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Gyronormed Function Spaces

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Abstract

In this paper, we introduce a gyrodistance on the gyrolinear space of functions (whose gyronorm is measurable) from a measure space to the Möbius disk $\mathbb{D}$. The gyrodistance in question can be expressed in terms of a modification of the Lebesgue integral, which we will call the Lebesgue gyrointegral. The gyronormed space generated in this manner, which we will call the L^1 gyrospace, is similar to the familiar L^1 function space in many aspects. We establish several properties of the latter, showing that many of them mirror those of classical L^1 spaces. Finally, we show that the gyrodistance in question induces a metric topology on the L^1 gyrospace.

Keywords: Gyrovector spaces, Gyrogroups, Gyrolinear spaces, Function gyrovector spaces.

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1. Introduction

Gyrogroups were introduced for the first time by Abraham A. Ungar in [1] and [2]; the concept arose naturally from the study of the nonassociative noncommutative algebraic structure of 3-dimensional relativistically admissible velocities, $(\mathbb{R}^3_c, \oplus)$, where here $\mathbb{R}^3_c$ denotes the set

$$\mathbb{R}^3_c = \{\mathbf{x} \in \mathbb{R}^3;\ |\mathbf{x}| < c\},$$

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and $\oplus$ denotes the relativistic velocity composition law given by the formula

$$\mathbf{x} \oplus \mathbf{y} = \frac{\mathbf{x} + \mathbf{y}}{1 + \frac{\mathbf{x} \cdot \mathbf{y}}{c^2}} + \frac{\gamma_{\mathbf{x}}}{c^2(\gamma_{\mathbf{x}} + 1)} \frac{\mathbf{x} \times (\mathbf{x} \times \mathbf{y})}{1 + \frac{\mathbf{x} \cdot \mathbf{y}}{c^2}},$$

inside of which, in turn, c represents the speed of light, $\cdot$ represents the canonical dot product of vectors, and $\times$ the canonical cross product of vectors. For two nonparallel velocities $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3_c$, it was observed that even though the velocities $\mathbf{x} \oplus \mathbf{y}$ and $\mathbf{y} \oplus \mathbf{x}$ were not equal, they had the same magnitude. The operator that Ungar employed to "fix" the nonassociativity and noncommutativity of the loop structure of $(\mathbb{R}^3_c, \oplus)$ was the unique rotation that transformed $\mathbf{y} \oplus \mathbf{x}$ into $\mathbf{x} \oplus \mathbf{y}$ by a rotation about a screw axis parallel to $\mathbf{x} \times \mathbf{y}$. This operator was called the Thomas rotation:

$$\mathrm{tom}[\mathbf{x}, \mathbf{y}] : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3_c.$$

The following weak-associative and weak-commutative relations in terms of the Thomas rotation operator were obtained by Ungar:

$$\mathbf{x} \oplus \mathbf{y} = \mathrm{tom}[\mathbf{x}, \mathbf{y}](\mathbf{y} \oplus \mathbf{x}), \tag{1}$$

$$\mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z}) = (\mathbf{x} \oplus \mathbf{y}) \oplus \mathrm{tom}[\mathbf{x}, \mathbf{y}]\mathbf{z}, \tag{2}$$

$$(\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = \mathbf{x} \oplus (\mathbf{y} \oplus \mathrm{tom}[\mathbf{y}, \mathbf{x}]\mathbf{z}). \tag{3}$$

Equation (1) was called the weak commutative law, whereas Equations (2) and (3) were called the right associative law, and the left associative law, respectively. It was soon noticed [3] that an analogous behaviour was exhibited by the automorphisms of the complex unit disk $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$, i.e., Möbius transformations of $\mathbb{D}$.

More precisely, by introducing the Möbius addition, as in [3] and [4, p.2], defined by the formula

$$z \oplus w = \frac{z + w}{1 + \bar{z}w} \ ,$$

and by defining an operator called gyration

$$\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} \ ,$$

the structure we obtain is weakly associative and weakly commutative, satisfying the following identities:

$$z \oplus w = \mathrm{gyr}[z, w](w \oplus z),$$

$$z \oplus (w \oplus u) = (z \oplus w) \oplus \mathrm{gyr}[z, w]u,$$

$$(z \oplus w) \oplus u = z \oplus (w \oplus \mathrm{gyr}[w, z]u).$$

This analogy was the foundation for the subsequent generalization of the concept; gyrogroups, which were introduced in [5], provided a more general framework for describing the behaviour of both Möbius transformations of $\mathbb{D}$ and relativistically admissible velocities.

From the notion of gyrogroup then emerged gyrovector spaces, in [6] and [4].

Gyrogroups and gyrovector spaces soon started to be interesting, other than from an algebraic perspective, from a geometric perspective as well; they provided a powerful framework for the study of hyperbolic geometry.

Several "gyroequivalents" of well-known notions were given: gyrolines [4, p.62], gyrometrics [4, p.61], gyrotranslations [4, p.64], gyromidpoints [4, p.69], and many more.

In this article, we will introduce a generalization of L^1 function spaces, called L^1 gyrospaces. Many of the properties of the classical L^1 vector space have "gyroanalogues" in L^1 gyrospaces. For example, the L^1 space is a normed vector space, and similarly, the L^1 gyrospace is a gyronormed gyrolinear space. Furthermore, both possess a natural metric topology induced by their underlying algebraic structures. Introducing a "gyroequivalent" of L^1 spaces is of interest, as such a structure can be used to quantitatively compare functions in a way that is intrinsically hyperbolic, just like classical L^1 spaces allow us to quantitatively compare functions (via a metric function) in a "Euclidean setting". L^1 spaces also play an important role in various fields, such as machine learning and statistics. In particular, the L^1 norm is used to improve the prediction accuracy and the interpretability of regression models, via feature selection (i.e., the removal of certain covariates, thus yielding a simplified model). The technique in question is known as Lasso (also known as L^1 regularization), which was originally introduced in [7], while the name was coined in [8]. Among other things, Lasso has been applied in economics and finance as well [9].

Furthermore, the L^1 norm has found applications in dynamical systems theory and ergodic theory [10], due to the properties of L^1 contractions and dilations, as shown in [11].

2. Preliminaries

In this section, we will recall some preliminary notions we will use throughout this paper. We start by presenting the definition of a gyrogroup [4, p.6].

Definition 2.1 (Gyrogroup)**.** A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms:

1. There is an identity element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$;
2. For any $a \in G$, there exists an element $\ominus a \in G$ such that $\ominus a \oplus a = 0$;
3. For any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a, b]c \in G$ such that $\oplus$ obeys the left gyroassociative law:

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$$

4. The map $\text{gyr}[a,b] : G \to G$ is an automorphism of the groupoid $(G, \oplus)$, i.e. $\text{gyr}[a,b] \in Aut(G, \oplus)$ for all $a, b \in G$. The map $\text{gyr}[a,b]$ is called the gyration of G generated by a, b;

5. The gyration $\text{gyr}[a,b]$ generated by any $a, b \in G$ obeys the left loop property:

$$\text{gyr}[a,b] = \text{gyr}[a \oplus b, b].$$

The operator $\text{gyr} : G \times G \to Aut(G, \oplus)$ is called the gyrator of G. A gyrogroup which has the following property (gyrocommutativity)

$$a \oplus b = \text{gyr}[a,b](b \oplus a),$$

for all $a, b \in G$ will be called a gyrocommutative gyrogroup.
Some gyrocommutative gyrogroups admit scalar multiplication; this gives rise to gyrovector spaces, just like abelian groups with a scalar multiplication give rise to vector spaces. Let us give the full definition of a gyrovector space [4, p.55-56].

Definition 2.2 (Gyrovector space)**.** A gyrovector space $(X, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(X, \oplus)$, with the addition of a scalar multiplication $\oplus : \mathbb{R} \times X \to X$ that obeys the following axioms:

1. X is a subset of a real inner product space V, which we will call the carrier of X, $X \subset V$.

2. X inherits the inner product and the norm of V; both are invariant under gyrations i.e.

$$\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \quad \forall \mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in X,$$

3. $1 \otimes \mathbf{a} = \mathbf{a}$,

4. $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$,

5. $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$,

6. $|r| \otimes \mathbf{a} / \|r \otimes \mathbf{a}\| = \mathbf{a} / \|\mathbf{a}\|$, $\mathbf{a} \neq \mathbf{0}, r \neq 0$,

7. $\text{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}$,

8. $\text{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = id_X$,

9. The set $\pm\|X\| := \{\pm\|\mathbf{a}\| : \mathbf{a} \in X\}$ forms a one-dimensional vector space under operations $\oplus', \otimes'$, with the following two properties:

$$\|r \otimes \mathbf{a}\| = |r| \otimes' \|\mathbf{a}\|,$$

$$\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus' \|\mathbf{b}\|,$$

for all $r, r_1, r_2 \in \mathbb{R}$ and for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in X$.

The requirement imposed on the carrier space V in the above definition is rather strong; for this reason, the notion of gyrovector space was generalized in [12], by letting V be just a real normed space rather than an inner product space; these new objects were called generalized gyrovector spaces.
Further generalizations of the concept of a gyrovector space were presented in [13]; we hereby present their definitions.

