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Characterization of Approximate a-Birkhoff-James Orthogonality in C^*-Algebras

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Abstract

Assume that $\mathcal{A}$ is a unital C^*-algebra and $a \in \mathcal{A}$ is a positive and invertible element. Set

$$\mathcal{S}_a(\mathcal{A}) = \{\frac{f}{f(a)} : f \in \mathcal{S}(\mathcal{A}),\ f(a) \neq 0\},$$

where $\mathcal{S}(\mathcal{A})$ is the state space of $\mathcal{A}$.

The main aim of this paper is to introduce and study the notions of approximate a-orthogonality and approximate a-Birkhoff-James orthogonality associated to the norm:

$$\|x\|_a = \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \sqrt{\varphi(x^*ax)} \quad (x \in \mathcal{A}),$$

in C^*-algebra $\mathcal{A}$. First, by providing some examples, we show that these approximate orthogonalities are generally incomparable in non-commutative C^*-algebras. Next, we will see that under what conditions, these orthogonality relationships are related. Also, two different characterizations of approximate a-Birkhoff-James orthogonality in terms of the elements of $\mathcal{S}_a(\mathcal{A})$ are obtained. Moreover, the strong version of approximate a-Birkhoff-James orthogonality is studied. Finally, we prove that if approximate a-Birkhoff-James orthogonality and its strong version coincide on $\mathcal{A}$, then $\mathcal{A}$ is commutative.

Keywords: C^*-algebra, State space of C^*-algebra, Approximate orthogonality, Approximate a-Birkhoff-James orthogonality.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$, and let $\mathcal{A}'$ be the topological dual space of $\mathcal{A}$. The adjoint of element $x \in \mathcal{A}$ is denoted by x^*. Also, the real part of x is denoted by $\mathrm{Re}(x) = \frac{1}{2}(x + x^*)$. We denote the cone of positive elements of $\mathcal{A}$ by $\mathcal{A}^+$. A linear functional $f \in \mathcal{A}'$ is called positive if $f(a) \geq 0$ for all $a \in \mathcal{A}^+$.

The set of all positive linear functionals $f \in \mathcal{A}'$ such that $\|f\| = 1$ is denoted by $\mathcal{S}(\mathcal{A})$ and it is called state space of $\mathcal{A}$. Let $a \in \mathcal{A}^+$. A generalization of $\mathcal{S}(\mathcal{A})$ is proposed in [1] as the set

$$\mathcal{S}_a(\mathcal{A}) := \{\varphi \in \mathcal{A}^{'} : \varphi \geq 0, \varphi(a) = 1\}.$$

Obviously, $\mathcal{S}_a(\mathcal{A}) = \mathcal{S}(\mathcal{A})$, whenever $a = 1_{\mathcal{A}}$. According to [1, Proposition 2.3], if a is invertible, then $\mathcal{S}_a(\mathcal{A})$ is w^*-compact and

$$\|x\|_a := \sup\{\sqrt{\varphi(x^*ax)} : \varphi \in \mathcal{S}_a(\mathcal{A})\}, \quad (x \in \mathcal{A}),$$

is a sub-multiplicative norm on $\mathcal{A}$. Moreover, it was proved in [1] the following result:

Proposition 1.1. *([1, Lemma 3.1]). For any $x \in \mathcal{A}$ and $a \in \mathcal{A}^+$ such that $xa = ax$, we have $\|x\|_a \leq \|x\|$.*

The a-adjoint of $x \in \mathcal{A}$ is the element $x^\sharp \in \mathcal{A}$ such that $ax^\sharp = x^*a$. It was proved in [1, Corollary 4.9] that

$$\|x\|_a^2 = \|xx^\sharp\|_a = \|x^\sharp x\|_a = \|x^\sharp\|_a^2. \tag{1}$$

Further details regarding these concepts can be found in previous studies by [1, 2].

The concept of Birkhoff-James orthogonality (briefly, BJ-orthogonality) provide a good framework for studying the geometry of operator spaces; see e.g., [3–7] and the references therein. In particular, BJ-orthogonality in C^*-algebras and Hilbert C^*-modules has been studied extensively in [8–13].

Let $\varepsilon \in [0, 1)$. In inner product space $(X, \langle\cdot,\cdot\rangle)$, a natural way to generalize orthogonality is to define the approximate orthogonality by: $x \perp_\varepsilon y$ if and only if $|\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|$ $(x, y \in X)$; see [14]. Based on this idea, for elements x and y in unital C^*-algebra $\mathcal{A}$, approximate orthogonality with respect to the $\mathcal{A}$-valued inner product $\langle x, y\rangle = x^*y$ (ε-orthogonality) is stablished by $\|\langle x, y\rangle\| \leq \varepsilon\|x\|\|y\|$ [12]. Chmieliński et al. in [14–17] introduced and studied the concept of approximate Birkhoff-James orthogonality (ε-BJ-orthogonality) in normed linear spaces. Accordingly, $x \in \mathcal{A}$ is said to be approximate Birkhoff-James orthogonal to $y \in \mathcal{A}$, written as $x \perp_{BJ-\varepsilon} y$, if

$$\|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\| \qquad (\forall\, \lambda \in \mathbb{C}).$$

Approximate BJ-orthogonality of Hilbert space operators, and operators on C^*-algebras and Hilbert C^*-modules are widely studied in [12, 17–19]. Also, approximate BJ-orthogonality of operators on semi-Hilbert spaces is investigated in [20, 21].

Recently, the concept of BJ-orthogonality associated to $\|\cdot\|_a$ in unital C^*-algebra $\mathcal{A}$, so called a-Birkhoff-James orthogonality, has been investigated in [22]. In this paper, we consider approximate a-orthogonality and approximate a-BJ-orthogonality in $\mathcal{A}$. By presenting some interesting examples, we describe the relation between these orthogonality relationships. In particular, we show that approximate a-orthogonality implies approximate a-Birkhoff-James orthogonality, provided that $a \geq 1_{\mathcal{A}}$. Next, two different characterizations of approximate a-BJ-orthogonality based on the elements of $\mathcal{S}_a(\mathcal{A})$ are obtained. Moreover, the strong version of approximate a-BJ-orthogonality in unital C^*-algebras is studied. In particular, we prove that if these two concepts of orthogonality are coincide on $\mathcal{A}$, then $\mathcal{A}$ is commutative.

