated by ψ^λ, $\lambda \in (0,\infty)$ such that for every continuous function $f : [0,\infty] \to [0,\infty]$,*

$$\begin{aligned}\int^{\sup} f \otimes \mathrm{d}m &= \lim_{\lambda\to+\infty} \int^{\oplus_\lambda} f \otimes \mathrm{d}m_\lambda \\ &= \lim_{\lambda\to+\infty} \left(\psi^\lambda\right)^{-1}\left(\int \psi^\lambda\big(f(u)\big)\mathrm{d}u\right).\end{aligned}$$

Remark 1. Consider the semiring $([0,\infty], \inf, \otimes)$, when $\otimes$ is generated by the decreasing and continuous function ψ. Similar to Theorem 2.5, the integral $\int^{\inf} f \otimes$

dm can be derived as:

$$\int^{\text{inf}} f \otimes \mathrm{d}m = \lim_{\lambda \to +\infty} \left(\psi^{\lambda}\right)^{-1} \left(\int \psi^{\lambda}\big(f(u)\big)\mathrm{d}u \right).$$

In order to present the pseudo-analytic exponential $Exp(u)$, it is necessary to introduce the pseudo-power. For $w \in [c,d]_+$ and $q \in (0,\infty)$, the pseudo-power $w_{\otimes}^{(q)}$ is defined in the following way in a few steps.

- for $n, m \in \mathbb{N}$ and $s = \frac{n}{m}$ $w_{\otimes}^{(n)} := \underbrace{w \otimes w \otimes ... \otimes w}_{n-times}$, $w_{\otimes}^{(0)} := \mathbf{1}_{\otimes}$,
  $w_{\otimes}^{(\frac{1}{m})} := \sup\left\{u \mid u_{\otimes}^{(m)} \leqslant w\right\}$, $w_{\otimes}^{(s)} = w_{\otimes}^{(\frac{n}{m})} = \left(w_{\otimes}^{(\frac{1}{m})}\right)^{(n)}$. Note that $w_{\otimes}^{(s)}$ is well defined for all rational $s \in (0,\infty)$, independently of the representation of s,
- if q is not rational, then according to the continuity of $\otimes$
$$w_{\otimes}^{(q)} := \sup\left\{w_{\otimes}^{(s)} \mid s \in]0,q[, s \in Q\right\}.$$

Obviously, if $u_1 \otimes u_2 = \psi^{-1}(\psi(u_1).\psi(u_2))$, then
$$u_{\otimes}^{(q)} = \psi^{-1}(\psi^{q}(u)).$$

On the other hand, if $\otimes$ is idempotent, then $u_{\otimes}^{(q)} = u$ for any $u \in [c,d]_+$ and $q \in (0,\infty)$.
In this paper, similar to [30] we suppose that the generator function $\psi : [0,\infty] \to [0,\infty]$ is strictly monotone, onto, $\psi(\mathbf{0}_{\oplus}) = 0$, $\psi'(u) \neq 0$ for all u , $\psi \in C^2$ and $\psi^{-1} \in C^2$. By applying this function, we shall introduce some new operations as follows: for all $u, v \in [c,d]$, and $n \in \mathbb{R}$

- Pseudo-division:
$$u \otimes^{-1} v := \psi^{-1}\left(\frac{\psi(u)}{\psi(v)}\right),$$
  provided $v \neq \mathbf{0}_{\oplus}$.
- Pseudo-scalar product:
$$n \odot u := \psi^{-1}(n.\psi(u)).$$
- Pseudo-analytic exponential:
$$Exp_{\oplus}(u) := \sum_{n=0}^{+\infty} \psi^{-1}\left(\frac{1}{n!}\right) \otimes u_{\otimes}^{(n)},$$
  that is
$$Exp_{\oplus}(u) = \psi^{-1}(\exp(\psi(u))),$$
  where $\exp(\psi(u))$ is the standard exponential function.

- Pseudo-logarithm:

$$Ln_{\oplus}(u) := \psi^{-1}(\ln(\psi(u))),$$

where $\ln(\psi(u))$ is the standard logarithm function.

Note that the pseudo-multiplication $\otimes$ and the pseudo-scalar product $\odot$ are different. Since the compatibility condition $1 \odot u = u$ is not satisfied by $\otimes$ [31].

3. Main results

We present and prove inequality generalizations (1) related to pseudo-integrals.

Theorem 3.1. *Let* $([0,+\infty), \oplus, \otimes)$ *be a semiring. Also consider the generator* $\psi : [0,+\infty) \to [0,+\infty)$ *of the pseudo-addition* $\oplus$ *such that the pseudo-multiplication* $\otimes$ *be a surjective and strictly increasing function. Then, for any* σ-$\oplus$-*measure* m, *the following inequality*

$$\begin{aligned}
&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
\leq&\psi^{-1}(e)\otimes\psi^{-1}(\int_0^{+\infty} f(u)du),
\end{aligned}\tag{4}$$

holds for any nonnegative Riemann integrable function f *on* $[0,+\infty)$ *which* $\int_0^{+\infty} f(u)du < \infty$.

