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Statistical Bounds for the Energy of Graphs

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Abstract

This paper proposes several new statistical bounds for graph energy derived from the eigenvalues of the adjacency matrix. Using inequalities involving the arithmetic, geometric, and generalized means, along with variance and standard deviation, we establish both upper and lower bounds for $E(G)$. These statistical bounds capture not only mean relationships but also eigenvalue variability, offering more flexible and accurate estimates than conventional deterministic inequalities. The approach integrates tools from inequality theory and spectral graph theory, with applying weighted means and Jensen-type inequalities. We also conjecture based on numerical evidence that the energy-to-geometric mean ratio converges to a constant value for large Erdös-Rényi random graphs. A detailed analysis of path graphs demonstrates the effectiveness of the proposed bounds, offering improved estimates.

Keywords: Graph energy, Eigenvalues, Statistical bounds, Variance, Random graphs.

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1. Introduction

The concept of graph energy is the study of conjugated hydrocarbons using a tight-binding method known in chemistry as the Hückel molecular orbital (HMO) method. It is introduced by Gutman and defined as the sum of the absolute values

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of the eigenvalues of the adjacency matrix. In this context, let G be a simple graph with n vertices and adjacency matrix $A(G)$. The energy of G, denoted by $E(G)$, is given by:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ which represents the total energy of conjugated molecule. Graph energy has been discussed in several references such as [1–7]. Some fundamental theorems in mathematics can also found in [8, 9] and [4].

Recent studies have proven some inequalities for graph energy using eigenvalues and determinant of adjacency matrix, topological indices and self-loops for finding graph energy bounds. For instance, in references [1, 5], Akbari et al. derived lower bounds by the determinant of the adjacency matrix, while Jahanbani [3] provided sharp upper bounds using Randić index, and Jianping Liu [10] proposed some inequalities via self-loops for graph energy.

In contrast to these approaches, the present work systematically connects statistical indices, such as the arithmetic-geometric mean inequality, Jensen's inequality, and variance, standard deviation and other tools to establish new bounds for graph energy. In particular, these bounds have some advantages: Statistical indices allow for different distributions of vertices or edges and finding new bounds on the energy index. Additionally, they enhance the predictive power of properties for random and structured graphs in conditions of uncertainty. Unlike classical bounds that rely only on fixed spectral parameters or deterministic inequalities, statistical indices incorporate the dispersion and distributional features of eigenvalues. This allows our method to produce tighter and more adaptive bounds, particularly for irregular or random graphs where traditional inequalities often lose accuracy. You could study some statistical indices in articles [8, 9] and [11].

2. Statistical bounds for graph energy

This section is devoted to deriving new bounds for graph energy using statistical indices and inequalities. We begin by recalling fundamental definitions and classical inequalities, such as the arithmetic-geometric mean inequality, and then extend these concepts to the spectral setting of graph eigenvalues. By incorporating variance, standard deviation, and weighted means, we establish a series of upper and lower bounds for $E(G)$. Furthermore, we explore the role of convexity and Jensen's inequality in obtaining generalized mean bounds. The results presented here not only refine existing estimates but also provide a unified framework for analyzing graph energy through statistical lens. We conclude the section with a detailed numerical example illustrating the applicability and sharpness of the derived bounds.

Definition 2.1. ([8]). For the nonnegative real numbers $x_1,\ x_2,\ \ldots,\ x_n$, the arithmetic mean is as $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$, and the geometric mean is as $R_g = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$.

Definition 2.2. ([4]). *Inequality of arithmetic and geometric mean* for the non-negative real numbers $x_1,\ x_2,\ \dots,\ x_n$ is as follows:

$$R_g = \prod_{i=1}^{n} x_i^{\frac{1}{n}} \le \frac{1}{n}\sum_{i=1}^{n} x_i = \mu. \tag{1}$$

Equality holds whenever $x_1 = x_2 = \dots = x_n$. Generally if $\alpha = (\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n)$, $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$, and $0 < s < 1$, then

$$\prod_{i=1}^{n} y_i^{\alpha_i} \le \left(\sum_{i=1}^{n} \alpha_i y_i^s\right)^{1/s} \le \sum_{i=1}^{n} \alpha_i y_i, \tag{2}$$

and considering $\alpha_i = \frac{1}{n}$, $s = \frac{1}{2}$ and $y_i = |\lambda_i|^2$, we conclude:

Theorem 2.3. *Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of its adjacency matrix, then*

$$n\big|\det(A)\big|^{1/n} = n\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{1/n} = n\Big(\prod_{i=1}^{n}|\lambda_i|^2\Big)^{\frac{1}{2n}} \le E(G) \le \Big(n\sum_{i=1}^{n}|\lambda_i|^2\Big)^{\frac{1}{2}} = \sqrt{2mn}. \tag{3}$$

Proof. Set $y_i := |\lambda_i|^2$ for $i = 1, \dots, n$. Apply the generalized arithmetic-geometric mean inequality (inequality (2) in the text) with $\alpha_i = \frac{1}{n}$ and $s = \frac{1}{2}$. This yields

$$\Big(\prod_{i=1}^{n} y_i\Big)^{1/n} \le \Big(\frac{1}{n}\sum_{i=1}^{n} y_i^{1/2}\Big)^2 \le \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Substituting $y_i = |\lambda_i|^2$ gives

$$\Big(\prod_{i=1}^{n}|\lambda_i|^2\Big)^{1/n} \le \Big(\frac{1}{n}\sum_{i=1}^{n}|\lambda_i|\Big)^2 \le \frac{1}{n}\sum_{i=1}^{n}|\lambda_i|^2.$$

Multiplying through by n^2 yields

$$n^2\Big(\prod_{i=1}^{n}|\lambda_i|^2\Big)^{1/n} \le \Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2 \le n\sum_{i=1}^{n}|\lambda_i|^2.$$

Taking square roots (all quantities are nonnegative) gives

$$n\Big(\prod_{i=1}^{n}|\lambda_i|^2\Big)^{\frac{1}{2n}} \le \sum_{i=1}^{n}|\lambda_i| \le \Big(n\sum_{i=1}^{n}|\lambda_i|^2\Big)^{1/2}.$$

Noting that $\left(\prod_{i=1}^{n}|\lambda_i|^2\right)^{1/(2n)} = \left(\prod_{i=1}^{n}|\lambda_i|\right)^{1/n} = |\det(A)|^{1/n}$ and that $\sum_{i=1}^{n}|\lambda_i|^2 = \operatorname{tr}(A^2) = 2m$, we obtain the displayed bounds

$$n\,|\det(A)|^{1/n} \le E(G) \le \left(n\sum_{i=1}^{n}|\lambda_i|^2\right)^{1/2} = \sqrt{2mn}.$$

This completes the proof. The structure of this proof follows the approach given in Remark 2.2 of [4]. □

Theorem 2.4. *([9, Th, 2.1]). Let for* $i = 1,\ 2,\ \ldots,\ n$, $x_i \ge 0$, *and* $\alpha_i > 0$, $\beta_i > 0$ *satisfy* $\sum_{i=1}^{n}\ \alpha_i = \sum_{i=1}^{n}\ \beta_i = 1$. *Writing* $\alpha_{min} := min\{\alpha_1,\ \ldots,\ \alpha_n\}$, $\alpha_{max} := max\{\alpha_1,\ \ldots,\ \alpha_n\}$, *and similarly for* β_{min} *and* β_{max}, *we have*

$$\min_{k=1,2,\ldots,n}\{\frac{\alpha_k}{\beta_k}\}(\sum_{i=1}^{n}\ \beta_i x_i - \prod_{i=1}^{n}\ x_i^{\beta_i}) \le \sum_{i=1}^{n}\ \alpha_i x_i - \prod_{i=1}^{n}\ x_i^{\alpha_i}$$
$$\le \max_{k=1,2,\ldots,n}\{\frac{\alpha_k}{\beta_k}\}(\sum_{i=1}^{n}\ \beta_i x_i - \prod_{i=1}^{n}\ x_i^{\beta_i}). \qquad (4)$$

Equality holds in either of the inequalities if and only if either $x_1 = \cdots = x_n$ *or* $\alpha_{max} = \beta_{min}$ *(or equivalently,* $\alpha_{min} = \beta_{max}$*).*

Theorem 2.5. *Let* $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ *with* $\beta_i > 0$ *and* $\sum_{i=1}^{n}\beta_i = 1$, *then*

$$\frac{1}{\beta_{\max}}\left(\sum_{i=1}^{n}\beta_i x_i - \prod_{i=1}^{n}x_i^{\beta_i}\right) + n\cdot R_g \ \le\ E(G)\ \le\ \frac{1}{\beta_{\min}}\left(\sum_{i=1}^{n}\beta_i x_i - \prod_{i=1}^{n}x_i^{\beta_i}\right) + n\cdot R_g, \qquad (5)$$

where $R_g = \left(\prod_{i=1}^{n}x_i\right)^{1/n}$ *is the geometric mean of the absolute eigenvalues, and* $\beta_{\min} = \min_i \beta_i$, $\beta_{\max} = \max_i \beta_i$.