2. Approximate a-Birkhoff-James orthogonality in C^*-algebras

Throughout the paper, we suppose that $\mathcal{A}$ is a unital C^*-algebra with unit $1_{\mathcal{A}}$ and $a \in \mathcal{A}^+$ is invertible. Also, for any $x, y \in \mathcal{A}$, we define $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_a : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $\langle x, y \rangle_a := x^* a y$.
First we introduce the notions of approximate a-orthogonality and approximate a-BJ-orthogonality related to $\|\cdot\|_a$ in $\mathcal{A}$.

Definition 2.1. For $\varepsilon \in [0, 1)$, we say that an element $x \in \mathcal{A}$ is approximate a-orthogonal ((ε, a)-orthogonal) to element $y \in \mathcal{A}$, denoted by $x \perp_\varepsilon^a y$, if

$$\|\langle x, y \rangle_a\|_a \leq \varepsilon\, \|x\|_a \|y\|_a.$$

Note that $(\varepsilon, 1_{\mathcal{A}})$-orthogonality coincides with ε-orthogonality.

Definition 2.2. For $\varepsilon \in [0, 1)$, we say that an element $x \in \mathcal{A}$ is approximate a-Birkhoff-James orthogonal ((ε, a)-BJ-orthogonal) to element $y \in \mathcal{A}$, in short $x \perp_{BJ-\varepsilon}^a y$, if

$$\|x + \lambda y\|_a^2 \geq \|x\|_a^2 - 2\varepsilon\|x\|_a\|\lambda y\|_a, \qquad (\forall \lambda \in \mathbb{C}).$$

Note that $(\varepsilon, 1_{\mathcal{A}})$-BJ-orthogonality matches with ε-BJ-orthogonality. Also, Clearly, if $\varepsilon = 0$, then the above definition coincides with the definition of a-Birkhoff-James orthogonality which is defined and studied in [22].

Proposition 2.3. *For any $x, y \in \mathcal{A}$, the following statements hold:*

(i) *For $\varepsilon \in [0, \frac{1}{2})$, (ε, a)-BJ-orthogonality is non-degenerated,*

(ii) *(ε, a)-BJ-orthogonality is homogenous,*

(iii) *$x \perp^a_{BJ-\varepsilon} y$ if and only if $x^\sharp \perp^a_{BJ-\varepsilon} y^\sharp$,*

(iv) *Let $x, y \in \mathcal{A}$ be nonzero elements. If $x \perp^a_{BJ-\varepsilon} y$, then x, y are linearly independent.*

Proof. (i) If $x \in \mathcal{A}$ such that $x \perp^a_{BJ-\varepsilon} x$, then $\|x + \lambda x\|^2_a \geq \|x\|^2_a - 2\varepsilon|\lambda|\|x\|^2_a$ for all $\lambda \in \mathbb{C}$. Let $\lambda = -1$. Then we get $\|x\|^2_a(1 - 2\varepsilon) \leq 0$, and hence $\|x\|^2_a = 0$, since $\varepsilon \in [0, \frac{1}{2})$. Thus $x = 0$.

(ii) Assume that $x \perp^a_{BJ-\varepsilon} y$. Let $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$. Then

$$\begin{aligned}\|\alpha x + \lambda\beta y\|^2_a = \|\alpha(x + \lambda\frac{\beta}{\alpha}y)\|^2_a &= |\alpha|^2\|x + \lambda\frac{\beta}{\alpha}y\|^2_a \\ &\geq |\alpha|^2(\|x\|^2_a - 2\varepsilon|\lambda|\|x\|_a\|\frac{\beta}{\alpha}y\|_a) \\ &= \|\alpha x\|^2_a - 2\varepsilon|\lambda|\|\alpha x\|_a\|\beta y\|_a,\end{aligned}$$

for all $\lambda \in \mathbb{C}$. It follows that $\alpha x \perp^a_{BJ-\varepsilon} \beta y$.

(iii) Assume that $x \perp^a_{BJ-\varepsilon} y$. So, by (1), we get

$$\begin{aligned}\|x^\sharp + \lambda y^\sharp\|^2_a &= \|(x + \overline{\lambda}y)^\sharp\|^2_a = \|x + \overline{\lambda}y\|^2_a \\ &\geq \|x\|^2_a - 2\varepsilon|\lambda|\|x\|_a\|y\|_a = \|x^\sharp\|^2_a - 2\varepsilon|\lambda|\|x^\sharp\|_a\|y^\sharp\|_a,\end{aligned} \tag{2}$$

for all $\lambda \in \mathbb{C}$. Therefore $x^\sharp \perp^a_{BJ-\varepsilon} y^\sharp$. Also, (2) immediately follows the converse.