Definition 2.3 (Gyrolinear space)**.** Let $(X, \oplus)$ be a gyrocommutative gyrogroup. Let $\otimes$ be a map $\otimes : \mathbb{R} \times X \to X$. We say that $(X, \oplus, \otimes)$ is a gyrolinear space if it satisfies the following axioms:

1. $1 \otimes \mathbf{x} = \mathbf{x}$,
2. $(r_1 + r_2) \otimes \mathbf{x} = (r_1 \otimes \mathbf{x}) \oplus (r_2 \otimes \mathbf{x})$,
3. $(r_1 r_2) \otimes \mathbf{x} = r_1 \otimes (r_2 \otimes \mathbf{x})$,
4. $\mathrm{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{x}) = r \otimes \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{x}$,
5. $\mathrm{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = id_X$,

For all r_1, r_2, r in $\mathbb{R}$ and all $\mathbf{x}, \mathbf{u}, \mathbf{v}$ in X.

Definition 2.4 (Normed gyrolinear space)**.** Let $(X, \oplus, \otimes)$ be a gyrolinear space and let $\|\cdot\|$ be a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$. Let ϕ be a strictly monotone increasing bijection, $\phi : \|X\| \to \mathbb{R}_{\geq 0}$ where $\|X\| := \{\|\mathbf{x}\| \in \mathbb{R}_{\geq 0} : \mathbf{x} \in X\}$. We say that $(X, \oplus, \otimes, \|\cdot\|, \phi)$ is a normed gyrolinear space if it satisfies the following conditions:

1. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{e}$,
2. $\phi(\|\mathbf{x} \oplus \mathbf{y}\|) \leq \phi(\|\mathbf{x}\|) + \phi(\|\mathbf{y}\|)$,
3. $\phi(\|r \otimes \mathbf{x}\|) = |r| f(\|\mathbf{x}\|)$,
4. $\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{x}\| = \|\mathbf{x}\|$,

for any $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in X$ and any $r \in \mathbb{R}$.

We conclude this preliminary section with some elucidations regarding the notation we will employ throughout this document. First, we will use the symbols $\oplus$ and $\otimes$ to denote both the gyrovector space operations on gyrovectors and the operations on the 1-dimensional real vector space $\pm\|V\|$ of their norms (and their negations). This is done since it will be evident from the context when we will use which, and also since when we will be working with the Möbius gyrovector space the operation on norms will just be the Möbius addition restricted to real numbers (strictly) between -1 and 1.
In the following section, we will only work with the Möbius normed gyrovector

space $(\mathbb{D}, \oplus, \otimes, \|\cdot\|, \operatorname{arctanh})$, so our considerations will be restricted to the latter. We recall that our Möbius addition is given by

$$u \oplus v = \frac{u+v}{1+\overline{u}v}, \qquad u, v \in \mathbb{D}.$$

We will use the terms "gyrosum" and "Möbius addition" interchangeably.
Our Möbius gyroscalar multiplication is given by

$$\lambda \otimes v = \tanh(\lambda \operatorname{arctanh}(\|v\|)) \frac{v}{\|v\|}, \qquad v \in \mathbb{D}, \quad \lambda \in \mathbb{R}.$$

Möbius addition is not associative and not commutative, but rather gyroassociative and gyrocommutative; however, Möbius addition restricted to $\|\mathbb{D}\| := (-1, 1)$ is both associative and commutative, as we can readily verify:

$$x \oplus y = \frac{x+y}{1+\overline{x}y} = \frac{x+y}{1+xy} = \frac{y+x}{1+\overline{y}x} = y \oplus x,$$

$$x \oplus (y \oplus z) = \frac{x + \frac{y+z}{1+yz}}{1 + x\frac{y+z}{1+yz}} = \frac{\frac{x+y+z+xyz}{1+yz}}{\frac{1+yz+xy+xz}{1+yz}} = \frac{x+y+z+xyz}{1+yz+xy+xz},$$

$$(x \oplus y) \oplus z = \frac{\frac{x+y}{1+xy} + z}{1 + \frac{x+y}{1+xy}z} = \frac{\frac{x+y+z+xyz}{1+xy}}{\frac{1+xy+xz+yz}{1+xy}} = \frac{x+y+z+xyz}{1+xy+xz+yz} = x \oplus (y \oplus z),$$

for any $x, y, z \in \|\mathbb{D}\|$. More precisely, as we also stated earlier, $(\|\mathbb{D}\|, \oplus, \otimes)$ will have a vector space structure, and $\dim_{\mathbb{R}}(\|\mathbb{D}\|) = 1$. The associativity of $\oplus$ on the norms (and their negations) allows us to write expressions like

$$\bigoplus_{i=1}^{n} x_i = x_1 \oplus x_2 \oplus \cdots \oplus x_n, \qquad x_i \in \|\mathbb{D}\|,$$

unambiguously. Furthermore, we will use

$$\bigoplus_{n=1}^{\infty} x_n := \lim_{k \to \infty} \bigoplus_{n=1}^{k} x_n, \qquad x_n \in \|D\|, \quad \forall n \in \mathbb{N},$$

to denote limits of partial gyrosums. If such a limit exists, we will write

$$\bigoplus_{n=1}^{\infty} x_n = l \in \|\mathbb{D}\|.$$

We will also employ the following convention; given a function f from a measurable space (X, Σ) to the extended real line $\overline{\mathbb{R}}$ we will say f is a Borel function if and only if $f^{-1}(\infty), f^{-1}(-\infty) \in \Sigma$ and

$$f^{-1}(A) \in \Sigma, \quad \forall A \in \mathcal{B}(\mathbb{R}),$$

where here $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma algebra of $\mathbb{R}$.
The last fact we will notice before we move on is the following:

$$\frac{\partial}{\partial x}(x \oplus y) = \frac{\partial}{\partial x}\left(\frac{x+y}{1+xy}\right) = \frac{1+xy-(x+y)y}{(1+xy)^2} = \frac{1-y^2}{(1+xy)^2}.$$

Since $(1+xy)^2 > 0$ for all $x, y \in (-1,1)$, we will have that $\partial_x(x \oplus y) > 0$ if and only if $1 - y^2 > 0$, that is:

$$\frac{\partial}{\partial x}(x \oplus y) > 0 \Leftrightarrow y \in (-1,1). \tag{4}$$

Similarly:

$$\frac{\partial}{\partial y}(x \oplus y) > 0 \Leftrightarrow x \in (-1,1). \tag{5}$$

Thus, just like our regular sum on real numbers $+$, $\oplus$ is a "lexicographically increasing binary operation" on $(-1,1)$, that is, it is increasing with respect to x while keeping y fixed and viceversa.

3. The Lebesgue gyrointegral

Let f be a function from a measure space (X, Σ, μ) to the Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes, \|\cdot\|, \text{arctanh})$. If $\|f\| : (X, \Sigma, \mu) \to [0,1]$ is a Borel function (where here $\|f\|$ denotes the function obtained by taking the gyronorm of f) and has a finite image set, then we will call f a **simply normed function**. This could be rephrased alternatively in the following way:

Definition 3.1 (Simply normed function)**.** A function from a measure space (X, Σ, μ) to the Möbius gyrovector space is a simply normed function if and only if $\|f\| : (X, \Sigma, \mu) \to [0,1]$ is a simple function in the classical sense of measure theory.

We note that any measurable function $f : (X, \Sigma, \mu) \to (\mathbb{D}, \oplus, \otimes, \|\cdot\|, \text{arctanh})$ with finite image set is simply normed, but the converse is in general false; in fact there can be functions whose image set is infinite in $\mathbb{D}$ but assumes a finite set of norms.
One explicit example of this behaviour is the function $\varphi : (\mathbb{R}, \mathfrak{B}(\mathbb{R}), m) \to \mathbb{D}$ (where here $\mathfrak{B}(\mathbb{R})$ denotes the real Borel sigma algebra and m is the Lebesgue measure) defined by

$$\varphi(x) := \frac{1}{2}e^{ix} \in \mathbb{D}.$$

If $\|f\|$ is a simple function, given a collection of disjoint sets $(A_i)_{i=1}^n$ such that $\bigsqcup_{i=1}^n A_i = X$, we will be able to express it as:

$$\|f(x)\| = \sum_{i=1}^{n} \|a_i\| \chi_{A_i}(x), \qquad \|a_i\| \in [0,1],$$

where χ_{A_i} is the characteristic function of the set A_i defined as:

$$\chi_{A_i}(x) := \begin{cases} 1, & \text{if } x \in A_i, \\ 0, & \text{if } x \notin A_i. \end{cases}$$

Proposition 3.2. *Given a simply normed function $f : X \to \mathbb{D}$, we will have that*

$$\|f(x)\| = \sum_{i=1}^{n} \|a_i\| \chi_{A_i}(x) = \bigoplus_{i=1}^{n} \chi_{A_i}(x) \otimes \|a_i\|,$$

for all $x \in X$, i.e., the function can be expressed as a weighted gyrolinear combination of characteristic functions equivalent to the linear combination provided before.