Proof. According to the definition of pseudo-division and by utilizing the equality $u_1 \otimes u_2 = \psi^{-1}(\psi(u_1)\psi(u_2))$, we can apply the definition of pseudo-analytic exponential along with Equation (3) from Definition 2.4 to derive

$$\begin{aligned}
&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{\psi(\psi^{-1}(1))}{\psi(\psi^{-1}(u))}\right)\otimes\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\right)\otimes\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\psi(\psi^{-1}(\frac{1}{u})).\psi(\psi^{-1}(\int_0^u \ln(f(t))dt))\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)\right)\otimes dm \qquad (5)\\
=&\int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\psi\left(\psi^{-1}(\frac{1}{u}\int_0^u \ln(f(t))dt)\right)\right)\right)\otimes dm
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)\right) \otimes dm \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \psi\left(\psi^{-1}\left(\exp(\frac{1}{u}\int_0^u \ln(f(t))dt)\right)\right)d\mu\right) \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)d\mu\right).
\end{aligned}
$$

Using Equation (2) and the fact that ψ^{-1} is increasing, we apply the classical Carleman's inequality (1) to get

$$
\begin{aligned}
&\psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)d\mu\right) \\
&= \psi^{-1}\left(\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du\right) \\
&\leq \psi^{-1}\left(e\int_0^{+\infty} f(u)du\right) \qquad (6)\\
&= \psi^{-1}\left(\psi\left(\psi^{-1}(e)\right).\psi\left(\psi^{-1}(\int_0^{+\infty} f(u)du)\right)\right) \\
&= \psi^{-1}(e) \otimes \psi^{-1}\left(\int_0^{+\infty} f(u)du\right).
\end{aligned}
$$

Hence, combining (5) and (6) yields inequality (4). The proof is now completed. □

Example 3.2. Let $[c,d)=[0,+\infty]$. By using Theorem 3.1 we get the Carleman type inequalities.

a) Let $\psi(u)=u$. The corresponding pseudo-operations are $u_1 \otimes u_2 = u_1u_2$ and $u_1 \oplus u_2 = u_1+u_2$. The inequality (4) produces the following form:

$$
\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du \leq e\int_0^{+\infty} f(u)du,
$$

which is the same as classical Carleman's inequality (1).

b) Let $\psi(u)=u^\alpha$, $\alpha \in (1,+\infty)$. The corresponding pseudo-operations are $u_1 \oplus u_2 = \sqrt[\alpha]{u_1^\alpha + u_2^\alpha}$ and $u_1 \otimes u_2 = u_1u_2$. The inequality (4) produces can be expressed in the following form:

$$
\sqrt[\alpha]{\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du} \leq \sqrt[\alpha]{e\int_0^{+\infty} f(u)du}.
$$

c) Let $\psi(u) = \ln(u+1)$. The corresponding pseudo-operations are $u_1 \oplus u_2 = (u_1+1)(u_2+1)-1$ and $u_1 \otimes u_2 = e^{\ln(u_1+1)\ln(u_2+1)-1}$. The inequality (4) results in the following expression:

$$e^{\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du} \leq e^{e\int_0^{+\infty} f(u)du}.$$

Theorem 3.3. *In Theorem 3.1, if ψ is a strictly decreasing function instead of a strictly increasing function, we have:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm$$
$$\geq \psi^{-1}(e)\otimes\psi^{-1}\left(\int_0^{+\infty} f(u)du\right).$$

Proof. Clearly, we can give a completely similar proof as in Theorem 3.1 for this case, except that ψ is a decreasing function and reverses the direction of the inequality (6). □

Example 3.4. Let $[c,d) = [0,+\infty)$ and $\psi(u) = \frac{1}{e^u}$. The corresponding pseudo-operations are $u_1 \otimes u_2 = u_1 + u_2$ and $u_1 \oplus u_2 = \ln(\frac{e^{u_1+u_2}}{e^{u_1}+e^{u_2}})$. Using Theorem 3.3 we have:

$$\ln\left(\frac{1}{\int_0^{+\infty}\exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du}\right) \geq \ln\left(\frac{1}{e\int_0^{+\infty} f(u)du}\right).$$

Theorem 3.5. *Let $([0,+\infty),\oplus,\otimes)$ be a semiring. Also consider the generator $\psi : [0,+\infty) \to [0,+\infty)$ of the pseudo-addition $\oplus$ such that the pseudo-multiplication $\otimes$ be a surjective and strictly increasing function. Then, for any σ-$\oplus$-measure m, the following inequality*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm$$
$$\leq \psi^{-1}(e)\otimes\int_{[0,+\infty)}^{\oplus} h\otimes dm, \tag{7}$$

holds for any nonnegative continuous function h on $[0,+\infty)$ which $\int_{[0,+\infty)}^{\oplus} h\otimes dm < \infty$.