Proof. This result follows from Theorem 2.4 and a similar method with Theorem 2.4 of [4] by setting $x_i = |\lambda_i|$, and $\alpha_i = \frac{1}{n}$. Therefore, we have,

$$\frac{1}{n\beta_{\max}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right) \le \frac{1}{n}\sum_{i=1}^{n}|\lambda_i| - \left(\prod_{i=1}^{n}|\lambda_i|\right)^{1/n}$$
$$\le \frac{1}{n\beta_{\min}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right).$$

Multiplying through by n, we get:

$$\frac{1}{\beta_{\max}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right) \le \sum_{i=1}^{n}|\lambda_i| - nR_g \le \frac{1}{\beta_{\min}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right).$$

Adding nR_g to all sides yields the desired result:

$$\frac{1}{\beta_{\max}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right)+nR_g \leq E(G) \leq \frac{1}{\beta_{\min}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right)+nR_g.$$

□

Definition 2.6. ([4, 9]). Suppose $\mathbf{X} = (x_1,\ x_2,\ \ldots,\ x_n)$ be a vector of the nonnegative numbers. Then define the variance of x_i as:

$$\sigma^2(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2. \tag{6}$$

Also variance of the square roots $\sqrt{x_i}$ as:

$$\sigma^2(\mathbf{X^{1/2}}) = \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{x_i} - \frac{1}{n}\sum_{k=1}^{n}\sqrt{x_k}\right)^2. \tag{7}$$

with respect to the discrete probability $\sum_{i=1}^{n}\alpha_i\delta_{x_i}$.

Theorem 2.7. *([8, Th.1]). Let $x_i \geq 0$ for $i = 1,\ 2,\ \ldots,\ n$, and $\alpha_i > 0$ satisfy $\sum_{i=1}^{n}\ \alpha_i = 1$. Then*

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n}\alpha_i x_i - \sum_{i=1}^{n}\alpha_i(x_i^{\frac{1}{2}} - \sum_{k=1}^{n}\alpha_k.x_k^{\frac{1}{2}})^2, \tag{8}$$

where $\sum_{i=1}^{n}\ \alpha_i(x_i^{\frac{1}{2}} - \sum_{k=1}^{n}\ \alpha_k x_k^{\frac{1}{2}})^2 = var(\mathbf{x}^{\frac{1}{2}})$ of the vector $\mathbf{X}^{\frac{1}{2}} = (x_1^{\frac{1}{2}},\ x_2^{\frac{1}{2}},\ \ldots,\ x_n^{\frac{1}{2}})$ with respect to the probability $\sum_{i=1}^{n}\alpha_i\delta_{x_i}$. Therefore a large variance (of $\mathbf{x}^{\frac{1}{2}}$) pushes the arithmetic and geometric means apart.

Theorem 2.8. *Let G be a graph with adjacency eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and define $x_i = |\lambda_i|$ for $i = 1, 2, \ldots, n$. Then the energy of G satisfies the inequality:*

$$E(G) \geq n\left(R_g + \sigma^2(\mathbf{X^{1/2}})\right), \tag{9}$$

where $\mathbf{X}^{\frac{1}{2}} = (x_1^{\frac{1}{2}},\ x_2^{\frac{1}{2}},\ \ldots,\ x_n^{\frac{1}{2}})$.