Proof. First, we recall that $1 \otimes \|a_i\| = \|a_i\|$ and $0 \otimes \|a_i\| = 0 \in \|\mathbb{D}\|$ by the algebraic properties of the Möbius gyrovector space.
Since the only values that $\chi_{A_i}(x)$ can assume are 1 and 0, from the previous observation it follows that $\chi_{A_i}(x) \otimes \|a_i\| = \chi_{A_i}(x)\|a_i\|$ for all $\|a_i\| \in \|\mathbb{D}\|$ and for all $x \in X$. From this it follows that

$$\bigoplus_{i=1}^{n} \chi_{A_i}(x) \otimes \|a_i\| = \bigoplus_{i=1}^{n} \chi_{A_i}(x)\|a_i\|.$$

We will now conclude the proof by proving that

$$\bigoplus_{i=1}^{n} \chi_{A_i}(x)\|a_i\| = \sum_{i=1}^{n} \|a_i\| \chi_{A_i}(x), \tag{6}$$

but this follows by the assumption we made about the sets $(A_i)_{i=1}^n \subset \Sigma$. In fact, being disjoint sets, we will have that, for all $x \in X$, x will be in one and one only of these sets, let's call it A_{k_x}, and thus $\chi_{A_i}(x) = \delta_{ik_x}$, where here δ_{ik_x} denotes the Kronecker delta. Therefore, for all $x \in X$, both sides of Equation (6) reduce to just $\|a_{k_x}\|$, and are thus equal. □

Proposition 3.3. *Two simply normed functions f, g can have their norms expressed as gyrolinear combinations of characteristic functions of the same family of sets.*

Proof. Let $\|f(x)\| = \bigoplus_{i=1}^{n} \chi_{A_i}(x) \otimes \|a_i\|$ and $\|g(x)\| = \bigoplus_{j=1}^{m} \chi_{B_j}(x) \otimes \|b_j\|$, where here $(B_j), (A_i) \subset \Sigma$ are collections of disjoint sets of Σ whose disjoint union yields X. By observing that $A_i = \bigsqcup_{j=1}^{m} (A_i \cap B_j)$ we will be able to write χ_{A_i} as:

$$\chi_{A_i}(x) = \bigoplus_{j=1}^{m} \chi_{A_i \cap B_j}(x) = \sum_{j=1}^{m} \chi_{A_i \cap B_j}(x).$$

By substituting the above identity in our expression for $\|f\|$, we get

$$\|f(x)\| = \bigoplus_{i=1}^{n} \left(\sum_{j=1}^{m} \chi_{A_i \cap B_j}(x) \right) \otimes \|a_i\|,$$

which, employing the distributivity of scalars of the 1-dimensional $\mathbb{R}$-vector space $\|\mathbb{D}\|$, reduces to

$$\|f(x)\| = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \chi_{A_i \cap B_j}(x) \otimes \|a_i\|.$$

In a completely analogous manner, we get

$$\|g(x)\| = \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \chi_{A_i \cap B_j}(x) \otimes \|b_j\|,$$

which is our desired result. □

This fact will prove useful when investigating the properties of the gyrointegral we will now define.

Definition 3.4 (Lebesgue gyrointegral of simply gyronormed functions)**.** Let $f : (X, \Sigma, \mu) \to (\mathbb{D}, \oplus, \otimes, \|\cdot\|, \operatorname{arctanh})$ be a simply normed function from a measure space to the Möbius gyrovector space. From what we have seen in the previous propositions, we know that we can express $\|f(x)\|$ as:

$$\|f(x)\| = \bigoplus_{i=1}^{n} \chi_{A_i}(x) \otimes \|a_i\|.$$

We now define the **Lebesgue gyrointegral** of the gyronorm of f on X with respect to μ to be

$$\oint_X \|f(x)\| d\mu(x) := \bigoplus_{i=1}^{n} \mu(A_i) \otimes \|a_i\|.$$

The first thing we need to check is the well-definedness of our gyrointegral.

Proposition 3.5. *The Lebesgue gyrointegral for simply gyronormed functions is well-defined, i.e., it is invariant with respect to the representation we choose for the gyronorm* $\|f\|$.

Proof. Let $(A_i)_{i=1}^n, (B_j)_{j=1}^m \subset \Sigma$ be collections of disjoint sets with $X = \bigsqcup_{i=1}^n A_i$ and $X = \bigsqcup_{j=1}^m B_j$. Let

$$\|f(x)\| = \bigoplus_{i=1}^{n} \chi_{A_i}(x) \otimes \|a_i\|, \qquad \|f(x)\| = \bigoplus_{j=1}^{m} \chi_{B_j}(x) \otimes \|b_j\|,$$

be two equivalent representations of the gyronorm of our function f.
We notice that

$$A_i = \bigsqcup_{j=1}^{m} (A_i \cap B_j), \quad B_j = \bigsqcup_{i=1}^{n} (A_i \cap B_j).$$

By the additivity of our measure μ we have

$$\mu(A_i) = \sum_{j=1}^{m} \mu(A_i \cap B_j), \quad \mu(B_j) = \sum_{i=1}^{n} \mu(A_i \cap B_j).$$

Thus

$$\begin{aligned}\bigoplus_{i=1}^{n} \mu(A_i) \otimes \|a_i\| &= \bigoplus_{i=1}^{n} \left(\sum_{j=1}^{m} \mu(A_i \cap B_j) \right) \otimes \|a_i\| = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mu(A_i \cap B_j) \otimes \|a_i\| \\ &= \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \mu(A_i \cap B_j) \otimes \|b_j\| = \bigoplus_{j=1}^{m} \left(\sum_{i=1}^{n} \mu(A_i \cap B_j) \right) \otimes \|b_j\| \\ &= \bigoplus_{j=1}^{m} \mu(B_j) \otimes \|b_j\|,\end{aligned}$$

where in the above chain of equations we used the fact that $A_i \cap B_j \neq \emptyset \implies \|a_i\| = \|b_j\|$ alongside the distributivity of the gyroscalar multiplication over field addition of scalars on $\|\mathbb{D}\|$, a consequence of its vector space structure. □

The Lebesgue gyrointegral, in contrast with the linearity of the classical Lebesgue integral, is a gyrolinear integral.

Proposition 3.6 (Gyrolinearity of the Lebesgue gyrointegral)**.** *Let f and g be simply gyronormed functions from a measure space (X, Σ, μ) to the Möbius gyrovector space, then*

$$\oint_X (\lambda \otimes \|f\| \oplus \gamma \otimes \|g\|) d\mu = \lambda \otimes \oint_X \|f\| d\mu \oplus \gamma \otimes \oint_X \|g\| d\mu. \tag{7}$$

Proof. Using the conclusions of Proposition 3.3 we will start by writing our functions $\|f(x)\|$ and $\|g(x)\|$ as gyrolinear combinations of characteristic functions of the same family of disjoint sets $(B_i)_{i=1}^{n} \subset \Sigma$

$$\|f(x)\| = \bigoplus_{i=1}^{n} \chi_{B_i}(x) \otimes \|a_i\|, \quad \|g(x)\| = \bigoplus_{i=1}^{n} \chi_{B_i}(x) \otimes \|b_i\|.$$

Furthermore we observe that, for the scalar product compatibility axiom of vector spaces, and the distributive property

$$\lambda \otimes \|f(x)\| = \bigoplus_{i=1}^{n} (\lambda \chi_{B_i}(x)) \otimes \|a_i\| = \bigoplus_{i=1}^{n} (\lambda \otimes \|a_i\|) \chi_{B_i}(x).$$

The same considerations can be applied to $\|g(x)\|$ and

$$\begin{aligned}\lambda \otimes \|f\| \oplus \gamma \otimes \|g\| &= \bigoplus_{i=1}^{n} [(\lambda \otimes \|a_i\|)\chi_{B_i}(x) \oplus (\gamma \otimes \|b_i\|)\chi_{B_i}(x)] \\ &= \bigoplus_{i=1}^{n} \chi_{B_i}(x) \otimes (\lambda \otimes \|a_i\|) \oplus \chi_{B_i}(x) \otimes (\gamma \otimes \|b_i\|) \\ &= \bigoplus_{i=1}^{n} \chi_{B_i}(x) \otimes (\lambda \otimes \|a_i\| \oplus \gamma \otimes \|b_i\|).\end{aligned}$$

From this, it follows that we will be able to write the Lebesgue gyrointegral of the above function as

$$\begin{aligned}\oint_X (\lambda \otimes \|f\| \oplus \gamma \otimes \|g\|)\, d\mu &= \bigoplus_{i=1}^{n} \mu(B_i) \otimes (\lambda \otimes \|a_i\| \oplus \gamma \otimes \|b_i\|) \\ &= \bigoplus_{i=1}^{n} \mu(B_i) \otimes (\lambda \otimes \|a_i\|) \oplus \bigoplus_{i=1}^{n} \mu(B_i) \otimes (\gamma \otimes \|b_i\|) \\ &= \bigoplus_{i=1}^{n} \lambda \otimes (\mu(B_i) \otimes \|a_i\|) \oplus \bigoplus_{i=1}^{n} \gamma \otimes (\mu(B_i) \otimes \|b_i\|) \\ &= \lambda \otimes \bigoplus_{i=1}^{n} \mu(B_i) \otimes \|a_i\| \oplus \gamma \otimes \bigoplus_{i=1}^{n} \mu(B_i) \otimes \|b_i\| \\ &= \lambda \otimes \oint_X \|f\| d\mu \oplus \gamma \otimes \oint_X \|g\| d\mu.\end{aligned}$$

□

By construction, we observe that the Lebesgue gyrointegral for simply-gyronormed functions also respects the following monotonicity condition:

$$\|f\| \leq \|g\| \Longrightarrow \oint_X \|f\| d\mu \leq \oint_X \|g\| d\mu.$$

We now extend our Lebesgue gyrointegral to functions whose gyronorm is measurable.

