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A Survey on Metallic Vector Fields

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Abstract

This document introduces the idea of metallic vector fields in the framework of semi-Riemannian manifolds. Then, we study the geometry of such vector fields on closed and compact manifolds. The existence of metallic fields on immersed submanifolds will also be investigated. Finally, we investigate metallic vector fields on warped product manifolds.

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1. Introduction

Geometric vector fields are essential in the realms of theoretical physics and differential geometry. Among these vector fields are conformal and conformal Killing fields. A vector field X on a semi-Riemannian manifold (M, g) is classified as a conformal vector field if there exists a smooth function φ such that the Lie derivative of g along X, denoted as $\mathcal{L}_X g$, equals $2\varphi g$. When φ is identically zero, X is referred to as a conformal Killing vector field or simply a Killing vector field. It is widely recognized that the set of 1-parameter local diffeomorphisms generated by a Killing vector field acts as isometries of the manifold (M, g). Consequently, in

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the context of spacetime manifolds, Killing vector fields characterize the symmetries present in the underlying manifold. Moreover, thorough investigations into conformal vector fields can be found in [1–3]. Also, the notion of 2-conformal vector fields has been defined. A vector field X on a semi-Riemannian manifold (M, g) is termed a 2-conformal whenever there exists a smooth function φ such that $\mathcal{L}_X\mathcal{L}_X g = 2\varphi g$. If φ vanishes, then X becomes 2-Killing. Considering a 2-conformal vector field as a potential of hyperbolic Ricci soliton [4], we obtained some rigidity results for such spaces [5]
Using the notion of the square root of the metric tensor g, Garcia-Parrado and Senovilla have defined another geometric vector field called a bi-conformal vector field [6]. A bi-conformal vector field X refers to a vector field X defined on the manifold M that satisfies the following conditions:

$$\mathcal{L}_X g = \alpha g + \beta h, \quad \mathcal{L}_X h = \alpha h + \beta g,$$

where α and β are smooth functions, with h representing a symmetric square root of g. The functions α and β are referred to as gauges of the symmetry [1, 6], serving a purpose akin to the factor φ found in the concept of conformal vector fields. Subsequently, De et al. provided a definition for Ricci bi-conformal vector fields as outlined in [7].
A vector field X defined on a semi-Riemannian manifold (M, g) is classified as a Ricci bi-conformal vector field [7–11] if the following conditions hold:

$$\mathcal{L}_X g = \alpha g + \beta \mathrm{Ric}, \quad \mathcal{L}_X \mathrm{Ric} = \alpha \mathrm{Ric} + \beta g,$$

in which α and β are differentiable functions and Ric represents the Ricci curvature tensor associated with the metric g.

On the other hand, spacetime geometry and its connection to physical phenomena are fascinating topics of research that have attracted numerous researchers from both physics and mathematics. Einstein, in his general theory of relativity, described gravity as a factor which leads curving spacetime manifold. Hence, gravity is described by the metric tensor of spacetime manifold. Therefore, geometric vector fields on a spacetime manifold also carry physical quantities and have physical interpretations. Motivated by this fact, we define a new geometric vector field here:

Definition 1.1. A vector field X defined on a semi-Riemannian manifold (M^n, g) is termed a metallic vector field if it fulfills the condition

$$\mathcal{L}_X\mathcal{L}_X g = a\mathcal{L}_X g + bg, \tag{1}$$

where a and b represent real numbers.

The reason behind this naming will be explained in what follows.
If X is a homothetic conformal vector field, i.e., $L_X g = \sigma g$, for some real constant σ, then X is a metallic with scalars a and b if and only if σ satisfies the so-called

metallic quadratic equation $\sigma^2 - a\sigma - b = 0$.
In particular case, when $a = b = 1$ the golden number $\frac{1+\sqrt{5}}{2}$ is a root of equation $\sigma^2 - \sigma - 1 = 0$, so X is called a golden vector field whenever we have $\mathcal{L}_X\mathcal{L}_X g = \mathcal{L}_X g + g$.