Proof. Using the definitions of pseudo-division and pseudo-logarithm, the Equation (3) of Definition 2.4, the equality $u_1\otimes u_2 = \psi^{-1}(\psi(u_1)\psi(u_2))$, and by applying the definition of pseudo-analytic exponential we have:

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm$$

$$
\begin{aligned}
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{\psi(\psi^{-1}(1))}{\psi(\psi^{-1}(u))}\right) \otimes \int_{[0,u]}^{\oplus} \psi^{-1}(\ln(\psi(h(t)))) \otimes dm\right) \otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\right) \otimes \psi^{-1}\left(\int_0^u \psi(\psi^{-1}(\ln(\psi(h(t)))))dt\right)\right) \otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\right) \otimes \psi^{-1}\left(\int_0^u \ln(\psi(h(t)))dt\right)\right) \otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\psi(\psi^{-1}(\frac{1}{u})).\psi(\psi^{-1}(\int_0^u \ln(\psi(h(t)))dt))\right)\right) \otimes dm \qquad (8) \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)\right) \otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\psi\left(\psi^{-1}(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)\right)\right) \otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)\right) \otimes dm \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \psi\left(\psi^{-1}\left(\exp(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt)\right)\right)d\mu\right) \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)d\mu\right).
\end{aligned}
$$

Since $\psi \circ h$ is a continuous nonnegative function on $[0,+\infty)$ and $\int_{[0,+\infty)}^{\oplus} h \otimes dm < \infty$, it follows from (2) that

$$
\int_0^{+\infty} \psi(h(u))du = \int_{[0,+\infty)} \psi \circ h d\mu < \infty.
$$

If we apply the classical Carleman's inequality (1) with $f = \psi \circ h$, we obtain:

$$
\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)du \le e\int_0^{+\infty} \psi(h(u))du. \qquad (9)
$$

According to Equation (2), utilizing the fact that ψ^{-1} is increasing and applying inequality (9), we obtain:

$$
\begin{aligned}
& \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)d\mu\right) \\
&= \psi^{-1}\left(\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)du\right) \\
&\le \psi^{-1}\left(e\int_0^{+\infty} \psi(h(u))du\right)
\end{aligned}
$$

$$= \psi^{-1}\left(\psi\left(\psi^{-1}(e)\right).\psi\left(\psi^{-1}(\int_0^{+\infty}\psi(h(u))du)\right)\right)$$
$$= \psi^{-1}(e)\otimes\psi^{-1}\left(\int_0^{+\infty}\psi(h(u))du\right) = \psi^{-1}(e)\otimes\int_{[0,+\infty)}^{\oplus} h\otimes dm.$$

Hence, combining the equality (8) and the inequality (10) yields inequality (7). The proof is now completed. □

Theorem 3.6. *In Theorem 3.5, if ψ is a strictly decreasing function rather than a strictly increasing, the following inequality holds:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm$$
$$\geq\psi^{-1}(e)\otimes\int_{[0,+\infty)}^{\oplus} h\otimes dm.$$

Proof. Clearly, we can give a completely similar proof as in Theorem 3.5 for this case, except that ψ is a decreasing function and reverses the direction of the inequality (10). □

Theorem 3.7. *Under the assumptions outlined in Theorem 3.1, the following inequality holds:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm \leq e\odot\psi^{-1}\left(\int_0^{+\infty} f(u)du\right). \tag{11}$$

Proof. By using the definitions of pseudo-scalar product and pseudo-analytic exponential, and applying Equation (3) of Definition 2.4, we have

$$\begin{aligned}
&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}.\psi(\psi^{-1}(\int_0^u \ln(f(t))dt))\right)\right)\otimes dm\\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)\right)\otimes dm \qquad (12)\\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\psi\left(\psi^{-1}(\frac{1}{u}\int_0^u \ln(f(t))dt)\right)\right)\right)\otimes dm\\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)\right)\otimes dm\\
&= \psi^{-1}\left(\int_{[0,+\infty)} \psi\left(\psi^{-1}\left(\exp(\frac{1}{u}\int_0^u \ln(f(t))dt)\right)\right)d\mu\right)
\end{aligned}$$

$$= \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)d\mu\right).$$

Consider Equation (2), the fact that ψ^{-1} is increasing, and the classical Carleman's inequality (1), we get:

$$\begin{aligned}
& \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)d\mu\right) \\
= & \psi^{-1}\left(\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right)du\right) \\
\leq & \psi^{-1}\left(e\int_0^{+\infty} f(u)du\right) \qquad (13)\\
= & \psi^{-1}\left(e.\psi\left(\psi^{-1}(\int_0^{+\infty} f(u)du)\right)\right) = e\odot\psi^{-1}\left(\int_0^{+\infty} f(u)du\right).
\end{aligned}$$

Hence, combining the equality (12) and the inequality (13) yields inequality (11). The proof is now completed. □