(iv) As a contrary, suppose that $x \perp^a_{BJ-\varepsilon} y$, but x, y are not linearly independent. Hence $x = ky$ for some $k \in \mathbb{C}$. Then

$$\|ky + \lambda y\|^2_a \geq \|ky\|^2_a - 2\varepsilon\|ky\|_a\|\lambda y\|_a, \quad (\forall\lambda \in \mathbb{C}),$$

and so

$$|k + \lambda|^2\|y\|^2_a \geq \|y\|^2_a(|k|^2 - 2\varepsilon|k|\,|\lambda|), \quad (\forall\lambda \in \mathbb{C}).$$

Since $\|y\|_a \neq 0$, we get $|k + \lambda|^2 \geq |k|^2 - 2\varepsilon|k|\,|\lambda|$ for all $\lambda \in \mathbb{C}$. Let $\lambda = \frac{-k}{2^n}$ $(n \in \mathbb{N})$. Hence

$$|k|^2(1 - \frac{1}{2^n})^2 \geq |k|^2(1 - \frac{\varepsilon}{2^{n-1}}).$$

Consequently,

$$\varepsilon \geq \big(1 - (1 - \frac{1}{2^n})^2\big)2^{n-1} = 1 - \frac{1}{2^{n+1}} \qquad (n \in \mathbb{N}). \tag{3}$$

On the other hand, $\lim_{n\to\infty}(1 - \frac{1}{2^{n+1}}) = 1$. Therefore (3) implies that $\varepsilon \geq 1$, which is impossible. $\square$

Remark 1. Let $x \in \mathcal{A}$. Then $x^\sharp = a^{-1}x^*a$ is a unique a-adjoint of x. Hence (1) implies that

$$\|x\|_a^2 = \|a^{-1}x^*ax\|_a^2 = \|xa^{-1}x^*a\|_a = \|a^{-1}x^*a\|_a^2.$$

As a consequence of this fact, if $\mathcal{A}$ is commutative, then

$$\|x\|_a^2 = \|x^*x\|_a^2 = \|xx^*\|_a = \|x^*\|_a^2.$$

Therefore $\|\cdot\|_a$ coincides with the C^*-norm of $\mathcal{A}$, and so (ε, a)-orthogonality and ε-orthogonality are the same. Also, (ε, a)-BJ-orthogonality and ε-BJ-orthogonality are matched.

It is known that $\perp_\varepsilon \subseteq \perp_{BJ-\varepsilon}$ (see [12, Proposition 3.1]). The following example demonstrates that there is no such a relationship between $\perp_\varepsilon^a$ and $\perp_{BJ-\varepsilon}^a$, in general.

Example 2.4. Let Tr be the trace functional on C^*-algebra of all 2×2 complex matrices $\mathbb{M}_2(\mathbb{C})$ with identity matrix I_2 as unit. Consider the positive linear functional φ_h is defined by

$$\varphi : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}, \quad \varphi_h(x) = \mathrm{Tr}(hx) \quad (h \in \mathbb{M}_2(\mathbb{C})^+).$$

Then for each $a \in \mathbb{M}_2(\mathbb{C})^+$, we have

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \,:\, h \in \mathbb{M}_2(\mathbb{C})^+ \quad \text{and} \quad \mathrm{Tr}(ha) = 1\}.$$

First we consider the matrix $a = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$ and assume that $\varepsilon \in [\frac{1}{4}, \frac{1}{2})$. We show that there are $x, y \in \mathbb{M}_2(\mathbb{C})$ such that $x \perp_\varepsilon^a y$, but $x \not\perp_{BJ-\varepsilon}^a y$. After some simple matrix computations, we conclude that

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \,:\, h \in \mathcal{K}_a\},$$

where

$$\mathcal{K}_a := \left\{ h = \begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})^+ \,:\, h_{12} \in \mathbb{C},\, h_{11}, h_{22} \geq 0,\ \frac{1}{4}h_{11} + \frac{1}{5}h_{22} = 1 \right\}.$$

Let $x = y = I_2$. Then $\|x\|_a = 1$ and $\|y\|_a = 1$. Moreover, we have

$$\begin{aligned}
\|\langle x, y\rangle_a\|_a^2 = \|x^*ay\|_a^2 = \|a\|_a^2 &= \sup_{\varphi_h \in \mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))} \varphi_h(a^3) \\
&= \sup_{h \in \mathcal{K}_a} \mathrm{Tr}\Big(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{64} & 0 \\ 0 & \frac{1}{125} \end{bmatrix} \Big) \\
&= \sup_{\frac{1}{4}h_{11} + \frac{1}{5}5h_{22} = 1, h_{11}, h_{22} \geq 0} \frac{1}{64}h_{11} + \frac{1}{125}h_{22} = \frac{1}{16}.
\end{aligned}$$

Then

$$\|\langle x,y\rangle_a\|_a=\frac{1}{4}\leq\varepsilon\|x\|_a\|y\|_a=\varepsilon.$$

Therefore $x\perp^a_\varepsilon y$. On the other hand, for $\lambda=-1$, we have

$$\|x+\lambda y\|_a^2=0<1-2\varepsilon|\lambda|\|x\|_a\|y\|_a=1-2\varepsilon.$$

It follows that $x\not\perp^a_{BJ-\varepsilon} y$.

Now, let $a=\begin{bmatrix}2&0\\0&1\end{bmatrix}$ and let $\varepsilon\in[0,1)$. We prove that there are $x,y\in\mathbb{M}_2(\mathbb{C})$ such that $x\perp^a_{BJ-\varepsilon} y$ while $x\not\perp^a_\varepsilon y$. To this end, note that

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))=\{\varphi_h\,:\,h\in\mathcal{L}_a\},$$

where

$$\mathcal{L}_a:=\left\{h=\begin{bmatrix}h_{11}&h_{12}\\\overline{h}_{12}&h_{22}\end{bmatrix}\in\mathbb{M}_2(\mathbb{C})^+\,:\,h_{12}\in\mathbb{C},\,h_{11},h_{22}\geq 0,\;\;2h_{11}+h_{22}=1\right\}.$$

Take $x=I_2$ and $y=\begin{bmatrix}0&0\\1&0\end{bmatrix}$. Then $\|x\|_a=1$ and

$$\begin{aligned}\|y\|_a^2&=\sup_{\varphi_h\in\mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))}\varphi_h(y^*ay)=\sup_{h\in\mathcal{L}_a}\mathrm{Tr}(h(y^*ay))\\&=\sup_{h\in\mathcal{L}_a}\mathrm{Tr}\left(\begin{bmatrix}h_{11}&0\\0&0\end{bmatrix}\right)=\sup_{2h_{11}+h_{22}=1,h_{11},h_{22}\geq 0}h_{11}=\frac{1}{2}.\end{aligned}$$