A metallic vector field is a bridge to connect hyperbolic Ricci solitons to Ricci solitons structures, so it can relate wave phenomena of semi-Riemannian metrics and their heat behaviour. In the subsequent segment, we examine various geometric characteristics of metallic vector fields.

2. Geometry of metallic vector fields

In the following, we establish certain findings about metallic vector fields on compact Riemannian manifolds.

The following formulas have been proven in [12, 13] for any arbitrary vector field W on (M, g):

$$\mathrm{div}(\mathcal{L}_W g) = 2\big(d(\mathrm{div}(W)) + i_{Rc(W)}g\big), \tag{2}$$

$$\mathrm{trace}(\mathcal{L}_W\mathcal{L}_W g) = 2\left(\|\nabla W\|^2 + \mathrm{div}(\nabla_W W) - \mathrm{Ric}(W, W)\right), \tag{3}$$

where Ric denotes the Ricci curvature, Rc represents the Ricci operator defined such that $g(RcW, Y) = \mathrm{Ric}(W, Y)$ for any smooth vector fields W, Y defined on M, and ∇ signifies the Levi-Civita connection associated with g. At this point, we will initially deduce:

Proposition 2.1. *Let W be a metallic vector field with parameters a and b.*
(i) If $\mathcal{L}_W\mathcal{L}_W g$ is traceless, then either $\mathrm{div}W$ *is constant, or W is a 2-Killing field. Moreover, if M is closed and $\int_M \mathrm{Ric}(W, W) \leq 0$, then $\nabla W = 0$, hence W is a parallel vector field.*
(ii) When M is connected, compact, and divergence of $\mathcal{L}_W\mathcal{L}_W g$ is equal to zero, then either W is 2-conformal, or $\mathrm{Ric}(W, W) = 0$.

Proof. (i) When $\mathrm{trace}(\mathcal{L}_W\mathcal{L}_W g) = 0$, tracing into the Equation (1) it follows that:

$$2a\mathrm{div}(W) + nb = 0,$$

therefore, if $a \neq 0$, then $divW = -\frac{nb}{2a}$. Also, from the above formula we can deduce if $a = 0$, then $b = 0$ and (1) can be rewrite as $\mathcal{L}_W\mathcal{L}_W g = 0$.
Also, we mentioned

$$\mathrm{trace}(L_W L_W g) = \|\nabla W\|^2 - \mathrm{Ric}(W, W) - \mathrm{div}(\nabla_W W),$$

hence

$$\int_M \|\nabla W\|^2 = \int_M \mathrm{Ric}(W, W) \leq 0,$$

and we get $\nabla W = 0$.

(ii) If $\mathrm{div}(\mathcal{L}_W\mathcal{L}_W g) = 0$, and considering the following equality (see [12])

$$(\mathrm{div}(\mathcal{L}_W g))(Y) = 2Y(\mathrm{div}(W)) + 2\mathrm{Ric}(W, Y),$$

for some vector field Y. Incorporating the divergence operator into (1), we obtain

$$a\mathrm{div}(\mathcal{L}_W g) = 0. \tag{4}$$

Clearly, if $a = 0$ then W is 2-conformal by (1). In the case that $a \neq 0$, we get

$$\mathrm{Ric}(W, W) = -g(\mathrm{div}(W), W) = -W(\mathrm{div}(W)).$$

Since $\mathrm{trace}(\mathcal{L}_W\mathcal{L}_W g) = 0$ and the manifold M is connected, it follows that $\mathrm{trace}(\mathcal{L}_W\mathcal{L}_W g)$ remains constant. By applying the trace operation to both sides of (1), we deduce

$$\mathrm{trace}(\mathcal{L}_W\mathcal{L}_W g) = 2a(\mathrm{div}(W)) + nb,$$

and then differentiating it in the direction of W, we get $W(\mathrm{div}(W)) = 0$. Consequently, $\mathrm{Ric}(W, W) = 0$, thus concluding the proof. □