Definition 3.7 (Lebesgue gyrointegral of gyronorm-measurable functions)**.** Let f be a function from a measure space (X, Σ, μ) to the Möbius gyrovector space, whose norm is a measurable function; then we define the **Lebesgue gyrointegral of** $\|f\|$ to be

$$\oint_X \|f\| d\mu := \sup \left\{ \oint_X \|\phi\| d\mu \;; \|\phi\| \leq \|f\|,\ \phi \in S_{\|\cdot\|_{\mathbb{D}}} \right\},$$

where here $S_{\|\cdot\|_{\mathbb{D}}}$ denotes the set of simply gyronormed functions, i.e. of functions from (X, Σ, μ) to the Möbius gyrovector space, whose norms are simple functions. We say that $\|f\|$ is μ-gyrointegrable if the gyrointegral is in $[0,1]$, and we say that $\|f\|$ is μ-gyrosummable if the gyrointegral is in $[0,1)$.

Proposition 3.8. *Let (X, Σ, μ) be a measure space and $f, g : X \to \mathbb{D}$ be functions from said measure space to the Möbius gyrovector space with measurable gyronorms, then the gyrointegral possesses the following monotonicity property*

$$\|f\| \leq \|g\| \Longrightarrow \oint_X \|f\| d\mu \leq \oint_X \|g\| d\mu.$$

Proof. This fact follows immediately from Definition 3.7 and by the properties of the supremum. □

We now wish to show this integral is gyrolinear; in order to do so we will prove an adaptation of the monotone convergence theorem to gyrointegrals.

Theorem 3.9 (Monotone convergence theorem)**.** *Let (X, Σ, μ) be a measure space. If $\{\|f_n\|\}_{n=1}^{\infty}$ is a monotone increasing sequence of measurable gyronorms of functions (from X to the Möbius gyrovector space) $\|f_n\| : X \to [0,1]$ and*

$$\|f\| = \lim_{n\to\infty} \|f_n\|,$$

then

$$\lim_{n\to\infty} \oint_X \|f_n\| d\mu = \oint_X \|f\| d\mu.$$

Proof. The pointwise limit of the sequence of function gyronorms exists since the latter is monotonically increasing. By the monotonicity of the gyrointegral (Proposition 3.8), we have

$$\oint_X \|f_n\| d\mu \leq \oint_X \|f_{n+1}\| d\mu \leq \oint_X \|f\| d\mu.$$

Thus the gyrointegrals are increasing, they admit a limit and

$$\lim_{n\to\infty} \oint_X \|f_n\| d\mu \leq \oint_X \|f\| d\mu.$$

For the other direction, let $\|\varphi\| : X \to [0,1]$ be a simple function with $\|\varphi\| \leq \|f\|$. We now fix a real value $t \in (0,1)$ and define

$$A_n := \{x \in X \ ; \ \|f_n(x)\| \geq t \otimes \|\varphi(x)\|\}.$$

Since A_n can be expressed as $\{x \in X \ ; \ \|f_n(x)\| - t \otimes \|\varphi(x)\| \geq 0\}$ and $\|f_n(x)\| - t \otimes \|\varphi(x)\|$ is measurable, we will have that such sets will be measurable for all n.

$\{A_n\}_n$ is thus an increasing sequence of measurable sets whose union is the whole set X; from this it follows that

$$\oint_X \|f_n\| d\mu \geq \oint_{A_n} \|f_n\| d\mu \geq t \otimes \oint_{A_n} \|\phi\| d\mu. \tag{8}$$

Let us now use our supposition that the gyronorm of ϕ was simple to express it as:

$$\|\phi(x)\| = \bigoplus_{i=1}^{m} \chi_{C_i}(x) \otimes \|c_i\|.$$

The gyrointegral of $\|\phi\|$ on A_n will be

$$\oint_{A_n} \|\phi(x)\| d\mu(x) = \oint_X \|\phi(x)\| d\mu_{\llcorner A_n} = \bigoplus_{i=1}^{m} \mu(C_i \cap A_n) \otimes \|c_i\|,$$

but since by the properties of measures, $\lim_{n\to\infty} \mu(C_i \cap A_n) = \mu(C_i)$ we will have that

$$\lim_{n\to\infty} \oint_{A_n} \|\phi(x)\| d\mu(x) = \bigoplus_{i=1}^{m} \mu(C_i) \otimes \|c_i\| = \oint_X \|\phi(x)\| d\mu.$$

By taking the limit as n goes to infinity of the inequality (8) we then get $\lim_{n\to\infty} \|f_n\| d\mu \geq t \otimes \oint_X \|\phi(x)\| d\mu$. By the arbitrairity of our constant real value $0 < t < 1$, we conclude that

$$\lim_{n\to\infty} \oint_X \|f_n\| d\mu \geq \oint_X \|\phi(x)\| d\mu,$$

and since this inequality is valid for any simple function $\|\phi\| \leq \|f\|$ we get in the end

$$\lim_{n\to\infty} \oint_X \|f_n\| d\mu \geq \oint_X \|f\| d\mu,$$

by taking the supremum. This, together with the other direction we proved earlier, implies that $\lim_{n\to\infty} \oint_X \|f_n\| d\mu = \oint_X \|f\| d\mu$. □

The gyrolinearity of the gyrointegral will follow as a corollary of the fact just proved.

Corollary 3.10. *Let $f, g : (X, \Sigma, \mu) \to (\mathbb{D}, \oplus, \otimes, \|\cdot\|, \operatorname{arctanh})$ be two gyronorm-measurable functions and let $\lambda \in \mathbb{R}$, then:*

$$\oint_X (\|f\| \oplus \|g\|) d\mu = \oint_X \|f\| d\mu \oplus \oint_X \|g\| d\mu,$$

$$\oint_X \lambda \otimes \|f\| d\mu = \lambda \otimes \oint_X \|f\| d\mu.$$

Proof. Since $\|f\|$ and $\|g\|$ are non-negative measurable functions, there are two increasing sequences of non-negative simple functions $\{\|f_n\|\}_{n\in\mathbb{N}}$ and $\{\|g_n\|\}_{n\in\mathbb{N}}$ such that [14, page 31, theorem 4.1]

$$\lim_{n\to\infty} \|f_n(x)\| = \|f(x)\|, \qquad \lim_{n\to\infty} \|g_n(x)\| = \|g(x)\|.$$

Then, by the properties of simple functions and pointwise limits of functions, we will have that $\|f_n\| \oplus \|g_n\|$ is an increasing sequence of non-negative simple functions converging pointwise to $\|f\| \oplus \|g\|$. As a consequence of the monotone convergence theorem proved above, we have

$$\oint_X (\|f\| \oplus \|g\|) d\mu = \lim_{n\to\infty} \oint_X (\|f_n\| \oplus \|g_n\|) d\mu.$$

Now, appling the gyrolinearity of the gyrointegral for simple functions yields

$$\begin{aligned}\oint_X (\|f\| \oplus \|g\|) d\mu &= \lim_{n\to\infty} \Big(\oint_X \|f_n\| d\mu \oplus \oint_X \|g_n\| d\mu \Big) \\ &= \lim_{n\to\infty} \oint_X \|f_n\| d\mu \oplus \lim_{n\to\infty} \oint_X \|g_n\| d\mu \\ &= \oint_X \|f\| d\mu \oplus \oint_X \|g\| d\mu,\end{aligned}$$

which is precisely the first point of our assertion.
For the second point, we select an increasing sequence of non-negative simple functions $\{\|f_n\|\}_{n\in\mathbb{N}}$ as before and notice that

$$\begin{aligned}\oint_X (\lambda \otimes \|f\|) d\mu &= \lim_{n\to\infty} \oint_X (\lambda \otimes \|f_n\|) d\mu \\ &= \lim_{n\to\infty} \lambda \otimes \oint_X \|f_n\| d\mu \\ &= \lambda \otimes \oint_X \|f(x)\| d\mu(x).\end{aligned}$$

The above chain of equalities follows by a combination of the monotone convergence theorem for gyrointegrals, properties of pointwise convergence of functions and the gyrolinearity of the gyrointegral for simple gyronorms. □

4. Relationship with the Lebesgue integral

Another important consequence of the monotone convergence theorem for the gyrointegral is the following:

Theorem 4.1 (Relationship between the Lebesgue integral and the Lebesgue gyrointegral)**.** *Let f be a function from a measure space (X, Σ, μ) to the Möbius gyrovector space, whose gyronorm is a measurable and μ-gyrointegrable function, then*

$$\oint_X \|f(x)\| d\mu(x) = \tanh\left(\int_X \operatorname{arctanh}(\|f(x)\|) d\mu(x) \right).$$

Proof. By [14, p.31, Theorem 4.1] there exists an increasing sequence of non-negative simple functions $\{\|f_n\|\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|f_n\| = \|f\|$. Since the functions in question are simple, their gyrointegrals will be equal to

$$\oint_X \|f_n(x)\| d\mu(x) = \bigoplus_{i=1}^{m_n} \chi_{A_{in}} \otimes \|a_{in}\|.$$