Proof. From Theorem 2.7, for non-negative real numbers x_i and $\alpha_i = \frac{1}{n}$, we have:

$$\prod_{i=1}^{n} x^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n}x_i - \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{x_i} - \frac{1}{n}\sum_{k=1}^{n}\sqrt{x_i}\right)^2.$$

This implies for $x_i = \lambda_i$:

$$R_g \leq \frac{1}{n}E(G) - \sigma^2(\mathbf{\Lambda^{1/2}}) \quad \Rightarrow \quad R_g + \sigma^2(\mathbf{\Lambda^{1/2}}) \leq \frac{1}{n}E(G).$$

Multiplying both sides by n gives:

$$E(G) \geq n\left(R_g + \sigma^2(\mathbf{X^{1/2}})\right),$$

where $\mathbf{\Lambda^{1/2}} = (\lambda_1^{\frac{1}{2}},\ \lambda_2^{\frac{1}{2}},\ \ldots,\ \lambda_m^{\frac{1}{2}}) = \mathbf{X^{1/2}}$. □

Remark 1. ([4, 8]). Let $x_i \geq 0$ and $\alpha_i > 0$ for $i = 1, 2, \ldots, n$. Additionally, let $\sum_{i=1}^{n} \alpha_i = 1$, $0 < M_1 = min\{x_1,\ x_2,\ \ldots,\ x_n\}$ and $M_2 = max\{x_1,\ x_2,\ \ldots,\ x_n\}$. Then

$$\frac{1}{2M_2}\sum_{i=1}^{n} \alpha_i(x_i - \sum_{k=1}^{n} \alpha_k x_k)^2 \leq \sum_{i=1}^{n}\alpha_i x_i - \prod_{i=1}^{n} x_i\alpha_i \leq \frac{1}{2M_1}\sum_{i=1}^{n} \alpha_i(x_i - \sum_{k=1}^{n}\alpha_k x_k)^2.$$

Theorem 2.9. *Let G be a graph with adjacency eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and define $x_i = |\lambda_i|$ for $i = 1, 2, \ldots, n$. Let $M_1 = \min\{x_1, x_2, \ldots, x_n\}$ and $M_2 = \max\{x_1, x_2, \ldots, x_n\}$. Then the energy $E(G) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} x_i$ satisfies the following inequality:*

$$n\left(R_g + \frac{\sigma^2(X)}{2M_2}\right) \leq E(G) \leq n\left(R_g + \frac{\sigma^2(X)}{2M_1}\right).$$

Proof. From Remark 1 and similar to Theorem 2.11 of [4], for nonnegative values x_i and uniform weights $\alpha_i = \frac{1}{n}$, we have:

$$\frac{1}{2M_2}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{k=1}^{n} x_k\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} x_i - \left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{2M_1}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{k=1}^{n} x_k\right)^2.$$

Then multiplying all sides by n/n, we could write it using the variance $\sigma^2(X)$ as:

$$\frac{\sigma^2(X)}{2M_2} \leq \frac{1}{n}E(G) - R_g \leq \frac{\sigma^2(X)}{2M_1}.$$

Adding R_g to all sides:

$$R_g + \frac{\sigma^2(X)}{2M_2} \leq \frac{1}{n}E(G) \leq R_g + \frac{\sigma^2(X)}{2M_1}.$$

Multiplying all sides by n yields:

$$n\left(R_g + \frac{\sigma^2(X)}{2M_2}\right) \leq E(G) \leq n\left(R_g + \frac{\sigma^2(X)}{2M_1}\right).$$

□

Lemma 2.10. *([11]). Let $n \geq 2$ and $x_1,\ x_2,\ \ldots,\ x_n$ be a sequance of $n \geq 2$ real numbers with mean $\mu > 0$ and variance σ^2.*

(a) If $0 \leq \frac{\sigma}{\mu} < \frac{1}{\sqrt{n-1}}$*, then each* x_i *is positive.*

(b) If every term of the sequence $x_1,\ x_2,\ \ldots,\ x_n$ *is positive, then* $0 \leq \frac{\sigma}{\mu} < \sqrt{n-1}$.

Corollary 2.11. *([11]). Let* $x_1,\ x_2,\ \ldots,\ x_n$ *is a positive sequence for* $n \geq 2$ *with mean* μ *and variance* σ^2*, then*

$$\mu - R_g = \mu - \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \sqrt{n-1}\ \sigma. \tag{10}$$

Theorem 2.12. *Let* $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{E(G)}{n}$*, where* $x_i = |\lambda_i|$ *for* $i = 1, 2, \ldots, n$ *and* σ *be the standard deviation of the sequence* $x_1, x_2, \ldots, x_n$*, then*

$$nR_g - n\sqrt{n-1}\,\sigma \ \leq\ E(G) \ \leq\ nR_g + n\sqrt{n-1}\,\sigma.$$

where $R_g = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ *is the geometric mean of the eigenvalue magnitudes.*