Corollary 2.2. *When W is a metallic vector field, $\mathrm{div}(\mathcal{L}_W\mathcal{L}_W g) = 0$, and $a \neq 0$ then, $\Delta(\mathrm{div}(W)) = -\mathrm{div}(RcW)$.*

Proof. We checked that $a\mathrm{div}(\mathcal{L}_W g) = 0$, and from (2) it follows that

$$2a\big(g(Rc(W), Y) + g(\nabla \mathrm{div} W, Y)\big) = 0,$$

for any arbitrary field Y, and we obtain

$$2a(\nabla \mathrm{div} W + Rc(W) = 0.$$

By analyzing the divergence mentioned above, we obtain

$$0 = 2a\big(\Delta(\mathrm{div}(W)) + \mathrm{div}(RcW)\big),$$

since $a \neq 0$ therefore, W satisfies $\Delta(\mathrm{div}(W)) = -\mathrm{div}(RcW)$. □

Proposition 2.3. *Let vector field W be a metallic, divergence-free, and $a \neq 0$. Then*

$$\int_M \|\mathcal{L}_W\mathcal{L}_W g\|^2 = nb^2 Vol(M) + 2a^2 \int_M \big(\|\nabla W\|^2 - \mathrm{Ric}(W, W)\big).$$

Moreover, if one of the following conditions hold:
(i) $\int_M \|\mathcal{L}_W\mathcal{L}_W g\|^2 \leq nb^2 Vol(M)$;
(ii) $2a^2 \int_M \big(\mathrm{Ric}(W, W) - \|\nabla W\|^2\big) \geq nb^2 Vol(M)$,
then W becomes Killing.

Proof. We calculate the Hilbert–Schmidt norms as shown in Equation (1), from which we deduce:

$$\begin{aligned}\|\mathcal{L}_W\mathcal{L}_W g\|^2 &= b^2\|\mathrm{Id}\|^2 + 2ab\langle g, \mathcal{L}_W g\rangle + a^2\|\mathcal{L}_W g\|^2\\ &= nb^2 + 4ab\mathrm{div}(W) + a^2\|\mathcal{L}_W g\|^2 = nb^2 + a^2\|\mathcal{L}_W g\|^2.\end{aligned}$$

Since by [14] we have:

$$\begin{aligned} 0 &= \int_M \left(\mathrm{Ric}(W,W) + \frac{1}{2}\|\mathcal{L}_W g\|^2 - \|\nabla W\|^2 - (\mathrm{div}(W))^2\right)\\ &= \int_M \left(\mathrm{Ric}(W,W) + \frac{1}{2}\|\mathcal{L}_W g\|^2 - \|\nabla W\|^2\right),\end{aligned}$$

we obtain

$$\int_M \big(\|\mathcal{L}_W\mathcal{L}_W g\|^2 - nb^2\big) = a^2\int_M \|\mathcal{L}_W g\|^2 = 2a^2\int_M \big(\|\nabla W\|^2 - \mathrm{Ric}(W,W)\big),$$

and we get the assertion.
If $\int_M \|\mathcal{L}_W\mathcal{L}_W g\|^2 \leq nb^2 Vol(M)$ and $a \neq 0$, then $\mathcal{L}_W g = 0$.
So, if $2a^2\int_M\big(\mathrm{Ric}(W,W) - \|\nabla W\|^2\big) \geq n\int_M nb^2 Vol(M)$, then $\mathcal{L}_W\mathcal{L}_W g = 0$. Hence $a\mathcal{L}_W g + bg = 0$. By taking the trace, we obtain $0 = 2a\mathrm{div}(W) = nb$, so $b = 0$ (constant) and $\mathcal{L}_W g = 0$. □

In the unique scenario where the metallic field is identified as a $W(\mathrm{Ric})$-vector field [15], we can establish the following proposition.