By applying arctanh to both sides of the equation and by employing the identity $\operatorname{arctanh}(x \oplus y) = \operatorname{arctanh}(x) + \operatorname{arctanh}(y)$ we get

$$\operatorname{arctanh}\left(\oint_X \|f_n(x)\| d\mu(x) \right) = \sum_{i=1}^{m_n} \operatorname{arctanh}(\mu(A_{in}) \otimes \|a_{in}\|),$$

but $\operatorname{arctanh}(r \otimes \|a\|) = \operatorname{arctanh}(\tanh(r \operatorname{arctanh}(\|a\|)) = \operatorname{arctanh}(\|a\|)$ and so

$$\operatorname{arctanh}\left(\oint_X \|f_n\| d\mu \right) = \sum_{i=1}^{m_n} \mu(A_{in}) \operatorname{arctanh}(\|a_{in}\|) = \int_X \operatorname{arctanh}(\|f_n(x)\|) d\mu, \tag{9}$$

which is equivalent to $\oint_X \|f_n\| d\mu = \tanh\left(\int_X \operatorname{arctanh}(\|f_n(x)\|) d\mu \right)$. Now, since $\|f_n\|$ is increasing and non-negative, by the monotone convergence theorem for gyrointegrals we have:

$$\lim_{n\to\infty} \oint_X \|f_n\| d\mu = \oint_X \|f\| d\mu.$$

If we substitute (9) for $\oint_X \|f_n\| d\mu$ in the above equation, we get

$$\oint_X \|f\| d\mu = \lim_{n\to\infty} \tanh\left(\int_X \operatorname{arctanh}(\|f_n(x)\|) d\mu \right).$$

Since tanh is continuous on all of $\mathbb{R}$, we have

$$\lim_{n\to\infty} \tanh\left(\int_X \operatorname{arctanh}(\|f_n(x)\|) d\mu \right) = \tanh\left(\lim_{n\to\infty} \int_X \operatorname{arctanh}(\|f_n(x)\|) d\mu \right).$$

Furthermore, since $\{\|f_n\|\}_{n\in\mathbb{N}}$ is an increasing sequence of functions, and arctanh is a strictly monotonically increasing function on all of $\mathbb{R}$, the sequence

$\{\text{arctanh}(\|f_n(x)\|)\}_{n\in\mathbb{N}}$ will also be increasing, and thus by using the classical monotone convergence theorem for the Lebesgue integral we have

$$\lim_{n\to\infty}\int_X \text{arctanh}(\|f_n(x)\|)d\mu = \int_X \text{arctanh}(\|f(x)\|)d\mu.$$

And thus

$$\oint_X \|f\|d\mu = \tanh\left(\int_X \text{arctanh}(\|f(x)\|)d\mu\right),$$

which is what we wanted to prove. □

An immediate corollary of the fact we just proved is that the gyrointegral of the gyronorm of a function $f : (X, \Sigma, \mu) \to \mathbb{D}$ will be 0 if and only if its gyronorm is almost everywhere 0 (that is, if it is 0 everywhere but possibly in sets of measure 0).

Corollary 4.2. *Let f be a function from a measure space to the Möbius gyrovector space, then*

$$\oint_X \|f\|d\mu = 0 \Leftrightarrow \|f\| = 0 \quad a.e.$$

where here a.e is a shorthand notation for "almost everywhere".

Proof. Let's start with the forward direction; if $\|f\| = 0$ almost everywhere, then $\text{arctanh}(\|f(x)\|) = 0$ almost everywhere, and by the properties of the classical Lebesgue integral

$$\int_X \text{arctanh}(\|f(x)\|)d\mu = 0,$$

thus $\oint_X \|f\|d\mu = \tanh(\int_X \text{arctanh}(\|f(x)\|)d\mu) = 0$.
For the reverse direction, let us suppose that

$$\oint_X \|f\|d\mu = \tanh\left(\int_X \text{arctanh}(\|f(x)\|)d\mu\right) = 0.$$

By the bijectivity of the tanh function, we deduce that $\int_X \text{arctanh}(\|f(x)\|)d\mu = 0$, and thus, by the properties of classical Lebesgue integrals, that $\text{arctanh}(\|f(x)\|)$ is 0 almost everywhere; this in turn implies that $\|f(x)\|$ is 0 almost everywhere, which is what we wanted to prove. □

5. Function gyrolinear spaces

The classical function spaces L^p for $p \in \mathbb{R}_{\geq 1}$ are one of the central objects of study of functional analysis; they form a complete metric topological vector space with respect to the distance

$$\delta(f, g) = \int_X |f(x) - g(x)|^p d\mu.$$

In this section, we will show how to construct a gyrolinear gyronormed equivalent of these function spaces.
We start our study of gyrolinear function spaces with the following observation.

Definition 5.1 ($\mathfrak{L}^1_{\mathbb{D}}$ gyronorm of a function and $\mathfrak{L}^1_{\mathbb{D}}$ functions)**.** Let f be a gyronorm measurable function from a measure space (X, Σ, μ) to the Möbius gyrovector space; we will call the $\mathfrak{L}^1_{\mathbb{D}}$ gyronorm of f the following expression

$$\|f\|_1^{\mathbb{D}} := \oint_X \|f(x)\| d\mu(x).$$

Furthermore, we say that f is a $\mathfrak{L}^1_{\mathbb{D}}$ function if $\|f\|_1^{\mathbb{D}} = \oint_X \|f(x)\| d\mu(x) < 1$. We denote the set of all $\mathfrak{L}^1_{\mathbb{D}}$ functions by $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$.

We will now prove that (an appropriate quotient of) $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$ will form a normed gyrolinear space; let's first prove the following more general fact.

Proposition 5.2. *The set of functions from an arbitrary set X to the Möbius gyrovector space $\mathbb{D}$, $\mathfrak{F}(X \to \mathbb{D})$, forms a gyrolinear space under the following operations:*

$$(f \oplus g)(x) := f(x) \oplus g(x), \quad \forall x \in X,$$
$$(r \otimes f)(x) := (r \otimes f(x)), \quad \forall x \in X,$$

where $r \in \mathbb{R}$ and $f, g : X \to (\mathbb{D}, \oplus, \otimes, \|\cdot\|, \operatorname{arctanh})$.

Proof. The identity element is given by the constant function defined as $\gamma_0(x) := 0$ for all $x \in X$, where here 0 denotes the gyroadditive identity of the Möbius gyrogroup.
For any function f, the function defined by $\tilde{f}(x) := \ominus f(x)$ will be its gyroinverse since $(\tilde{f} \oplus f)(x) = \ominus f(x) \oplus f(x) = 0 = \gamma_0(x)$ for all $x \in X$.
We furthermore notice that function addition is gyroassociative and gyrocommutative:

$$(f \oplus (g \oplus h))(x) = f(x) \oplus (g(x) \oplus h(x)) = (f(x) \oplus g(x)) \oplus \operatorname{gyr}[f(x), g(x)]h(x),$$

The above chain of equalities follows by the definition of function gyrosum we gave and the gyroassociative property of the classic gyrosum in the Möbius gyrovector space. We now rewrite said expression as:

$$(f \oplus (g \oplus h)) = (f \oplus g) \oplus \operatorname{gyr}[f, g]h.$$

Similarly,

$$f \oplus g = f(x) \oplus g(x) = \operatorname{gyr}[f(x), g(x)](g(x) \oplus f(x)) = \operatorname{gyr}[f, g](g \oplus f).$$

Here the expression $\operatorname{gyr}[f, g]h$ is defined as:

$$\operatorname{gyr}[f, g]h := \operatorname{gyr}[f(x), g(x)]h(x).$$

For any two functions $f, g : X \to \mathbb{D}$, the map $\mathrm{gyr}[f, g]$ is an automorphism of the groupoid of functions from X to $\mathbb{D}$ under gyrosum of functions. The inverse of the latter is given by

$$\mathrm{gyr}^{-1}[f, g] = \mathrm{gyr}[g, f],$$

and thus the map is injective; furthermore, for all $x \in X$:

$$\begin{aligned}\mathrm{gyr}[f, g](h \oplus k) &:= \mathrm{gyr}[f(x), g(x)](h(x) \oplus k(x)) \\ &= \mathrm{gyr}[f(x), g(x)]h(x) \oplus \mathrm{gyr}[f(x), g(x)]k(x) \\ &= \mathrm{gyr}[f, g]h \oplus \mathrm{gyr}[f, g]k.\end{aligned}$$