Theorem 3.8. *In Theorem 3.7, if ψ is a strictly decreasing function rather than a strictly increasing function, we have:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm \geq e\odot\psi^{-1}\left(\int_0^{+\infty} f(u)du\right),$$

Proof. Clearly, we can give a completely similar proof as in Theorem 3.7 for this case, except that ψ is a decreasing function and reverses the direction of the inequality (13). □

Theorem 3.9. *Based on the assumptions outlined in Theorem 3.5, the following inequality is valid:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm \leq e\odot\int_{[0,+\infty)}^{\oplus} h\otimes dm. \qquad (14)$$

Proof. By the definition of pseudo-logarithm, Equation (3) of Definition 2.4, and by applying the definitions of pseudo-scalar product and pseudo-analytic exponential, we have:

$$\begin{aligned}
& \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm \\
= & \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\int_{[0,u]}^{\oplus} \psi^{-1}(\ln(\psi(h(t))))\otimes dm\right)\otimes dm
\end{aligned}$$

$$\begin{aligned}
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \psi(\psi^{-1}(\ln(\psi(h(t)))))dt\right)\right)\otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(\psi(h(t)))dt\right)\right)\otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}.\psi(\psi^{-1}(\int_0^u \ln(\psi(h(t)))dt))\right)\right)\otimes dm \qquad (15)\\
&= \int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\psi^{-1}\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)\right)\otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\psi\left(\psi^{-1}(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt)\right)\right)\right)\otimes dm \\
&= \int_{[0,+\infty)}^{\oplus} \psi^{-1}\left(\exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)\right)\otimes dm \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \psi\left(\psi^{-1}\left(\exp(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt)\right)\right)d\mu\right) \\
&= \psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)d\mu\right).
\end{aligned}$$

Since $\psi\circ h$ is a continuous nonnegative function on $[0,+\infty)$ and $\int_{[0,+\infty)}^{\oplus} h\otimes dm < \infty$, it follows from (2) that

$$\int_0^{+\infty}\psi(h(u))du = \int_{[0,+\infty)}\psi\circ h d\mu < \infty.$$

Now using Equation (2), the fact that ψ^{-1} is increasing, and the inequality (9), we have

$$\begin{aligned}
&\psi^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)d\mu\right) \\
=\ &\psi^{-1}\left(\int_0^{+\infty} \exp\left(\frac{1}{u}\int_0^u \ln(\psi(h(t)))dt\right)du\right) \\
\leq\ &\psi^{-1}\left(e\int_0^{+\infty}\psi(h(u))du\right) \qquad (16)\\
=\ &\psi^{-1}\left(e.\psi\left(\psi^{-1}(\int_0^{+\infty}\psi(h(u))du)\right)\right) \\
=\ &e\odot\psi^{-1}\left(\int_0^{+\infty}\psi\left(h(u)\right)du\right) = e\odot\int_{[0,+\infty)}^{\oplus} h\otimes dm.
\end{aligned}$$

Hence, combining the equality (15) and the inequality (16) yields inequality (14). The proof is now completed. □

Theorem 3.10. *In Theorem 3.9, if ψ is a strictly decreasing function instead of a strictly increasing function, the following inequality holds:*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\frac{1}{u} \odot \int_{[0,u]}^{\oplus} Ln_{\oplus}(h(t)) \otimes dm\right) \otimes dm \geq e \odot \int_{[0,+\infty)}^{\oplus} h \otimes dm.$$

Proof. We can provide a similar proof as in Theorem 3.9 for this case, noting that ψ is a decreasing function, which reverses the direction of inequality (16). □

Theorem 3.11. *Let* $([0,\infty), \sup, \otimes)$ *be a semiring. Also consider the generator* $\psi : [0,+\infty) \to [0,+\infty)$ *of the pseudo-addition* $\oplus$ *such that the pseudo-multiplication* $\otimes$ *be a surjective and strictly increasing function. If m is a complete* sup-*measure on* $[0,\infty)$*, then*

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\left(\psi^{-1}(1) \otimes^{-1} \psi^{-1}(u)\right) \otimes \psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right) \otimes dm$$
$$\leq \lim_{\lambda\to\infty} \left(\psi^{\lambda}\right)^{-1}(e) \otimes \lim_{\lambda\to\infty} \left(\psi^{\lambda}\right)^{-1}\left(\int_0^{+\infty} f(u)du\right),$$

holds for any nonnegative continuous function f on $[0,+\infty)$ *which* $\int_0^{+\infty} f(u)du < \infty$.