Proposition 2.4. *A metallic $W(\mathrm{Ric})$-vector field on a manifold (M^n, g) that meets the conditions $\nabla W = \lambda Rc$, $\lambda \in \mathbb{R}^*$, and trace$(\mathcal{L}_W Ric) = 0$, such that $a\lambda \neq 0$, is characterized as Ricci-flat, and W is a parallel vector field. Furthermore, the converse is also valid.*

Proof. One can easily check

$$\mathrm{div}(W) = \lambda R, \quad \mathcal{L}_W g = 2\lambda\mathrm{Ric}, \quad \mathcal{L}_W\mathrm{Ric} = 2\lambda\mathcal{L}_W(\mathrm{Ric}),$$

and the Equation (1) becomes

$$2\lambda\mathcal{L}_W Ric = 2a\lambda Ric + 2g.$$

Since trace$(\mathcal{L}_W\mathrm{Ric}) = 0$, we get $a\lambda R + nb = 0$. It follows that R is a constant. Since trace$(\mathcal{L}_W\mathcal{L}_W g) = 0$, (3) gives

$$\|\nabla W\|^2 + \mathrm{div}(\nabla_W W) - \mathrm{Ric}(W,W) = 0,$$

and from (2), we get

$$\mathrm{Ric}(W,W) = \frac{1}{2}\big(\mathrm{div}(\mathcal{L}_W g)(W) - W(\mathrm{div}(W))\big) = 0.$$

As

$$\|\nabla W\|^2 = \lambda^2\|Rc\|^2, \quad \mathrm{div}(\nabla W) = \lambda \mathrm{div}(RcW),$$

we infer

$$\lambda^2\|Rc\|^2 + \lambda div(RcW) = 0.$$

By integrating the above equation, we get $Rc = 0$, then $\nabla W = 0$, so we get the conclusion. □

3. Metallic vector fields on submanifolds

Let N represent a Riemannian manifold equipped with the metric $\bar{g}$, while M denotes an isometrically immersed submanifold within N, possessing the induced metric g. For any smooth vector fields W and Y defined on M, along with any normal vector field V, the Gauss and Weingarten equations are expressed as follows:

$$\bar{\nabla}_W Y = \nabla_W Y + h(W,Y), \quad \bar{\nabla}_W W = -A_W W + \nabla^{\perp}_W W,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection corresponding to $\bar{g}$, ∇ signifies the Levi-Civita connection associated with g, h represents the second fundamental form, A_W indicates the shape operator linked to W, and $\nabla^{\perp}$ is the normal connection.

In the following discussion, we will assume that M contains a metallic vector field, specifically the tangential component $W^{\top}$ of a concurrent vector field W existing on the manifold $(\bar{M}, \bar{g})$. Consequently, it holds that $\bar{\nabla} W = I$, where I denotes the identity map. Furthermore, for any vector fields U and V that are tangent to M, we find [16]

$$\begin{aligned}
\nabla_U W^{\top} = &\quad U + A_{W^{\top}} U,\\
(\mathcal{L}_{W^{\top}} g)(U,V) = &\quad 2\big(g(U,V) + g(A_{W^{\perp}}U, U)\big),\\
(\mathcal{L}_{W^{\top}}\mathcal{L}_{W^{\top}} g)(U,V) = &\quad 2\Big(2g(U,V) + 4g(A_{W^{\perp}}U,V) + 2g(A^2_{W^{\perp}}U,V)\\
&\quad + g\big((\nabla_{W^{\top}} A_{W^{\perp}})U,V\big)\Big).
\end{aligned}$$

We would like to remind [17, 18] that a hypersurface is classified as a *metallic shaped hypersurface* if its shape operator A adheres to the equation

$$A^2 = rA + sI,$$

where r and s are real constants. It has been established that for a hypersurface situated within a space of constant curvature, if the aforementioned condition holds at a point, the hypersurface qualifies as pseudosymmetric (for further information, refer to [14, 19]).

At this point, we can present the subsequent findings.