Hence for every $\lambda\in\mathbb{C}$, we have

$$\begin{aligned}\|x+\lambda y\|_a^2&=\sup_{\varphi_h\in\mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))}\varphi_h((x+\lambda y)^*a(x+\lambda y))\\&=\sup_{h\in\mathcal{L}_a}\mathrm{Tr}\left(\begin{bmatrix}h_{11}&h_{12}\\\overline{h}_{12}&h_{22}\end{bmatrix}\begin{bmatrix}2+|\lambda|^2&\overline{\lambda}\\\lambda&1\end{bmatrix}\right)\\&=\sup_{2h_{11}+h_{22}=1,h_{11},h_{22}\geq 0}\big((2+|\lambda|^2)h_{11}+2\mathrm{Re}(\lambda h_{12})+h_{22}\big)\\&\geq 1+\frac{|\lambda|^2}{2}\geq 1=\|x\|_a^2,\end{aligned}$$

since $h_0=\begin{bmatrix}\frac{1}{2}&0\\0&0\end{bmatrix}\in\mathcal{L}_a$. Then

$$\|x+\lambda y\|_a^2\geq\|x\|_a^2\geq\|x\|_a^2-2\varepsilon|\lambda|\|x\|_a\|y\|_a,$$

for all $\varepsilon \in [0,1)$, and so $x \perp^a_{BJ-\varepsilon} y$. But $x \not\perp^a_\varepsilon y$. In fact,

$$\begin{aligned}\|\langle x,y\rangle_a\|_a^2 &= \|x^*ay\|_a^2 = \| \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \|_a^2 \\ &= \sup_{h\in\mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \sup_{2h_{11}+h_{22}=1, h_{11},h_{22}\geq 0} h_{11} = \frac{1}{2}.\end{aligned}$$

Therefore, if

$$\|\langle x,y\rangle_a\|_a = \frac{1}{\sqrt{2}} \leq \varepsilon \frac{1}{\sqrt{2}},$$

then $\varepsilon \geq 1$, which is not possible.

In the next result, we will see under what circumstances, the concept of (ε,a)-orthogonality and (ε,a)-BJ-orthogonality are related.

Theorem 2.5. *Assume that $\varepsilon \in (0,1)$. Let $x,y \in \mathcal{A}$ and let $a \in \mathcal{A}$ such that $a \geq 1_\mathcal{A}$. If $x \perp^a_\varepsilon y$, then $x \perp^a_{BJ-\varepsilon} ya$.*

Proof. Note that since $aa^{-1} = a^{-1}a = 1_\mathcal{A}$ and $a \geq 1_\mathcal{A}$, by Proposition 1.1, we conclude that $\|a^{-1}\|_a \leq \|a^{-1}\| \leq 1$. Hence

$$\|y\|_a = \|yaa^{-1}\|_a \leq \|ya\|_a\|a^{-1}\|_a \leq \|ya\|_a. \tag{4}$$

Now, let $\varphi \in \mathcal{S}_a(\mathcal{A})$ so that $\varphi(\langle x,x\rangle_a) = \|x\|_a^2$ and let $b \in \mathcal{A}$ be an arbitrary element. So by (4) and the Cauchy-Schwartz inequality, for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\|x+\lambda ya\|_a^2 &\geq \varphi\big((x+\lambda ya)^*a(x+\lambda ya)\big) \\ &= \varphi(\langle x,x\rangle_a) + \varphi(\langle x,\lambda ya\rangle_a) + \varphi(\langle \lambda ya,x\rangle_a) + |\lambda|^2\varphi(\langle ya,ya\rangle_a) \\ &\geq \varphi(x^*ax) + 2\mathrm{Re}\varphi(\langle x,\lambda ya\rangle_a) \\ &= \|x\|_a^2 + 2\mathrm{Re}\varphi(\langle x,\lambda ya\rangle_a) \\ &\geq \|x\|_a^2 - 2|\mathrm{Re}\varphi(\langle x,\lambda ya\rangle_a)| \\ &\geq \|x\|_a^2 - 2|\varphi(\langle x,\lambda ya\rangle_a)| \\ &\geq \|x\|_a^2 - 2|\lambda|\varphi^{\frac{1}{2}}(\langle x,y\rangle_a a\langle y,x\rangle_a)\varphi^{\frac{1}{2}}(a) \\ &\geq \|x\|_a^2 - 2|\lambda|\|\langle x,y\rangle_a\|_a \\ &\geq \|x\|_a^2 - 2\varepsilon|\lambda|\|x\|_a\|y\|_a \\ &\geq \|x\|_a^2 - 2\varepsilon|\lambda|\|x\|_a\|ya\|_a.\end{aligned}$$

Therefore $x \perp^a_{BJ-\varepsilon} ya$. □

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Any $A \in \mathcal{B}(\mathcal{H})^+$ produces a positive semi-definite sesquilinear form on H as follows:

$$\langle\cdot,\cdot\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Also, the semi-norm

$$\|x\|_A = \sqrt{\langle Ax, x\rangle} \quad (x \in \mathcal{H}),$$

is induced on H by $\langle\cdot,\cdot\rangle_A$ cf. [23]. In addition, the set

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \,:\, \exists\, c > 0,\, \|Tx\|_A \leq c\|x\|_A, \quad \forall x \in \mathcal{H}\},$$

is a unital subalgebra of $\mathcal{B}(\mathcal{H})$ furnished with the semi-norm

$$\gamma_A(T) := \sup_{\|x\|_A=1} \sqrt{\langle ATx, Tx\rangle} \quad (T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})).$$