Proof. By Theorem 2.5, there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $\psi^\lambda, \lambda \in (0,\infty)$, such that

$$\begin{aligned} & \int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\left(\psi^{-1}(1) \otimes^{-1} \psi^{-1}(u)\right) \otimes \psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right) \otimes dm \\ = & \lim_{\lambda\to\infty} \int_{[0,+\infty)}^{\oplus_\lambda} Exp_{\oplus}\left(\left(\left(\psi^{\lambda}\right)^{-1}(1) \otimes^{-1} \left(\psi^{\lambda}\right)^{-1}(u)\right) \otimes \left(\psi^{\lambda}\right)^{-1}\left(\int_0^u \ln(f(t))dt\right)\right) \otimes dm_\lambda \\ = & \lim_{\lambda\to\infty} \left(\psi^{\lambda}\right)^{-1}\left(\int_{[0,+\infty)} \psi^{\lambda}\left(Exp_{\oplus}\left(\left(\left(\psi^{\lambda}\right)^{-1}(1) \otimes^{-1} \left(\psi^{\lambda}\right)^{-1}(u)\right) \otimes \left(\psi^{\lambda}\right)^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\right) d\mu\right). \end{aligned} \tag{17}$$

In an analogous way as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} & \int_{[0,+\infty)}^{\oplus_\lambda} Exp_{\oplus}\left(\left(\left(\psi^{\lambda}\right)^{-1}(1) \otimes^{-1} \left(\psi^{\lambda}\right)^{-1}(u)\right) \otimes \left(\psi^{\lambda}\right)^{-1}\left(\int_0^u \ln(f(t))dt\right)\right) \otimes dm_\lambda \\ = & \left(\psi^{\lambda}\right)^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right) d\mu\right). \end{aligned}$$

Now by applying Equation (2), the fact that $\left(\psi^{\lambda}\right)^{-1}$ is increasing, and the classical Carleman's inequality (1), we obtain:

$$\left(\psi^{\lambda}\right)^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{u}\int_0^u \ln(f(t))dt\right) d\mu\right) \leq \left(\psi^{\lambda}\right)^{-1}(e) \otimes \left(\psi^{\lambda}\right)^{-1}\left(\int_0^{+\infty} f(u)du\right). \tag{18}$$

Tending λ to $+\infty$ in (18), we have

$$\lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} \left(\int_{[0,+\infty)} \exp\left(\frac{1}{u} \int_0^u \ln(f(t))dt \right) d\mu \right) \tag{19}$$
$$\leq \lim_{\lambda\to\infty} \left((\psi^{\lambda})^{-1} (e) \otimes (\psi^{\lambda})^{-1} (\int_0^{+\infty} f(u)du) \right).$$

Thus, applying Theorem 2.5, combining (17) and (19) and by the continuity of $\otimes$, we get

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus} \left((\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)) \otimes \psi^{-1} \left(\int_0^u \ln(f(t))dt \right) \right) \otimes dm$$
$$\leq \lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} (e) \otimes \lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} \left(\int_0^{+\infty} f(u)du \right).$$

The proof is now completed. □

Remark 2. The inequality dependent on inf-measure (Remark 1) can be obtained in a completely similar way to Theorem 3.11 as follows:

$$\int_{[0,+\infty)}^{\inf} Exp_{\oplus} \left((\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)) \otimes \psi^{-1} \left(\int_0^u \ln(f(t))dt \right) \right) \otimes dm$$
$$\geq \lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} (e) \otimes \lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} \left(\int_0^{+\infty} f(u)du \right),$$

where the generator $\psi : [0,+\infty) \to [0,\infty)$ of the pseudo-multiplication $\otimes$ is a surjective and strictly decreasing function, f is nonnegative continuous function on $[0,+\infty)$ which $\int_0^{+\infty} f(u)du < \infty$, and m is a complete inf-measure on $[0,\infty)$.

4. Further results

Theorem 4.1. *Let* $([0,\infty), \sup, \otimes)$ *be a semiring. Also consider the generator* $\psi : [0,+\infty) \to [0,+\infty)$ *of the pseudo-addition* $\oplus$ *such that the pseudo-multiplication* $\otimes$ *be a surjective and strictly increasing function. If* m *is a complete* sup*-measure on* $[0,\infty)$*, then*

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus} \left((\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)) \otimes \int_{[0,u]}^{\sup} Ln_{\oplus}(h(t)) \otimes dm \right) \otimes dm$$
$$\leq \lim_{\lambda\to\infty} (\psi^{\lambda})^{-1} (e) \otimes \int_{[0,+\infty)}^{\sup} h \otimes dm,$$

holds for any nonnegative continuous function h *on* $[0,+\infty)$ *which* $\int_{[0,+\infty)}^{\sup} h\otimes dm < \infty$.