The surjectivity follows by treating $\mathrm{gyr}[f, g]$ as a parametrized family of gyrations (automorphisms), and given a function h construct point-wise a function $\tilde{h}$ by utilizing the surjectivity of the individual gyrations such that $\mathrm{gyr}[f, g]\tilde{h} = h$. From these considerations it follows that, for any $f, g \in \mathfrak{F}(X \to \mathbb{D})$:

$$\mathrm{gyr}[f, g] \in \mathrm{Aut}(\mathfrak{F}(X \to \mathbb{D}), \oplus).$$

The left loop property follows from the left loop property for the gyrator of the Möbius gyrogroup. Thus, we have so far proven that $(\mathfrak{F}(X \to \mathbb{D}), \oplus)$ is a gyrocommutative gyrogroup; let us now prove that it is a gyrolinear space.
We notice that $1 \otimes f(x) = f(x)$ for all $x \in X$, and thus property 1 is satisfied. Furthermore, $(r_1 + r_2) \otimes f(x) = (r_1 \otimes f(x)) \oplus (r_2 \otimes f(x)) = (r_1 \otimes f) \oplus (r_2 \otimes f)$, and thus property 2 is satisfied. Property 3 similarly follows from the gyrolinear space structure of the Möbius disk, $(r_1 r_2) \otimes f(x) = r_1 \otimes (r_2 \otimes f(x)) = r_1 \otimes (r_2 \otimes f)$.
$\mathrm{gyr}[f, g](r \otimes h) = \mathrm{gyr}[f(x), g(x)](r \otimes h(x)) = r \otimes \mathrm{gyr}[f(x), g(x)]h(x)$ for all $x \in X$ and thus property 4 is satisfied, i.e. $\mathrm{gyr}[f, g](r \otimes h) = r \otimes \mathrm{gyr}[f, g]h$.
Property 5 is satisfied since it will be satisfied for each x in the domain of the input functions of the gyrator as a consequence of the gyrolinear space structure of the Möbius disk, that is, for all $x \in X$:

$$\mathrm{gyr}[r_1 \otimes f(x), r_2 \otimes f(x)] = id_{\mathbb{D}},$$

and thus $\mathrm{gyr}[r_1 \otimes f, r_2 \otimes f] = id_{\mathfrak{F}(X \to \mathbb{D})}$. In other words, we have just shown that $\mathfrak{F}(X \to \mathbb{D})$ is a gyrolinear space, and our proof is complete. □

Proposition 5.3. *Let (X, Σ) be a sigma algebra, then the subset of $\mathfrak{F}(X \to \mathbb{D})$ given by the Σ-measurable functions from X to the Möbius gyrovector space, which we will denote as $\mathfrak{F}_m^{\Sigma}(X \to \mathbb{D})$ is a gyrolinear subspace of $\mathfrak{F}(X \to \mathbb{D})$.*

Proof. We start by noticing that the additive identity of $\mathfrak{F}(X \to \mathbb{D})$, $\gamma_0(x) := 0$ (defined in this way for all $x \in X$) is in $\mathfrak{F}_m^{\Sigma}(X \to \mathbb{D})$ since it is a constant function, and constant functions are measurable.
We now just need to show that given two measurable functions $f, g \in \mathfrak{F}_m^{\Sigma}(X \to \mathbb{D})$, and for any $r \in \mathbb{R}$, $f \oplus g$ and $r \otimes f$ are measurable. The latter is immediate, since the

function $r \otimes f = \tanh(r \operatorname{arctanh}(\|f(x)\|))\frac{f(x)}{\|f(x)\|}$ is a composition of the continuos (and thus measurable) function

$$\mathbb{D} \ni z \to \begin{cases} \tanh(r \operatorname{arctanh}(z))\frac{z}{\|z\|}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

and the measurable function $f(x)$.
For the second, it suffices to observe that since the function $F : X \to \mathbb{D}^2$ defined by $F(x) := (f(x), g(x))$ is measurable, and the gyrosum $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is continuous, $f(x) \oplus g(x) = \oplus \circ F(x)$ is a composition of measurable functions, and it is thus measurable. □

We notice that if $f, g \in \mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$ and $r \in \mathbb{R}$, then $f \oplus g, r \otimes f, 0 \in \mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$; this is because $\oint_X 0 d\mu = 0 < 1$,

$$\oint_X \|f \oplus g\| d\mu \leq \oint_X \|f\| d\mu \oplus \oint_X \|g\| d\mu < 1,$$

and

$$\oint_X \|r \otimes f\| d\mu = |r| \otimes \oint_X \|f\| d\mu < 1.$$

From this, we deduce that also $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$ is a gyrolinear space, and it is a gyrolinear subspace of both $\mathfrak{F}^{\Sigma}_m(X \to \mathbb{D})$ and $\mathfrak{F}(X \to \mathbb{D})$. To give the latter the structure of a gyronormed gyrolinear space, we need to consider an appropriate quotient with respect to an equivalence relation, just like in classical Lebesgue space theory. In particular, we will define our relation $\sim$ as:

$$f, g \in \mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu), \qquad f \sim g \quad \Leftrightarrow \quad f = g \ \ a.e.$$

where here by a.e. we mean "almost everywhere" in the classical sense of measure theory, i.e. $f(x) = g(x)$, for all $x \in X \setminus E$, with $\mu(E) = 0$.
We will now consider the set of equivalence classes of $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$ with respect to this relation $\sim$ and call it $L^1_{\mathbb{D}}(X, \Sigma, \mu)$

$$L^1_{\mathbb{D}}(X, \Sigma, \mu) := \mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)/\sim.$$

We define the gyrosum of two elements of $L^1_{\mathbb{D}}(X, \Sigma, \mu)$, $[f]_\sim, [g]_\sim \in L^1_{\mathbb{D}}(X, \Sigma, \mu)$ as the equivalence class containing the gyrosum of two gyronorm-finite almost everywhere representatives (we can assert this since every μ-gyrosummable function will be gyronorm-finite almost everywhere) $\tilde{f} \in [f]_\sim$ and $\tilde{g} \in [g]_\sim$

$$[f]_\sim \oplus [g]_\sim := [\tilde{f} \oplus \tilde{g}]_\sim \in L^1_{\mathbb{D}}(X, \Sigma, \mu).$$

Similarly, we define the gyroscalar multiplication of $[f]_\sim \in L^1_{\mathbb{D}}(X, \Sigma, \mu)$ with the scalar $r \in \mathbb{R}$ as the equivalence class containing $r \otimes \tilde{f}$, for a gyronorm-finite almost everywhere representative of $[f]_\sim$

$$r \otimes [f]_\sim := [r \otimes \tilde{f}]_\sim.$$

Let's prove that these 2 operations are well defined, i.e. that they don't depend on the choice of representatives. Let $\tilde{f}_1, \tilde{f}_2 \in [f]_\sim$ and $\tilde{g}_1, \tilde{g}_2 \in [g]_\sim$; by the definition of the equivalence relation $\sim$, we have that:

$$\tilde{f}_1(x) = f(x), \quad \forall x \in X \setminus E_{f_1}, \quad \tilde{f}_2(x) = f(x), \quad \forall x \in X \setminus E_{f_2},$$

$$\tilde{g}_1(x) = g(x), \quad \forall x \in X \setminus E_{g_1}, \quad \tilde{g}_2(x) = g(x), \quad \forall x \in X \setminus E_{g_2},$$

where here $\mu(E_{f_1}), \mu(E_{f_2}), \mu(E_{g_1}), \mu(E_{g_2}) = 0$. From the above equations, we can then deduce that

$$\tilde{f}_1(x) \oplus \tilde{g}_1(x) = \tilde{f}_2(x) \oplus \tilde{g}_2(x),$$

for all $x \in X \setminus (E_{f_1} \cap E_{g_1} \cap E_{f_2} \cap E_{g_2})$, i.e. $\tilde{f}_1 \oplus \tilde{g}_1 \sim \tilde{f}_2 \oplus \tilde{g}_2$, and so the $\oplus$ operation between equivalence classes is well-defined.
For the $\otimes$ operation, let $r \in \mathbb{R}$ be given, and let $\tilde{f}_1, \tilde{f}_2 \in [f]_\sim$ be two representatives of the class $[f]_\sim$. As before, by using the definition of $\sim$, we note that for all $x \in X \setminus A_{f_1}$:

$$\tilde{f}_1(x) = f(x),$$

and for all $x \in X \setminus A_{f_2}$

$$\tilde{f}_2(x) = f(x),$$

with $\mu(E_{f_1}), \mu(E_{f_2}) = 0$. As a consequence of the above equality, we have $r \otimes \tilde{f}_1(x) = r \otimes f(x)$ for all $x \in X \setminus A_{f_1}$ and $r \otimes \tilde{f}_2(x) = r \otimes f(x)$ for all $x \in X \setminus A_{f_2}$ and thus

$$r \otimes \tilde{f}_1(x) = r \otimes \tilde{f}_2(x), \qquad \text{for all } x \in X \setminus (A_{f_1} \cap A_{f_2}).$$

Since $\mu(A_{f_1} \cap A_{f_2}) = 0$, $r \otimes \tilde{f}_1 \sim r \otimes \tilde{f}_2$, and so $\otimes$ is well defined as well.
From these considerations, and the gyrolinear space structure of $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$, it follows that $L^1_{\mathbb{D}}(X, \Sigma, \mu) := \mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)/\sim$ forms a gyrolinear space with respect to the operation $\oplus$ and $\otimes$ on equivalence classes just introduced.
In light of Corollary 4.2, the quotient we took was necessary in order to create a normed gyrolinear space structure with respect to the $\mathfrak{L}^1_{\mathbb{D}}$ gyronorm. We are now ready to show that $L^1_{\mathbb{D}}$ is a normed gyrolinear space, with gyronorm given by $\|\cdot\|^{\mathbb{D}}_1$ (i.e. the $L^1_{\mathbb{D}}$ gyronorm).
For simplicity, throughout the proof, we will drop the notation $[f]_\sim$ for equivalence classes of functions and just denote them as f.