Let $(\mathcal{H}_f, \pi_f)$ be the GNS representation associated to $f \in \mathcal{S}_a(\mathcal{A})$; see e.g., [24, 25]. In [1] the authors presented the unital faithful $*$-representation π_a for $\mathcal{A}$ as the orthogonal direct sum of all $(\mathcal{H}_f, \pi_f)$, where f ranges over $\mathcal{S}_a(\mathcal{A})$; i.e.,

$$\pi_a = \bigoplus_{f\in\mathcal{S}_a(\mathcal{A})} \pi_f : \mathcal{A} \mapsto \mathcal{B}\Big(\bigoplus_{f\in\mathcal{S}_a(\mathcal{A})} \mathcal{H}_f\Big).$$

In particular, it was proved in [1, Theorem 3.5] that

$$\|x\|_a = \gamma_{\pi_a(a)}(\pi_a(x)) \quad (x \in \mathcal{A}). \tag{5}$$

Sen et al. in [20, 21] introduced the notion of approximate orthogonality with respect to the semi-norm $\gamma_A(\cdot)$ for positive operator $A \in \mathcal{B}(\mathcal{H})$. The following characterization of (ε, A)-BJ-approximate orthogonality is obtained in [20, 21].

Theorem 2.6. *Let $T, S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ and $\varepsilon \in [0, 1)$. Then $T \perp^A_{BJ-\varepsilon} S$ if and only if for each $\theta \in [0, 2\pi)$, there is a sequence $\{h_n\} \subset \mathcal{H}$ of A-unit vectors ($\|h_n\|_A = 1$) such that the following conditions hold:*
(i) $\lim_{n\to\infty} \|Th_n\|_A = \gamma_A(T)$,
(ii) $\lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\langle Th_n, Sh_n\rangle_A) \geq -\varepsilon\gamma_A(T)\gamma_A(S)$.

Theorem 2.7. *For any $x, y \in \mathcal{A}$, the following statements are equivalent:*

(i) $x \perp^a_{BJ-\varepsilon} y$,

(ii) There exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that

$(ii-1)$ $\varphi(x^*ax) = \|x\|_a^2$,

$(ii-2)$ $\mathrm{Re}(e^{-i\theta}\varphi(y^*ax)) \geq -\varepsilon\|x\|_a\|y\|_a$ $(\mathrm{Re}(e^{-i\theta}\varphi(x^*ay)) \geq -\varepsilon\|x\|_a\|y\|_a)$.

Proof. $(i) \Rightarrow (ii)$ Assume that $x \perp^{a}_{BJ-\varepsilon} y$. Hence $\pi_a(x), \pi_a(y) \in \mathcal{B}_{\pi_a(a)^{\frac{1}{2}}}(\mathcal{H})$, and so $\pi_a(x) \perp^{\pi_a(a)}_{BJ-\varepsilon} \pi_a(y)$. So, Theorem 2.6 yields that for each $\theta \in [0, 2\pi)$ there exists a sequence of $\pi_a(a)$-unit vectors $\{h_n\} \subset \mathcal{H}$ such that

$$\lim_{n\to\infty} \|\pi_a(x)h_n\|_{\pi_a(a)} = \gamma_{\pi_a(a)}(\pi_a(x)), \tag{6}$$

and

$$\lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\langle \pi_a(x)h_n, \pi_a(y)h_n\rangle_{\pi_a(a)}) \geq -\varepsilon\gamma_{\pi_a(a)}(\pi_a(x))\gamma_{\pi_a(a)}(\pi_a(y)). \tag{7}$$

The linear functionals

$$\varphi_n(z) = \langle \pi_a(z)h_n, h_n\rangle \quad (n \in \mathbb{N}),$$

belong to $\mathcal{S}_a(\mathcal{A})$. Now, (6) and (7), respectively, imply that

$$\begin{aligned}\lim_{n\to\infty} \varphi_n(x^*ax) &= \lim_{n\to\infty} \langle \pi_a(x^*ax)h_n, h_n\rangle = \lim_{n\to\infty} \langle \pi_a(a)\pi_a(x)h_n, \pi_a(x)h_n\rangle \\ &= \lim_{n\to\infty} \|\pi_a(x)(h_n)\|^2_{\pi_a(a)} = \gamma^2_{\pi_a(a)}(\pi_a(x)) = \|x\|^2_a,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\varphi_n(y^*ax)) &= \lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\langle \pi_a(y^*ax)h_n, h_n\rangle) \\ &= \lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\langle \pi_a(a)\pi(x)h_n, \pi_a(y)h_n\rangle) \\ &= \lim_{n\to\infty} \mathrm{Re}(e^{-i\theta}\langle \pi_a(x)h_n, \pi_a(y)h_n\rangle_{\pi_a(a)}) \\ &\geq -\varepsilon\gamma_{\pi_a(a)}(\pi_a(x))\gamma_{\pi_a(a)}(\pi_a(y)) = -\varepsilon\|x\|_a\|y\|_a.\end{aligned}$$

On the other hand, invertibility of a implies that $\mathcal{S}_a(\mathcal{A})$ is w^*-compact. So, one can find $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi_n \xrightarrow{w^*} \varphi$. Therefore $\varphi(x^*ax) = \|x\|^2_a$ and $\mathtt{Re}(e^{-i\theta}\varphi(y^*ax)) \geq -\varepsilon\|x\|_a\|y\|_a$.