Proof. By Theorem 2.5, there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is a generated by $\psi^\lambda, \lambda \in (0,\infty)$, such that

$$\begin{aligned}
&\int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\sup} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm \\
=\ &\lim_{\lambda\to\infty}\int_{[0,+\infty)}^{\oplus_\lambda} Exp_{\oplus}\left(\left(\left(\psi^\lambda\right)^{-1}(1)\otimes^{-1}\left(\psi^\lambda\right)^{-1}(u)\right)\otimes\lim_{\lambda\to\infty}\int_{[0,u]}^{\oplus_\lambda} Ln_{\oplus}(h(t))\otimes dm_\lambda\right)\otimes dm_\lambda \qquad (20)\\
=\ &\lim_{\lambda\to\infty}\left(\psi^\lambda\right)^{-1}\left(\int_{[0,+\infty)}\psi^\lambda\left(Exp_{\oplus}\left(\left(\left(\psi^\lambda\right)^{-1}(1)\otimes^{-1}\left(\psi^\lambda\right)^{-1}(u)\right)\otimes\left(\psi^\lambda\right)^{-1}\left(\int_0^u\psi^\lambda(Ln_{\oplus}(h(t)))dt\right)\right)\right)d\mu\right).
\end{aligned}$$

Like as in the proof of Theorem 3.5, we obtain

$$\begin{aligned}
&\int_{[0,+\infty)}^{\oplus_\lambda} Exp_{\oplus}\left(\left(\left(\psi^\lambda\right)^{-1}(1)\otimes^{-1}\left(\psi^\lambda\right)^{-1}(u)\right)\otimes\int_{[0,u]}^{\oplus_\lambda} Ln_{\oplus}(h(t))\otimes dm_\lambda\right)\otimes dm_\lambda \\
=\ &\left(\psi^\lambda\right)^{-1}\left(\int_{[0,+\infty)}\exp\left(\frac{1}{u}\int_0^u\ln(\psi^\lambda(h(t)))dt\right)d\mu\right).
\end{aligned}$$

Since $\psi^\lambda\circ h$ for any $\lambda\in(0,+\infty)$ is nonnegative continuous function on $[0,+\infty)$ and $\int_{[0,+\infty)}^{\oplus_\lambda} h\otimes dm_\lambda<\infty$, it follows from (2) that

$$\int_0^{+\infty}\psi^\lambda(h(u))du=\int_{[0,+\infty)}\psi^\lambda\circ h d\mu<\infty.$$

We apply now the classical Carleman's inequality (1) with $f=\psi^\lambda\circ h$. Then we obtain:

$$\int_0^{+\infty}\exp\left(\frac{1}{u}\int_0^u\ln(\psi^\lambda(h(t)))dt\right)du\le e\int_0^{+\infty}\psi^\lambda(h(u))du.$$

Now by Equation (2), using the fact that $\left(\psi^\lambda\right)^{-1}$ is increasing and applying the classical Carleman's inequality (1), we have:

$$\begin{aligned}
&\left(\psi^\lambda\right)^{-1}\left(\int_{[0,+\infty)}\exp\left(\frac{1}{u}\int_0^u\ln(\psi^\lambda(h(t)))dt\right)d\mu\right)\\
&\le\left(\psi^\lambda\right)^{-1}(e)\otimes\left(\psi^\lambda\right)^{-1}\left(\int_0^{+\infty}\left(\psi^\lambda\right)^{-1}(h(u))du\right).
\end{aligned}\qquad (21)$$

As we approach λ to $+\infty$ in Equation (21), we get:

$$\lim_{\lambda\to\infty}\left(\psi^\lambda\right)^{-1}\left(\int_{[0,+\infty)}\exp\left(\frac{1}{u}\int_0^u\ln(\psi^\lambda(h(t)))dt\right)d\mu\right)\qquad (22)$$

$$\leq \lim_{\lambda\to\infty}\left(\left(\psi^{\lambda}\right)^{-1}(e)\otimes\left(\psi^{\lambda}\right)^{-1}\left(\int_0^{+\infty}\left(\psi^{\lambda}\right)^{-1}(h(u))du\right)\right).$$

By applying Theorem 2.5 and combining Equations (20) and (22), along with the by the continuity of $\otimes$, we obtain:

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\sup} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm$$
$$\leq \lim_{\lambda\to\infty}\left(\psi^{\lambda}\right)^{-1}(e)\otimes\int_{[0,+\infty)}^{\sup} h\otimes dm.$$

This completes the proof. □

Remark 3. The inequality dependent on inf-measure (Remark 1) can be obtained in a completely similar way to Theorem 4.1 as follows:

$$\int_{[0,+\infty)}^{\inf} Exp_{\oplus}\left(\left(\psi^{-1}(1)\otimes^{-1}\psi^{-1}(u)\right)\otimes\int_{[0,u]}^{\inf} Ln_{\oplus}(h(t))\otimes dm\right)\otimes dm$$
$$\geq \lim_{\lambda\to\infty}\left(\psi^{\lambda}\right)^{-1}(e)\otimes\int_{[0,+\infty)}^{\inf} h\otimes dm,$$

where the generator $\psi:[0,+\infty)\to[0,\infty)$ of the pseudo-multiplication $\otimes$ is a surjective and strictly decreasing function, h is nonnegative continuous function on $[0,+\infty)$ which $\int_{[0,+\infty)}^{\inf} h\otimes dm<\infty$, m is a complete inf-measure on $[0,\infty)$.