Proposition 3.1. *In the event that $W^\top$ represents a metallic vector field situated on the hypersurface (M, g) exhibiting a parallel shape operator (specifically, $\nabla A_{W^\perp} = 0$), it can be classified as a metallic shaped hypersurface.*

Proof. From Equation (1) and utilizing the aforementioned relations, we derive

$$(4-2a)g(U,E) + (8-2a)g(A_{W^\perp}U,E) + 4g(A^2_{W^\perp}U,E) = bg(U,E),$$

for any vector fields U, E tangent to M. Hence

$$A^2_{\zeta^\perp} = \frac{4-a}{2}A_{\zeta^\perp} + \frac{2a+b-4}{4}I,$$

and we reach the conclusion. □

Proposition 3.2. *Let the vector field denoted as $W^\top$ be metallic, and consider (M, g) as a $W^\top$-totally umbilical submanifold characterized by the condition $A_{W^\perp} = fI$, where f represents a smooth real-valued function defined on M. Then f satisfies*

$$W^\top(W^\top(f)) + (4f - a + 4)W^\top(f) = 0.$$

Proof. In this case,

$$A^2_{W^\perp}U = f^2U, \quad (\nabla_{W^\top}A_{W^\perp})U = W^\top(f)U,$$

where U is arbitrary vector field, and we reach

$$4f^2 + 2(4-a)f + 2W^\top(f) - 2a - b + 4 = 0.$$

Taking the derivative along $W^\top$ results in

$$-2W^\top(W^\top(f)) - 2(4f - a + 4)W^\top(f) = 0,$$

which leads us to our conclusion. □

It is important to note that a *totally geodesic submanifold* refers to a submanifold characterized by a vanishing shape operator; hence, as a consequence, we deduce:

Proposition 3.3. *In a totally geodesic submanifold, $W^\top$ is metallic vector field with $2a + b = 4$.*

Proposition 3.4. *If (M^n, g) denotes a compact minimal submanifold and $W^\top$ is metallic with $a \neq 0$ and the divergence of $\mathcal{L}_{W^\top}\mathcal{L}_{W^\top}g$ is zero, then*

$$\int_M \|A_{W^\perp}\|^2 = n(n-1)Vol(M).$$

Proof. Given that M is compact, we can conclude that [3]

$$\int_M \left(\mathrm{Ric}(W^\top, W^\top) + \frac{1}{2}\|\mathcal{L}_{W^\top} g\|^2 - \|\nabla W^\top\|^2 - (\mathrm{div}(W^\top))^2 \right) = 0.$$

Direct computations give

$$\|\mathcal{L}_{W^\top} g\|^2 = 4\left(\|A_{W^\perp}\|^2 + 2\mathrm{trace}(A_{W^\perp}) + n\right) = 4\left(\|A_{W^\perp}\|^2 + n\right),$$

$$\|\nabla W^\top\|^2 = \|A_{W^\perp}\|^2 - 2\mathrm{trace}(A_{W^\perp}) + n = \|A_{W^\perp}\|^2 + n,$$

$$(\mathrm{div}(W^\top))^2 = (\mathrm{trace}(A_{W^\perp}))^2 + 2n\mathrm{trace}(A_{W^\perp}) + n^2 = n^2,$$

Given that M represents a minimal submanifold. Additionally, from the equation $\mathrm{div}(\mathcal{L}_{W^\top}\mathcal{L}_{W^\top} g) = 0$ and the condition $a \neq 0$, it follows that $\mathrm{div}(\mathcal{L}_{W^\top} g) = 0$. From Equation (2), we derive

$$\mathrm{Ric}(W^\top, W^\top) = 0,$$

hence we get the result. □

Proposition 3.5. *Consider a minimal submanifold denoted as (M^n, g) where $W^\top = \nabla\psi$ is metallic, $a \neq 0$, and it is known that $\mathcal{L}_{W^\top}\mathcal{L}_{W^\top} g$ exhibits divergence-free properties. Under these conditions, we have*

$$\frac{1}{2}\Delta(\|\nabla\psi\|^2) = \|A_{\nabla\psi}\|^2 + n.$$

Thus, it follows that $\|\nabla\psi\|^2$ qualifies as a subharmonic function (specifically, $\Delta(\|\nabla\psi\|^2) \geq 0$). Additionally, if the manifold M be closed, then $\nabla\psi = 0$ is a concurrent vector field.