$(ii) \Rightarrow (i)$ Let $\lambda = |\lambda|e^{-i\theta}$ for some $\theta \in [0, 2\pi)$. Then there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ for which $\varphi(x^*ax) = \|x\|^2_a$ and $\mathtt{Re}(e^{-i\theta}\varphi(y^*ax)) \geq -\varepsilon\|x\|_a\|y\|_a$. Therefore

$$\begin{aligned}\|x + \lambda y\|^2_a &\geq \varphi((x + \lambda y)^*a(x + \lambda y)) \\ &= \varphi(\langle x, x\rangle_a) + 2|\lambda|e^{-i\theta}\mathrm{Re}(\varphi\langle x, y\rangle_a) + |\lambda|^2\varphi(\langle y, y\rangle_a) \\ &\geq \varphi(\langle x, x\rangle_a) + 2|\lambda|\mathrm{Re}(e^{-i\theta}\varphi\langle x, y\rangle_a) \\ &\geq \varphi(\langle x, x\rangle_a) - 2\varepsilon|\lambda|\|x\|_a\|y\|_a = \|x\|^2_a - 2\varepsilon|\lambda|\|x\|_a\|y\|_a.\end{aligned}$$

Thus $x \perp^{a}_{BJ-\varepsilon} y$. □

Zamani in [7] was shown that if $T, S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, then the set

$$W_A(T, S) := \{\lambda \in \mathbb{C} : \exists\{h_n\} \subset \mathcal{H}, \|h_n\|_A = 1, \langle Th_n, Sh_n\rangle_A \to \lambda, \|Th_n\|_A \to \gamma_A(T)\},$$

is a nonempty compact and convex subset of $\mathbb{C}$. Moreover, it was proved in [20] that

Theorem 2.8. *([20, Theorem 2.1]). Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})^+$ and $T, S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$. Then for each $\varepsilon \in [0, 1)$, the following statements are equivalent:*
(i) $T \perp^A_{BJ-\varepsilon} S$,
(ii) $W_A(T, S) \cap B(0, \varepsilon\, \gamma_A(T)\gamma_A(S)) \neq \emptyset$.

We end this section by presenting the following characterization of the (ε, a)-BJ-orthogonality in terms of the elements of $\mathcal{S}_a(\mathcal{A})$.

Theorem 2.9. *Let $x, y \in \mathcal{A}$ and let $\varepsilon \in [0, 1)$. Then the following statements are equivalent:*

(i) $x \perp^a_{BJ-\varepsilon} y$,

(ii) There is $\varphi \in \mathcal{S}_a(\mathcal{A})$ so that $\varphi(x^*ax) = \|x\|_a^2$ and $|\varphi(\langle x, y\rangle_a)| \leq \varepsilon\|x\|_a\|y\|_a$.

Proof. Let

$$W_a(x, y) := \{\lambda \in \mathbb{C} : \exists \varphi \in \mathcal{S}_a(\mathcal{A}),\ \varphi(y^*ax) = \lambda,\ \varphi(x^*ax) = \|x\|_a^2\}.$$

By (5) and Theorem 2.8, it is sufficient to show that $W_a(x, y) = W_{\pi_a(a)}(\pi_a(x), \pi_a(y))$. Assume that $\lambda \in W_{\pi_a(a)}(\pi_a(x), \pi_a(y))$. Then there exists a sequence $\{h_n\} \subset \mathcal{H}$ of $\pi_a(a)$-unit vectors such that

$$\langle \pi_a(x)h_n, \pi_a(y)h_n\rangle_{\pi_a(a)} = \lambda,\ \|\pi_a(x)h_n\|_{\pi_a(a)} \to \gamma_{\pi_a(a)}(\pi_a(x)).$$

So

$$\langle \pi_a(y^*ax)h_n, h_n\rangle = \lambda,\ \langle \pi_a(x^*ax)h_n, h_n\rangle \to \gamma_{\pi_a(a)}(\pi_a(x)).$$

Hence for the linear functionals $\varphi_n(z) = \langle \pi_a(z)h_n, h_n\rangle$ defined on $\mathcal{A}$, we have

$$\varphi_n(y^*ax) \to \lambda \text{ and } \varphi_n(x^*ax) = \|x\|_a^2.$$

But, one can find $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi_n \xrightarrow{w^*} \varphi$, which follows that $\lambda \in W_a(x, y)$.

Now, let $\lambda \in W_a(x, y)$. So there is $f \in \mathcal{S}_a(\mathcal{A})$ such that

$$f(y^*ax) = \lambda \text{ and } f(x^*ax) = \|x\|_a^2.$$

Also, Lemma 2.4 of [1] implies that there exists a $*$-representation $(\mathcal{H}_f, \pi_f)$ and a unique cyclic vector $h_f \in \mathcal{H}_f$ such that $\langle \pi_f(a)h_f, h_f\rangle = 1$ and $f(z) = \langle \pi_f(z)h_f, h_f\rangle$ for all $z \in \mathcal{A}$. Let $h := \oplus_{g\in\mathcal{S}_a(\mathcal{A})} h_g \in \oplus_{g\in\mathcal{S}_a(\mathcal{A})}\mathcal{H}_g$ be such that all h_g are zero, except h_f. Then we have

$$\|h\|_{\pi_a(a)} = \langle \pi_a(a)h, h\rangle = \sum_{g\in\mathcal{S}_a(\mathcal{A})} \langle \pi_g(a)h_g, h_g\rangle = \langle \pi_f(a)h_f, h_f\rangle = 1,$$

and

$$\begin{aligned}\langle \pi_a(x)h, \pi_a(y)h\rangle_{\pi_a(a)} &= \langle \pi_a(a)\pi_a(x)h, \pi_a(y)h\rangle = \langle \pi_a(y^*ax)h, h\rangle \\ &= \sum_{g\in\mathcal{S}_a(\mathcal{A})} \langle \pi_a(y^*ax)h_g, h_g\rangle = \langle \pi_a(y^*ax)h_f, h_f\rangle = f(y^*ax) = \lambda.\end{aligned}$$

Moreover, by (5), we get

$$\begin{aligned}\|\pi_a(x)h\|^2_{\pi_a(a)} &= \langle \pi_a(a)\pi_a(x)h, \pi_a(x)h\rangle = \langle \pi_a(x^*ax)h, h\rangle \\ &= \sum_{g\in\mathcal{S}_a(\mathcal{A})} \langle \pi_a(x^*ax)h_g, h_g\rangle = \langle \pi_a(x^*ax)h_f, h_f\rangle \\ &= f(x^*ax) = \|x\|^2_a = \gamma_{\pi_a(a)}(\pi(x)).\end{aligned}$$

So, $W_a(x,y) \subseteq W_{\pi_a(a)}(\pi_a(x), \pi_a(y))$. □

Remark 2. For $\varepsilon = 0$ in Theorem 2.9, we derive the characterization of a-BJ-orthogonality which is obtained in Theorem 2.6 of [22].