Theorem 4.2. *Let* $([0,\infty),\sup,\otimes)$ *be a semiring. Also consider the generator* $\psi:[0,+\infty)\to[0,+\infty)$ *of the pseudo-addition* $\oplus$ *such that the pseudo-multiplication* $\otimes$ *be a surjective and strictly increasing function. If* m *is a complete* sup*-measure on* $[0,\infty)$*, then*

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm$$
$$\leq e\odot\lim_{\lambda\to\infty}\left(\psi^{\lambda}\right)^{-1}\left(\int_0^{+\infty} f(u)du\right),$$

holds for any nonnegative continuous function f *on* $[0,+\infty)$ *which* $\int_0^{+\infty} f(u)du<\infty$.

Proof. Clearly, we can give a completely similar proof as in Theorem 3.11 for this case. □

Remark 4. The inequality dependent on inf-measure (Remark 1) can be obtained in a completely similar way to Theorem 4.2 as follows:

$$\int_{[0,+\infty)}^{\inf} Exp_{\oplus}\left(\frac{1}{u}\odot\psi^{-1}\left(\int_0^u \ln(f(t))dt\right)\right)\otimes dm$$

$$\geq e \odot \lim_{\lambda\to\infty} \left(\psi^{\lambda}\right)^{-1} \left(\int_0^{+\infty} f(u)du\right),$$

where the generator $\psi : [0,+\infty) \to [0,\infty)$ of the pseudo-multiplication $\otimes$ is a surjective and strictly decreasing function, f is nonnegative continuous function on $[0,+\infty)$ which $\int_0^{+\infty} f(u)du < \infty$, and m is a complete inf-measure on $[0,\infty)$.

Theorem 4.3. *Let* $([0,\infty), \sup, \otimes)$ *be a semiring. Also consider the generator* $\psi : [0,+\infty) \to [0,+\infty)$ *of the pseudo-addition* $\oplus$ *such that the pseudo-multiplication* $\otimes$ *be a surjective and strictly increasing function. If* m *is a complete* sup*-measure on* $[0,\infty)$*, then*

$$\int_{[0,+\infty)}^{\sup} Exp_{\oplus}\left(\frac{1}{u} \odot \int_{[0,u]}^{\sup} Ln_{\oplus}(h(t)) \otimes dm\right) \otimes dm \leq e \odot \int_{[0,+\infty)}^{\sup} h \otimes dm,$$

holds for any nonnegative continuous function h *on* $[0,+\infty)$ *which* $\int_{[0,+\infty)}^{\sup} h \otimes dm < \infty$*.*

Proof. Clearly, we can give a completely similar proof as in Theorem 4.1 for this case. □

Remark 5. The inequality dependent on inf-measure (Remark 1) can be obtained in a completely similar way to Theorem 4.3 as follows:

$$\int_{[0,+\infty)}^{\inf} Exp_{\oplus}\left(\frac{1}{u} \odot \int_{[0,u]}^{\inf} Ln_{\oplus}(h(t)) \otimes dm\right) \otimes dm \geq e \odot \int_{[0,+\infty)}^{\inf} h \otimes dm,$$

where the generator $\psi : [0,+\infty) \to [0,\infty)$ of the pseudo-multiplication $\otimes$ is a surjective and strictly decreasing function, h is nonnegative continuous function on $[0,+\infty)$ which $\int_{[0,+\infty)}^{\inf} h \otimes dm < \infty$, and m is a complete inf-measure on $[0,\infty)$.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] E. Pap, Pseudo-additive measures and their aplications, *Handbook of Measure Theory* **II** (2002) 1403 − 1468, https://doi.org/10.1016/B978-044450263-6/50036-1.

[2] E. Pap, Generalized real analysis and its applications, *Internat. J. Approx. Reason.* **47** (2008) 368 − 386, https://doi.org/10.1016/j.ijar.2007.05.015.

[3] E. Pap, M. Štrboja and I. Rudas, Pseudo-L^p space and convergence, *Fuzzy Sets and Systems* **238** (2014) 113 – 128, https://doi.org/10.1016/j.fss.2013.06.010.

[4] M. Takács, Approximate reasoning in fuzzy systems based on pseudo-analysis and uninorm residuum, *Acta Polytech. Hung.* **1** (2004) 49 – 62.

[5] J. -L. Marichal, The influence of variables on pseudo-Boolean functions with applications to game theory and multicriteria decision making, *Discrete Appl. Math.* **107** (2000) 139 – 164, https://doi.org/10.1016/S0166-218X(00)00264-X.

[6] E. Pap and N. Ralević, Pseudo-Laplace transform, *Nonlinear Anal.* **33** (1998) 533 – 550, https://doi.org/10.1016/S0362-546X(97)00568-3.

[7] R. J. Donovan, D. R. Hicks, J. A. Kryka, D. J. Lambert and R. R. Roediger, Optimizing apparatus and method for defining visibility boundaries in compiled code, U.S. Patent 6,090,155, issued July 18, 2000.

[8] M. Sugeno and T. Murofushi, Pseudo-additive measures and integrals, *J. Math. Anal. Appl.* **122** (1987) 197 – 222, https://doi.org/10.1016/0022-247X(87)90354-4.