Proof. Applying Bochner's formula yields

$$\frac{1}{2}\Delta(|\nabla\psi|^2) = \|\nabla W^\top\|^2 + W^\top(\mathrm{div}(W^\top)) + \mathrm{Ric}(W^\top, W^\top),$$

and through the calculations conducted previously, we arrive at the initial conclusion. Assuming that M is closed, and taking into account that $n = \Delta(\psi)$, by integrating this equation, we derive

$$\int_M \|A_{\nabla\psi}\|^2 = 0,$$

which shows $A_{\nabla\psi} = 0$, and the proof is completed. □

4. Metallic vector fields on warped product manifolds

In the following, we present two outcomes concerning metallic vector fields within warped product manifolds. Throughout the remainder of this section, (M_i, g_i) denotes semi-Riemannian manifolds for $i = 1, 2$, while $M = M_1 \times_f M_2$ represents a warped product semi-Riemannian manifold characterized by the metric tensor $g = g_1 + f^2 g_2$ along with the warping function $f : M_1 \to \mathbb{R}$. A smooth vector field U defined on M can be expressed as $U = U_1 + U_2$, where each U_i symbolizes smooth vector fields that are tangent to their respective manifolds M_i.

Proposition 4.1. *Let vector field $W = W_1 + W_2$ be 2-Killing on $M_1 \times_f M_2$, then:*
(i) W_1 is a metallic vector field on (M_1, g_1) with $a \neq 0$ iff W_1 be Killing.

$$\mathcal{L}_{W_1} g_1 = 0,$$

(ii) W_2 is a metallic field on (M_2, g_2) if and only if there exist $a, b \in \mathbb{R}$ such that

$$\mathcal{L}_{W_2} g_2 = -\frac{bf^2 + W_1(W_1(f^2))}{2W_1(f^2) + af^2 g_2},$$

provided that $2W_1(f^2) + af^2 \neq 0$.

Proof. From [13], for all $U = U_1 + U_2$ and $V = V_1 + V_2$ we have

$$\begin{aligned}(\mathcal{L}_W \mathcal{L}_W g)(U, V) &= (\mathcal{L}_{W_1} \mathcal{L}_{W_1} g_1)(U_1, V_1) + f^2(\mathcal{L}_{W_2} \mathcal{L}_{W_2} g_2)(U_2, V_2) \\ &+ 2W_1(f^2)(\mathcal{L}_{W_2} g_2)(U_2, V_2) + W_1(W_1(f^2))g_2(U_2, V_2).\end{aligned}$$

Since, W is a 2-Killing vector field, then

$$\begin{aligned}(\mathcal{L}_{W_1} \mathcal{L}_{W_1} g_1) &= 0, \\ f^2(\mathcal{L}_{W_2} \mathcal{L}_{W_2} g_2) + 2W_1(f^2)(\mathcal{L}_{W_2} g_2) + W_1 W_1(f)) &= 0.\end{aligned} \tag{5}$$

If W_1 be a metallic vector field, then there exist real scalars a and b, such that

$$\mathcal{L}_{W_1} \mathcal{L}_{W_1} g_1 = a\mathcal{L}_{W_1} g_1 + bg_1,$$

hence, we have $a\mathcal{L}_{W_1} g_1 + bg_1 = 0$. If $a \neq 0$, then

$$\mathcal{L}_{W_1} g_1 = \frac{-b}{a} g_1 \Rightarrow \mathcal{L}_{W_1} \mathcal{L}_{W_1} g_1 = \frac{b^2}{a^2} g_1.$$