Proof. From Corollary 2.11, we have:

$$\mu - R_g \leq \sqrt{n-1}\,\sigma.$$

For upper bound, multiplying by n and using $E(G) = n\mu$:

$$E(G) \leq nR_g + n\sqrt{n-1}\,\sigma.$$

For the lower bound, since all $x_i > 0$ and $n \geq 2$, we have $\sigma \geq 0$ and $\sqrt{n-1} > 0$, thus $n\sqrt{n-1}\,\sigma \geq 0$. From the AM-GM inequality we have $\mu \geq R_g$, which implies:

$$E(G) = n\mu \geq nR_g \geq nR_g - n\sqrt{n-1}\,\sigma.$$

Combining these bounds yields:

$$nR_g - n\sqrt{n-1}\,\sigma \ \leq\ E(G) \ \leq\ nR_g + n\sqrt{n-1}\,\sigma.$$

This completes the proof. □

Definition 2.13. ([12]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a norm, and $0 \leq \theta \leq 1$, then

$$f(\theta x + (1-\theta)y) \leq f(\theta x) + f((1-\theta)y) = \theta f(x) + (1-\theta) f(y),$$

is named Jensen's inequality. The equality follows from the homogeneity of a norm.
We could extend the inequality to infinite sums, integrals, and expected values.

Definition 2.14. ([12]). Let f be twice differentiable, that is, the second derivative $\nabla^2 f$ exists at each point in **dom** f, which is open, then

1. f is convex if and only if its **dom** is convex and the second derivative $\nabla^2 f(x) \geq 0$ for all $x \in \mathbf{dom} f$. For functions on R, this reduces to $f''(x) \geq 0$, implying the derivative is nondecreasing.

2. f is concave if $\nabla f(x) \leq 0$ for all $x \in \mathbf{dom} f$. Strict convexity can be partially characterized by the condition $\nabla^2 f(x) > 0$, but the converse is not always true.

Theorem 2.15. *For any real number $r \geq 1$, the energy of the graph G with $x_i = |\lambda_i|$ for $i = 1, 2, \ldots, n$ satisfies:*

$$E(G) = \sum_{i=1}^{n} x_i \leq n \cdot \left(\frac{1}{n} \sum_{i=1}^{n} x_i^r \right)^{1/r} = n \cdot M_r(x_1, x_2, \ldots, x_n),$$

where $M_r(x_1, \ldots, x_n)$ is the generalized mean of order r.

Proof. Consider the function $f(x) = x^r$ for $x \geq 0$ and $r \geq 1$. The second derivative is:

$$f''(x) = r(r-1)x^{r-2}.$$

For $r \geq 1$ and $x \geq 0$, we have $f''(x) \geq 0$, so f is convex.

By Jensen's inequality for the convex function f, applied to the values $x_1, \ldots, x_n$, we have:

$$f\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i),$$

which gives:

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^r \leq \frac{1}{n} \sum_{i=1}^{n} x_i^r.$$

Taking the r-th root of both sides (which preserves inequality since $r \geq 1$):

$$\frac{1}{n} \sum_{i=1}^{n} x_i \leq \left(\frac{1}{n} \sum_{i=1}^{n} x_i^r \right)^{1/r}.$$

Multiplying both sides by n yields:

$$\sum_{i=1}^{n} x_i \leq n \cdot \left(\frac{1}{n} \sum_{i=1}^{n} x_i^r \right)^{1/r},$$

that is, $E(G) \leq n \cdot M_r(x_1, \ldots, x_n)$. □

Finally, it is worth noting that our framework extends naturally from deterministic to random graphs. Among various models, the classical Erdős–Rényi random graph $G(n,p)$—where each of the $\binom{n}{2}$ possible edges appears independently with probability $p \in (0,1)$ (see [13])-provides a compelling probabilistic setting.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the adjacency eigenvalues of $G(n,p)$ and define $x_i = |\lambda_i|$. We conjecture that as $n \to \infty$, the ratio

$$\frac{R_g}{E(G)/n} \to c,$$

for some universal constant $c > 0$, where $R_g = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ is the geometric mean of the absolute eigenvalues.