Proposition 5.4 ($L^1_{\mathbb{D}}$ is a normed gyrolinear space)**.** *Let* $(L^1_{\mathbb{D}}(X, \Sigma, \mu), \oplus, \otimes)$ *be the gyrolinear space of equivalence classes modulo* $\sim$ *of functions of* $\mathfrak{L}^1_{\mathbb{D}}(X, \Sigma, \mu)$*, then* $(L^1_{\mathbb{D}}(X, \Sigma, \mu), \oplus, \otimes, \|\cdot\|^{\mathbb{D}}_1, \text{arctanh})$ *is a normed gyrolinear space.*

Proof. $\|\cdot\|^{\mathbb{D}}_1 : L^1_{\mathbb{D}} \to \mathbb{R}$ is always non-negative by construction, and arctanh is a strictly monotone increasing bijection from $\|L^1_{\mathbb{D}}\|$ to $\mathbb{R}_{\geq 0}$. Let us now prove the other required properties:

- $\|f\|^{\mathbb{D}}_1 = \oint_X \|f\| d\mu = 0$ if and only if $f = [0]_\sim \in L^1_{\mathbb{D}}$, which is the gyroadditive identity of $L^1_{\mathbb{D}}$.

- We now show that $\phi(\|\mathbf{x} \oplus \mathbf{y}\|) \le \phi(\|\mathbf{x}\|) + \phi(\|\mathbf{y}\|)$ (the second property of Definition 2.4):

$$\begin{aligned}
\operatorname{arctanh}(\|f \oplus g\|_1^{\mathbb{D}}) &= \operatorname{arctanh}(\tanh(\int_X \operatorname{arctanh}\|f \oplus g\| d\mu)) \\
&= \int_X \operatorname{arctanh}(\|f \oplus g\|) d\mu \le \int_X \operatorname{arctanh}(\|f\| \oplus \|g\|) d\mu \\
&= \int_X \operatorname{arctanh}(\|f\|) d\mu + \int_X \operatorname{arctanh}(\|g\|) d\mu \\
&= \tanh(\operatorname{arctanh}(\int_X \operatorname{arctanh}(\|f\|) d\mu)) \\
&+ \tanh(\operatorname{arctanh}(\int_X \operatorname{arctanh}(\|g\|) d\mu)) \\
&= \operatorname{arctanh}(\|f\|_1^{\mathbb{D}}) + \operatorname{arctanh}(\|g\|_1^{\mathbb{D}}).
\end{aligned}$$

- The third property of Definition 2.4 holds as well:

$$\begin{aligned}
\operatorname{arctanh}(\|r \otimes f|_1^{\mathbb{D}}) &= \operatorname{arctanh}(\tanh(\int_X \operatorname{arctanh}(|r| \otimes \|f\|) d\mu)) \\
&= \operatorname{arctanh}(\tanh(\int_X |r| \operatorname{arctanh}(\|f\|) d\mu)) \\
&= \operatorname{arctanh}(\tanh(|r| \int_X \operatorname{arctanh}(\|f\|) d\mu)) \\
&= \operatorname{arctanh}(|r| \otimes \tanh(\int_X \operatorname{arctanh}(\|f\|) d\mu)) \\
&= |r| \operatorname{arctanh}(\otimes \tanh(\int_X \operatorname{arctanh}(\|f\|) d\mu)) \\
&= |r| \operatorname{arctanh}(\|f\|_1^{\mathbb{D}}).
\end{aligned}$$

- And finally, for the fourth property of Definition 2.4, we observe that:

$$\begin{aligned}
\|\operatorname{gyr}[f, g]h\|_1^{\mathbb{D}} &= \underset{X}{\oint\!\!\!\!\oplus} \|\operatorname{gyr}[f(x), g(x)]h(x)\| d\mu(x) \\
&= \underset{X}{\oint\!\!\!\!\oplus} \|h(x)\| d\mu(x) = \|h\|_1^{\mathbb{D}}.
\end{aligned}$$

From these properties, it follows that $(L^1_{\mathbb{D}}(X, \Sigma, \mu), \oplus, \otimes, \|\cdot\|_1^{\mathbb{D}}, \operatorname{arctanh})$ is a normed gyrolinear space. □

We will call the gyrolinear space $L^1_{\mathbb{D}}$ the **L-1 gyrospace**. The gyronorm $\|\cdot\|_1^{\mathbb{D}}$ induces a gyrodistance between functions defined by

$$d_{\oplus}(f, g) := \|\ominus g \oplus f\|_1^{\mathbb{D}} = \underset{X}{\oint\!\!\!\!\oplus} \|\ominus g \oplus f\| d\mu.$$

The strictly monotone increasing bijection arctanh "transforms" the gyrodistance $d_\oplus$ into a metric, thus endowing $L^1_{\mathbb{D}}$ with a metric topology.

Proposition 5.5. *The function* $\delta : L^1_{\mathbb{D}} \to \mathbb{R}_{\geq 0}$ *defined by*

$$\delta(f,g) := \operatorname{arctanh}(d_\oplus(f,g)) = \int_X \operatorname{arctanh}(\| \ominus g \oplus f\|)d\mu, \tag{10}$$

is a metric, and thus $(L^1_{\mathbb{D}}, \delta)$ *is a metric space.*

Proof. We show the assertion by proving that the metric axioms are satisfied:

- $\delta(f,g) \geq 0$ by construction.
- $\delta(f,g) = \delta(g,f)$ since, by the properties of the gyronorm $\|\cdot\|$ of the Möbius gyrovector space:

$$\begin{aligned}\delta(f,g) &= \int_X \operatorname{arctanh}(\| \ominus g \oplus f\|)d\mu \\ &= \int_X \operatorname{arctanh}(\| \ominus \operatorname{gyr}[\ominus g, f](\ominus f \oplus g)\|)d\mu \\ &= \int_X \operatorname{arctanh}(\| \ominus f \oplus g\|)d\mu.\end{aligned}$$

- $\delta(f,g) = 0 \Longrightarrow \operatorname{arctanh}(\| \ominus g \oplus f\|) = 0$ almost everywhere, and thus $\| \ominus g \oplus f\| = 0$ almost everywhere; by the properties of the gyronorm $\|\cdot\|$ on the Möbius gyrovector space, we then deduce that $f = g$ almost everywhere (i.e. $[f]_\sim = [g]_\sim$ in $L^1_{\mathbb{D}}$ and thus we have obtained our desired result).
  For the converse, observe that if $f = g$ almost everywhere then

$$\operatorname{arctanh}(\| \ominus g \oplus f\|) = 0,$$

  almost everywhere and thus $\int_X \operatorname{arctanh}(\| \ominus g \oplus f\|)d\mu = 0$.
- $\delta(f,h) = \int_X \operatorname{arctanh}(\| \ominus h \oplus f\|)d\mu$. But by [4, theorem 3.11 page 61]:

$$\begin{aligned}\int_X \operatorname{arctanh}(\| \ominus h \oplus f\|)d\mu &\leq \int_X \operatorname{arctanh}(\| \ominus h \oplus g\| \oplus \| \ominus g \oplus f\|)d\mu \\ &= \delta(f,g) + \delta(g,h).\end{aligned}$$

□

We will now provide some particularly interesting instances of $L^1_{\mathbb{D}}(X, \Sigma, \mu)$ spaces, and explain how we can operatively use the theoretical framework we just built in said spaces.
The measurement of the distance between functions from $\mathbb{R}$ to $\mathbb{R}$ or $\mathbb{C}$ to $\mathbb{C}$ is of

great importance in the theory of signals, since such quantitative metrics regarding signals can help assess which one is closest to another arbitrary signal.
One practical goal we would want to achieve is that of measuring distances between functions from the Möbius disk to itself, in a way that is intrinsic to the hyperbolic nature of the space. In the next example we will see that $L^1_{\mathbb{D}}$ spaces allow us to achieve this goal (for $\mathfrak{L}^1_{\mathbb{D}}$ functions).