Therefore $b = 0$ and we get the conclusion.
Similarly, if W_2 be a metallic vector field, then there exist real scalars a and b, such that

$$\mathcal{L}_{W_2} \mathcal{L}_{W_2} g_2 = a\mathcal{L}_{W_2} g_2 + bg_2.$$

Now, this formula beside (5), gives

$$f^2(a\mathcal{L}_{W_2} g_2 + bg_2) + 2W_1(f^2)(\mathcal{L}_{W_2} g_2) + W_1 W_1(f))g_2 = 0,$$

so,

$$\mathcal{L}_{W_2} g_2 = -\frac{bf^2 + W_1(W_1(f^2))}{2W_1(f^2) + af^2} g_2.$$

□

Proposition 4.2. *When manifold $M_1 \times_f M_2$ admits a metallic vector field $W = W_1 + W_2$ with parameters a, b, then:*
(i) W_1 is a metallic vector field on (M_1, g_1).
(ii) W_2 is a metallic vector field on (M_2, g_2) if and only if

$$W_1(\ln f) \qquad \text{and} \qquad \tfrac{W_1(W_1(f))}{f},$$

be constants.

Proof. From [13], we can write

$$(\mathcal{L}_W g)(U, V) = (\mathcal{L}_{W_1} g_1)(U_1, V_1) + f^2 (\mathcal{L}_{W_2} g_2)(U_2, V_2) + W_1(f^2) g_2(U_2, V_2),$$

for every vector field U_1, V_i tangent to M_i. At this point, the conclusion derives from (1), along with the formulation of the second Lie derivative of g as outlined below.
There exist scalars a and b such that

$$(\mathcal{L}_W \mathcal{L}_W g)(U, V) = a(\mathcal{L}_W g)(U, V) + bg(U, V),$$

so, we get

$$\begin{gathered}(\mathcal{L}_{W_1} \mathcal{L}_{W_1} g_1)(U_1, V_1) + f^2 (\mathcal{L}_{W_2} \mathcal{L}_{W_2} g_2)(U_2, V_2) + 2X_1(f^2)(\mathcal{L}_{W_2} g_2)(U_2, V_2) \\ + W_1(W_1(f^2)) g_2(U_2, V_2) = \\ a\big((\mathcal{L}_{W_1} g_1)(U_1, V_1) + f^2 (\mathcal{L}_{W_2} g_2)(U_2, V_2) + W_1(f^2) g_2(U_2, V_2)\big) \\ + bg_1(U_1, V_1) + bf^2 g_2(U_2, V_2),\end{gathered}$$

for all vector fields U_i, V_i tangent to M_i. Hence, we have

$$\begin{aligned} \mathcal{L}_{W_1} \mathcal{L}_{W_1} g &= a \mathcal{L}_{W_1} g + bg, \\ f^2 \mathcal{L}_{W_2} \mathcal{L}_{W_2} g_2 &= (af^2 - 2X_1(f^2)) \mathcal{L}_{W_2} g_2 + \big(bf^2 + W_1(f^2) - W_1(W_1(f^2))\big) g_2. \end{aligned}$$

The above equations indicate that W_1 is a metallic vector field on (M_1, g_1), and W_2 is metallic field on (M_2, g_2) if and only if

$$\frac{W_1(f^2)}{f^2} = W_1(\ln f), \quad \frac{W_1(W_1(f))}{f},$$

be constants. □

Remark 1. Let the warping function f be constant. When $M_1 \times M_2$ admits a metallic vector field $W = W_1 + W_2$, then W_i are metallic fields over M_i.

Example 4.3. If vector field $W = k\frac{\partial}{\partial t} + U$ is metallic on the warped product Robertson-Walker spacetime $(I \times_f \mathbb{R}^3)$, where $k \in \mathbb{R}$ and

$$f : I \to \mathbb{R}, \qquad f(t) = e^{c_1 t + c_2} \qquad c_1, c_2 \in \mathbb{R},$$

then U is a metallic vector field on $(\mathbb{R}^3, g_{can})$, by means of Proposition 4.2.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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