This ratio arises from the classical AM-GM inequality $R_g \leq \frac{E(G)}{n} = \mu$ applied to the absolute eigenvalues. In the case of independent random variables, Aldaz [9] defined the ratio

$$r_n(x) = \frac{(\prod_{i=1}^{n} |x_i|)^{1/n}}{\frac{1}{n}\sum_{i=1}^{n} |x_i|},$$

and proved that $r_n(x) \to e^{-\gamma} \approx 0.5615$ as $n \to \infty$, where γ is the Euler–Mascheroni constant. Our framework generalizes this concept to the dependent eigenvalue structure of random graphs.

Numerical simulations for Erdős–Rényi graphs with n up to 800 suggest that $\frac{nR_g}{E(G)}$ converges to approximately 0.707 for large n. This observed limit, while based on computational evidence, provides a basis for conjecturing asymptotic behavior in random graphs. Determining this limit analytically remains an open problem that would significantly advance our understanding of spectral properties in large random networks.

Example 2.16. Consider the path graph P_4. To illustrate the flexibility of Theorem 2.5, we interpret the coefficients β_i as vertex weights in a specific application. Let $\beta = (0.3, 0.5, 0.1, 0.1)$ and $\beta' = (0.4, 0.4, 0.1, 0.1)$ represent different vertex importance distributions. The eigenvalues are:

• — • — • — •

β: 0.3 0.5 0.1 0.1

β': 0.4 0.4 0.1 0.1

Figure 1: Path graph P_4 with vertex weight vectors β and β'

$$\lambda_k = 2\cos\left(\frac{k\pi}{5}\right), \quad k = 1, 2, 3, 4,$$

yielding:

$$|\lambda_1| = |\lambda_4| \approx 1.618034, \quad |\lambda_2| = |\lambda_3| \approx 0.618034.$$

The energy is:

$$E(P_4) \approx 2(1.618034) + 2(0.618034) = 4.472136.$$

The geometric mean is $R_g \approx R_g = 0.999999 \approx 1$ and $n = 4$. We now apply the following inequality of Theorem 2.5:

$$\frac{1}{\beta_{\max}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right)+nR_g \le E(G) \le \frac{1}{\beta_{\min}}\left(\sum_{i=1}^{n}\beta_i|\lambda_i| - \prod_{i=1}^{n}|\lambda_i|^{\beta_i}\right)+nR_g,$$

Case 1: $\beta = (0.3, 0.5, 0.1, 0.1)$

$$\sum \beta_i|\lambda_i| \approx 1.018034, \quad \prod |\lambda_i|^{\beta_i} \approx 0.908244,$$

$$\text{Lower bound} = \frac{1}{0.5}(1.018034 - 0.908244) + 4 \approx 4.219580,$$

$$\text{Upper bound} = \frac{1}{0.1}(1.018034 - 0.908244) + 4 \approx 5.097901,$$

$$\boxed{4.220 \le E(P_4) \le 5.098.}$$

Case 2: $\beta' = (0.4, 0.4, 0.1, 0.1)$

$$\sum \beta_i|\lambda_i| \approx 1.118034, \quad \prod |\lambda_i|^{\beta_i} = 1.000000,$$

$$\text{Lower bound} = \frac{1}{0.4}(1.118034 - 1.000000) + 4 \approx 4.295085,$$

$$\text{Upper bound} = \frac{1}{0.1}(1.118034 - 1.000000) + 4 \approx 5.180340,$$

$$\boxed{4.295 \le E(P_4) \le 5.180.}$$

In both cases, the actual energy $E(P_4) = 4.472$ lies within the established bounds, validating our theoretical results.

3. Conclusion

In this paper, we introduce a comprehensive framework for bounding graph energy using statistical indices derived from the eigenvalues of the adjacency matrix. By using classical inequalities such as the arithmetic-geometric mean inequality, weighted mean inequalities, and variance-based bounds, we establish both upper and lower estimates that refine and extend existing results. The use of Jensen's inequality further allows us to derive bounds involving generalized means, broadening the applicability of our approach. Our investigation also extend to random graphs, where we conjectured the convergence of the ratio between geometric mean and average energy to a universal constant, supported by numerical evidence. This bridges spectral graph theory with probability theory and random matrix theory, opening avenues for future research. The numerical validation on path graphs confirme the practicality and accuracy of our bounds. Overall, this work underscores the power of statistical tools in spectral graph theory and provides a foundation for further exploration of energy bounds in both deterministic and random graph models.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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