Example 5.6 (Distance between functions from the Möbius disk to itself.)**.** The Möbius disk $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$ together with the Riemannian metric:

$$ds^2 = \frac{4\|dx\|^2}{(1-\|x\|^2)^2} = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2},$$

is a Riemannian manifold; this Riemannian metric induces a natural measure-theoretic structure on $\mathbb{D}$, thanks to the volume form:

$$dV = \left(\frac{2}{1-x^2-y^2}\right)^2 dx\ dy.$$

In fact, the volume form we defined above generates a positive linear functional on $C_c(\mathbb{D})$ (the space of compactly supported complex-valued functions), ψ, defined as:

$$\psi : f \to \psi(f) := \int_{\mathbb{D}} f dV.$$

But by the Riesz-Markov-Kakutani representation theorem, there exists a unique positive Borel measure $\mu_{\mathbb{D}}$ on $\mathbb{D}$ such that:

$$\psi(f) = \int_{\mathbb{D}} f(z) d\mu_{\mathbb{D}}(z), \qquad \forall f \in C_c(\mathbb{D}).$$

Since the Riemannian manifold given by $\mathbb{D}$ together with the Riemannian metric ds we defined above is σ-compact, the measure $\mu_{\mathbb{D}}$ will be a Radon measure.
Then, for functions $f : \mathbb{D} \to \mathbb{D}$, with measurable gyronorm, we will be able to compute their gyronorm, given by:

$$\|f\|^{\mathbb{D}}_1 = \underset{\mathbb{D}}{\oint\!\!\!\!\!\oplus} \|f(z)\| d\mu_{\mathbb{D}}(z).$$

More concretely, by using Theorem 4.1, we can compute the gyronorm of said function by computing:

$$\|f\|^{\mathbb{D}}_1 = \tanh\left(\int_{\mathbb{D}} \operatorname{arctanh}(\|f(z)\|)\left(\frac{2}{1-x^2-y^2}\right)^2 dx\ dy\right).$$

If the function f is a $\mathfrak{L}^1_{\mathbb{D}}$ function, the gyronorm will be strictly less than 1, and we will be able to compute and make sense of distances between functions since

$L^1_{\mathbb{D}}(\mathbb{D}, \mathcal{B}(\mathbb{D}), \mu_{\mathbb{D}}, \oplus, \otimes, \|\cdot\|^{\mathbb{D}}_1, \operatorname{arctanh})$ has a metric space structure.
In particular, given two functions $f : \mathbb{D} \to \mathbb{D}$ and $g : \mathbb{D} \to \mathbb{D}$, both members of $L^1_{\mathbb{D}}(\mathbb{D}, \mathcal{B}(\mathbb{D}), \mu_{\mathbb{D}}, \oplus, \otimes, \|\cdot\|^{\mathbb{D}}_1, \operatorname{arctanh})$, we will be able to compute their gyrodistance by computing the following integral:

$$d_{\oplus}(f, g) = \tanh\left(\int_{\mathbb{D}} \operatorname{arctanh}(\| \ominus g(z) \oplus f(z)\|)\left(\frac{2}{1 - x^2 - y^2}\right)^2 dx\ dy\right).$$

Whereas, for computing their distance, it suffices to take the hyperbolic arctangent of the above expression, that is:

$$\begin{aligned}\delta(f, g) &= \operatorname{arctanh}(d_{\oplus}(f, g))\\ &= \int_{\mathbb{D}} \operatorname{arctanh}(\| \ominus g(z) \oplus f(z)\|)\left(\frac{2}{1 - x^2 - y^2}\right)^2 dx\ dy.\end{aligned}$$

In other words, $L^1_{\mathbb{D}}$ allow us to quantitatively compare $\mathfrak{L}^1_{\mathbb{D}}$ functions from the Möbius disk to itself, by considering the natural measure space structure of $\mathbb{D}$, $(\mathbb{D}, \mathcal{B}(\mathbb{D}), \mu_{\mathbb{D}})$, in a way that is native to its hyperbolic geometry.

$L^1_{\mathbb{D}}$ spaces can also be used to compute distances between functions from $\mathbb{H}$ to $\mathbb{H}$ (where here $\mathbb{H}$ denotes the upper half plane, $\mathbb{H} := \{z \in \mathbb{C}\ ,\ \operatorname{Im}(z) > 0\}$) in the Poincaré half-plane model of hyperbolic geometry. This is achieved through the use of the Cayley transform, $C : \mathbb{H} \to \mathbb{D}$.

Example 5.7 (Distance between functions from $\mathbb{H}$ to $\mathbb{H}$)**.** Let us consider the upper half plane $\mathbb{H}$ together with the Riemannian metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The volume form induced by the metric in question is:

$$dV = \frac{dx\ dy}{y^2}.$$

In a manner similar to Example 5.6, the above volume form will, thanks to the Riesz-Markov-Kakutani representation theorem, induce a measure on $\mathbb{H}$, which we will denote as $\mu_{\mathbb{H}}$.
We will, from now on, consider $\mathbb{H}$ together with the following measure space structure: $(\mathbb{H}, \mathcal{B}(\mathbb{H}), \mu_{\mathbb{H}})$.
Let $C : \mathbb{H} \to \mathbb{D}$ denote the Cayley transform, given by:

$$C(z) = \frac{z - i}{z + i}.$$

Let $f : \mathbb{H} \to \mathbb{H}$ be a function from the upper half plane to itself; by composing said function with the Cayley transform, we will obtain a new function g from the

upper half plane to the Möbius disk, $g : \mathbb{H} \to \mathbb{D}$.
At this point, by assuming the property of gyronorm measurability, we will be able to compute the gyronorm of g, given by:

$$\|g\|_1^{\mathbb{D}} = \oint\!\!\!\!\oplus_{\mathbb{H}} \|g(z)\| d\mu_{\mathbb{H}}(z).$$

More concretely, by using Theorem 4.1, we can compute the gyronorm of said function by computing:

$$\|g\|_1^{\mathbb{D}} = \tanh\left(\int_{\mathbb{H}} \operatorname{arctanh}(\|g(z)\|)\frac{dx\ dy}{y^2}\right),$$

which, given in terms of the original function $f : \mathbb{H} \to \mathbb{H}$ is:

$$\|g\|_1^{\mathbb{D}} = \tanh\left(\int_{\mathbb{H}} \operatorname{arctanh}\left(\left\|\frac{f(z)-i}{f(z)+i}\right\|\right)\frac{dx\ dy}{y^2}\right).$$

We will be more interested in functions $f : \mathbb{H} \to \mathbb{H}$ for which $\|g\|_1^{\mathbb{D}} < 1$, for $g = C(f(z))$. For this reason, we will give a special name to this class of functions.

Definition 5.8. Let $f : \mathbb{H} \to \mathbb{H}$ be a function from the upper half plane to itself, and let $C(z)$ denote the Cayley transform; we will say that f is a **Möbius gyrosummable function** if the composite function $C(f(z))$ is a $\mathfrak{L}_{\mathbb{D}}^1$ function, i.e. if $\|C(f(z))\|_1^{\mathbb{D}} < 1$.

It is possible to compute the gyrodistance between two Möbius gyrosummable functions, $\varphi_1 : \mathbb{H} \to \mathbb{H}$ and $\varphi_2 : \mathbb{H} \to \mathbb{H}$ by using the $L_{\mathbb{D}}^1$ space as an intermediary, through the following integral:

$$\| \ominus C(\varphi_1(z)) \oplus C(\varphi_2(z))\|_1^{\mathbb{D}} = \oint\!\!\!\!\oplus_{\mathbb{H}} \| \ominus C(\varphi_1(z)) \oplus C(\varphi_2(z))\| d\mu_{\mathbb{H}}.$$

We will use the symbol $d_{\oplus}^{\mathbb{H}\to\mathbb{D}}(\varphi_1, \varphi_2)$ to denote such gyrodistance. Operationally, this value is given by the following integral:

$$d_{\oplus}^{\mathbb{H}\to\mathbb{D}}(\varphi_1, \varphi_2) = \tanh\left(\int_{\mathbb{H}} \operatorname{arctanh}(\| \ominus C(\varphi_1(z)) \oplus C(\varphi_2(z))\|)\frac{dx\ dy}{y^2}\right).$$

There are also two other noteworthy cases that we will explore; the first one is that of functions from $\mathbb{C}$ to $\mathbb{D}$. In this case, we will consider the complex numbers together with the 2-dimensional Lebesgue measure, and to compute the gyrodistance between two functions we will compute the following integral:

$$d_{\oplus}(f, g) = \tanh\left(\int_{\mathbb{C}} \operatorname{arctanh}(\| \ominus g(z) \oplus f(z)\|) dx\ dy\right).$$

The second one is that of functions from the real number to the Möbius disk. In this case, we will consider the real numbers together with the Lebesgue measure on $\mathbb{R}$, and to compute the gyrodistance between two functions we will compute the following integral:

$$d_{\oplus}(f,g) = \tanh\left(\int_{\mathbb{R}} \operatorname{arctanh}(\| \ominus g(x) \oplus f(x)\|)dx\right).$$

6. Conclusions

In this paper, we showed that the set of functions from an arbitrary measure space to the Möbius disk $\mathbb{D}$ is a gyrolinear space, and that furthermore it is possible to endow the latter with a gyrodistance, which in turn induces a metric topology on the space. This gyrodistance was constructed through the introduction of a new operator, called the Lebesgue gyrointegral. Furthermore, we showed how, through the use of the gyrodistance we defined, we can calculate distances between functions from the Möbius disk to itself, as well as functions from other spaces to the Möbius disk, such as functions from the upper half plane to $\mathbb{D}$, from $\mathbb{C}$ to $\mathbb{D}$ or from $\mathbb{R}$ to $\mathbb{D}$.
Several directions for future work naturally arise. One possibility is to extend the analysis to higher-dimensional analogues of the disk, by considering functions from an arbitrary measure space to the Möbius ball $\mathbb{V}_s^n := \{v \in \mathbb{R}^n \ , \ \|v\| < s\}$, endowed with its natural gyrovector space structure.
Another direction involves considering gyrospaces of functions from an arbitrary measure space to the Einstein gyrovector spaces, thus allowing for the measurement of distances and the quantitative comparison of functions within a Beltrami–Klein hyperbolic framework. Finally, identifying a generalization of L^p spaces to a hyperbolic setting could yield further insights.

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