3. Approximate strong a-Birkhoff-James orthogonality in C^*-algebras

In this section we investigate the concept of approximate strong a-Birkhoff-James orthogonality in $\mathcal{A}$.

Definition 3.1. For $\varepsilon \in [0,1)$, we say that an element $x \in \mathcal{A}$ is approximate strong a-Birkhoff-James orthogonal (strongly (ε, a)-BJ-orthogonal) to element $y \in \mathcal{A}$, in short $x \perp^a_{SBJ-\varepsilon} y$, if

$$\|x + yb\|^2_a \geq \|x\|^2_a - 2\varepsilon\|x\|_a\|yb\|_a, \qquad (\forall b \in \mathcal{A}).$$

Proposition 3.2. *Let $x, y \in \mathcal{A}$ and let $\varepsilon \in [0,1)$. Then the following statements hold:*

(i) For $\varepsilon \in [0, \frac{1}{2})$, strongly (ε, a)-BJ-orthogonality is non-degenerated,

(ii) Strongly (ε, a)-BJ-orthogonality is homogenous,

(iii) If $x \perp^a_{SBJ-\varepsilon} y$, then $x \perp^a_{BJ-\varepsilon} y$,

(iv) $x \perp^a_{SBJ-\varepsilon} y$ if and only if $x \perp^a_{BJ-\varepsilon} yb$ for all $b \in \mathcal{A}$.

Proof. (i) Let $x \in \mathcal{A}$ and $x \perp^a_{SBJ-\varepsilon} x$. So $\|x + xb\|^2_a \geq \|x\|^2_a - 2\varepsilon\|x\|_a\|xb\|_a$ for all $b \in \mathcal{A}$. Let $b = -1_{\mathcal{A}}$. Then $\|x - x1_{\mathcal{A}}\|^2_a \geq \|x\|^2_a - 2\varepsilon\|x\|^2_a$, and so $\|x\|^2_a(1 - 2\varepsilon) \leq 0$. Therefore $x = 0$, since $\varepsilon \in [0, \frac{1}{2})$.

(ii) Assume that $x \perp^a_{SBJ-\varepsilon} y$. Let $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$. Then

$$\begin{aligned}\|\alpha x + \beta yb\|^2_a &= \|\alpha(x + \frac{\beta}{\alpha}yb)\|^2_a = |\alpha|^2\|x + \frac{\beta}{\alpha}yb\|^2_a \\ &\geq |\alpha|^2(\|x\|^2_a - 2\varepsilon\|x\|_a\|\frac{\beta}{\alpha}yb\|_a) \\ &= \|\alpha x\|^2_a - 2\varepsilon\|\alpha x\|_a\|\beta yb\|_a,\end{aligned}$$

for all $b \in \mathcal{A}$. It follows that $\alpha x \perp^a_{SBJ-\varepsilon} \beta y$.

(iii) It is enough to take $b = \lambda 1_{\mathcal{A}}$ for $\lambda \in \mathbb{C}$ in Definition 3.1.
(iv) Assume that $x \perp^a_{SBJ-\varepsilon} y$. Then

$$\|x + yb\|_a^2 \geq \|x\|_a^2 - 2\varepsilon\|x\|_a\|yb\|_a \quad (\forall b \in \mathcal{A}). \tag{8}$$

Substituting b with λb ($\lambda \in \mathbb{C}$) in (8), we concludes that $\|x + \lambda yb\|_a^2 \geq \|x\|_a^2 - 2\varepsilon|\lambda|\|x\|_a\|yb\|_a$, and so $x \perp^a_{BJ-\varepsilon} yb$. Now, assume that $x \perp^a_{BJ-\varepsilon} yb$ for all $b \in \mathcal{A}$. So

$$\|x + \lambda yb\|_a^2 \geq \|x\|_a^2 - 2\varepsilon|\lambda|\|x\|_a\|yb\|_a, \tag{9}$$

for all $\lambda \in \mathbb{C}$. Taking $\lambda = 1$ in (9), we conclude that $x \perp^a_{SBJ-\varepsilon} y$. □

In Proposition 3.2, we have shown that $\perp^a_{SBJ-\varepsilon} \subseteq \perp^a_{BJ-\varepsilon}$. But the converse is not true in general. The following example illustrate this fact.

Example 3.3. Let $\varepsilon \in [0, \frac{1}{2})$ and $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \,:\, h \in \mathcal{L}_a\}.$$

Take $x = I_2$, $y = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$. By the same argument as in Example 2.4, we get $\|x\|_a = 1$ and $\|y\|_a = \dfrac{1}{\sqrt{2}}$. Hence for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\|x + \lambda y\|_a^2 &= \sup_{\varphi_h \in \mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))} \varphi_h((x + \lambda y)^* a (x + \lambda y)) \\
&= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda} \\ \frac{1}{2}\overline{\lambda} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2}\lambda \\ \lambda & 1 \end{bmatrix} \right) \\
&= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} 2 + |\lambda|^2 & 2\mathrm{Re}\lambda \\ 2\mathrm{Re}\lambda & 1 + \frac{1}{2}|\lambda|^2 \end{bmatrix} \right) \\
&= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}((2 + |\lambda|^2)h_{11} + 4\mathrm{Re}(\lambda h_{12}) + (1 + \frac{1}{2}|\lambda|^2)h_{22}) \\
&\geq 1 + \frac{|\lambda|^2}{2} \geq 1 = \|x\|_a^2,
\end{aligned}$$

since $h_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}_a$. Therefore

$$\|x + \lambda y\|_a^2 \geq \|x\|_a^2 \geq \|x\|_a^2 - 2\varepsilon|\lambda|\|x\|_a\|y\|_a,$$

for all $\varepsilon \in [0, \frac{1}{2})$. So $x \perp^a_{BJ-\varepsilon} y$. But for $\varepsilon \in [0, \frac{1}{2})$, $x \not\perp^a_{SBJ-\varepsilon} y$. Indeed, take $b = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$. By a similar argument, we get $\|yb\|_a = 1$. Moreover, since

$x + yb = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, if

$$\|x + yb\|_a^2 = 0 \geq 1 - 2\varepsilon = \|x\|_a^2 - 2\varepsilon\|x\|_a\|yb\|_a,$$

then we conclude that $\varepsilon \in [\frac{1}{2}, 1)$, which is impossible.