[9] E. Pap and M. Štrboja, Generalization of the Jensen inequality for pseudo-integral, *Inform. Sci.* **180** (2010) 543 – 548, https://doi.org/10.1016/j.ins.2009.10.014.

[10] S. Abbaszadeh, A. Ebadian and M. Jaddi, Hölder type integral inequalities with different pseudo-operations, *Asian-Eur. J. Math.* **12** (2019) #1950032, https://doi.org/10.1142/S1793557119500323.

[11] S. Abbaszadeh and A. Ebadian, Nonlinear integrals and Hadamard-type inequalities, *Soft Comput.* **22** (2018) 2843 – 2849, https://doi.org/10.1007/s00500-017-2776-3.

[12] H. Agahi, R. Mesiar and Y. Ouyang, Chebyshev type inequalities for pseudo-integrals, *Nonlinear Anal.* **72** (2010) 2737 – 2743, https://doi.org/10.1016/j.na.2009.11.017.

[13] H. Agahi, Y. Ouyang, R. Mesiar, E. Pap and M. Štrboja, Hölder and Minkowski type inequalities for pseudo-integral, *Appl. Math. Comput.* **217** (2011) 8630 – 8639, https://doi.org/10.1016/j.amc.2011.03.100.

[14] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, London, 1964.

[15] H. S. Bear, *A Primer of Lebesgue Integration*, Academic Press, 2002.

[16] R. G. Bartle, *The Elements of Integration and Lebesgue Measure*, John Wiley & Sons, 2014.

[17] K. Knopp, Uber reihen mit positivern gliedern, *J. Lond. Math. Soc.* **3** (1928) 205 – 211, https://doi.org/10.1112/jlms/s1-3.3.205.

[18] M. L. Johansson, L. E. Persson and A. Wedestig, Carleman's inequality - history, proofs and some new generalizations, *J. Inequal. Pure Appl. Math.* **4** (2003) #53.

[19] A. Belghazi, F. Smadhi, N. Zaidi and O. Zair, Carleman inequalities for the heat equation in singular domains, *C. R. Math.* **348** (2010) 277 – 282, https://doi.org/10.1016/j.crma.2010.02.007.

[20] M. Z. Sarikaya and H. Yildirim, Some Hardy type integral inequalities, *J. Inequal. Pure Appl. Math.* **7** (2006) #178.

[21] S. Kaijser, L. -E. Persson and A. Öberg, On Carleman's and Knopp's inequalities, *J. Approx. Theory* **117** (2002) 140 – 151, https://doi.org/10.1006/jath.2002.3684.

[22] J. Pečarić and K. B. Stolarsky, Carleman's inequality: history and new generalizations, *Aequationes Math.* **61** (2001) 49 – 62, https://doi.org/10.1007/s000100050160.

[23] Y. X. Ma and T. F. Guo, A Carleman type inequality for sugeno integral, *Adv. Mater. Res.* **694** (2013) 2874 – 2876, https://doi.org/10.4028/www.scientific.net/AMR.694-697.2874.

[24] H. Román-Flores, A. Flores-Franulič, I. Aguirre-Cipe and M. Romero-MartÍnez, A Sugeno integral inequality of Carleman-Knopp type and some refinements, *Fuzzy Sets and Systems* **396** (2020) 72 – 81, https://doi.org/10.1016/j.fss.2019.05.013.

[25] P. Benvenuti, R. Mesiar and D. Vivona, Monotone set functions-based integrals, *Handbook of Measure Theory* eds. E. Pap (Elsevier, Amsterdam, (2002) 1329 – 1379, https://doi.org/10.1016/B978-044450263-6/50034-8.

[26] W. Kuich and A. Salomaa, *Semirings, Automata, Languages*, Springer-Verlag, Berlin, 1986.

[27] E. Pap, *Null-Additive Set Functions*, Springer Dordrecht, London, 1995.

[28] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966.

[29] R. Mesiar and E. Pap, Idempotent integral as limit of g-integrals, *Fuzzy Sets and Systems* **102** (1999) 385 – 392, https://doi.org/10.1016/S0165-0114(98)00213-9.

[30] P. Jain, Classical inequalities for pseudo-integral, *Georgian Math. J.* **29** (2022) 373 − 385, https://doi.org/10.1515/gmj-2021-2136.

[31] A. Boccuto and D. Candeloro, Differential calculus in Riesz spaces and applications to g-calculus, *Mediterr. J. Math.* **8** (2011) 315 − 329, https://doi.org/10.1007/s00009-010-0072-x.

Mohammadreza Yasamian
Department of Basic Sciences,
Technical and Vocational University (TVU),
Tehran, Iran
e-mail: myasamian@yahoo.com

Mohsen Jaddi
Department of Basic Sciences,
Technical and Vocational University (TVU),
Tehran, Iran
e-mail: mohsenjaddi@gmail.com