Proposition 3.4. *Let $x, y \in \mathcal{A}$ and let $\varepsilon \in [0, 1)$. If $x^\sharp x \perp_{SBJ-\varepsilon}^a x^\sharp y$, then $x \perp_{SBJ-\varepsilon}^a y$.*

Proof. Assume that $x \neq 0$. Since $x^\sharp x \perp_{SBJ-\varepsilon}^a x^\sharp y$, by the definition, we get

$$\|x^\sharp x + x^\sharp yb\|_a^2 \geq \|x^\sharp x\|_a^2 - 2\varepsilon\|x^\sharp x\|_a\|x^\sharp yb\|_a \quad (\forall b \in \mathcal{A}).$$

By (1) and the sub-multiplicative property of $\|\cdot\|_a$, we have

$$\|x^\sharp\|_a^2\|x + yb\|_a^2 \geq \|x^\sharp x + x^\sharp yb\|_a^2 \geq \|x\|_a^4 - 2\varepsilon\|x\|_a^3\|yb\|_a.$$

But $\|x\|_a \neq 0$. Then

$$\|x + yb\|_a^2 \geq \|x\|_a^2 - 2\varepsilon\|x\|_a\|yb\|_a,$$

and so $x \perp_{SBJ-\varepsilon}^a y$. □

Our next results give us characterization of strong (ε, a)-BJ-orthogonality based on Proposition 3.2, (iv) and Theorems 2.7 and 2.9.

Theorem 3.5. *Let $x, y \in \mathcal{A}$ and let $\varepsilon \in [0, 1)$. Then $x \perp_{SBJ-\varepsilon}^a y$ if and only if for each $\theta \in [0, 2\pi)$ there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\mathtt{Re}(e^{-i\theta}\varphi(\langle x, y\rangle_a b)) \geq -\varepsilon\|x\|_a\|yb\|_a$ for all $b \in \mathcal{A}$.*

Theorem 3.6. *Let $x, y \in \mathcal{A}$ and let $\varepsilon \in [0, 1)$. Then $x \perp_{SBJ-\varepsilon}^a y$ if and only if there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $|\varphi(\langle x, y\rangle_a b)| \leq \varepsilon\|x\|_a\|yb\|_a$ for all $b \in \mathcal{A}$.*

Finally, in the last result, we investigate the condition of equivalence between (ε, a)-BJ-orthogonality and its strong version on $\mathcal{A}$ implies that $\mathcal{A}$ must be a commutative C^*-algebra.

Theorem 3.7. *Let $\varepsilon \in [0, \frac{1}{2})$. If*

$$x \perp_{SBJ-\varepsilon}^a y \Leftrightarrow x \perp_{BJ-\varepsilon}^a y \quad (\forall x, y \in \mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. First, we prove that for all $x, b \in \mathcal{A}$ there exists $0 \neq \alpha \in \mathbb{C}$ so that

$$xb \perp_{SBJ-\varepsilon}^a (xb^2 + \alpha xb). \tag{10}$$

If $xb = 0$, then clearly (10) holds. So, let $x \in \mathcal{A}$ and $xb \neq 0$. Then $xb \not\perp^a_{BJ-\varepsilon} x$. In fact, if $xb \perp^a_{BJ-\varepsilon} x$, then $xb \perp^a_{SBJ-\varepsilon} x$, and so $xb \perp^a_{BJ-\varepsilon} xb$, by Proposition 3.2, part (iv). Since (ε, a)-BJ-orthogonality is non-degenerated for $\varepsilon \in [0, \frac{1}{2})$, we conclude that $xb = 0$, which is incredible. Since a is invertible, one can find $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(\langle xb, xb\rangle_a) = \|xb\|_a^2$. On the other hand, since $xb \not\perp^a_{BJ-\varepsilon} x$, by Theorem 2.9, we get $|\varphi(\langle xb, x\rangle_a)| > \varepsilon\|xb\|_a\|x\|_a > 0$, and hence $\varphi(\langle xb, x\rangle_a) \neq 0$. Now, let $\alpha = \dfrac{-\|xb\|_a}{\varphi(\langle xb, x\rangle_a)}$. Therefore

$$\begin{aligned}|\varphi(\langle xb, xb + \alpha x\rangle_a)| &= |\|xb\|_a^2 - \frac{\|xb\|_a^2}{\varphi(\langle xb, x\rangle_a)}\varphi(\langle xb, x\rangle_a)| \\ &= 0 \leq \varepsilon\|xb\|_a\|xb + \alpha x\|_a.\end{aligned}$$

Hence Theorem 2.9 yields that $xb \perp^a_{BJ-\varepsilon} (xb+\alpha x)$, and so $xb \perp^a_{SBJ-\varepsilon} (xb^2+\alpha xb)$, by the assumption and Proposition 3.2.

It is known that in non-commutative C^*-algebras, there is a nonzero $b \in \mathcal{A}$ with $b^2 = 0$ (see [24], p.68). If $x = b^*$, then there is $\alpha \neq 0$ such that $xb \perp^a_{SBJ-\varepsilon} \alpha xb$, by (10). Therefore $b^*b = xb = 0$, and hence $b = 0$. This is a contradiction, and so $\mathcal{A}$ is commutative. □

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