Mathematics Interdisciplinary Research **11** (1) (2026) 93 – 111

Original Scientific Paper

Solving Distributed-Order Fractional Equations via Genocchi Wavelets and Weighted Residual Method

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Abstract

In this work, a novel method to finding the numerical solution of distributed-order fractional differential equations (DFDEs) is introduced. This method is based on the Genocchi wavelets (GWs), and weighted residual method (collocations method). For this aim, an exact mathematical formula that incorporates regularized beta functions is meticulously formulated to ascertain the Riemann-Liouville fractional integral operator (R-LFIO) corresponding to these specific wavelets. By employing the aforementioned integral operator and leveraging the capabilities of Gauss-Legendre numerical integration, the original problem is adeptly transformed into a comprehensive system of algebraic equations, thereby facilitating a more manageable analysis and solution process. Also, the error analysis is investigated and examples are given to demonstrate the effectiveness and accuracy of the method.

Keywords: Distributed-order fractional differential equations, Fractional integral operator, Beta function, Collocation method, Genocchi wavelets.

2020 Mathematics Subject Classification: 34A08; 41A35; 65T60.

How to cite this article
P. Rahimkhani, E. Keshavarz and R. Moeti, Solving distributed-order fractional equations via Genocchi wavelets and weighted residual method, *Math. Interdisc. Res.* **11** (1) (2026) 93-111.

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Academic Editor: Abbas Saadatmandi
Received 9 February 2025, Accepted 28 July 2025
DOI: 10.22052/mir.2025.256348.1503

1. Introduction

The fractional differential equations (FDEs) have proven to be tools of great necessity in modelling complex systems within the purview of science and engineering because they manifest about nonlocal behaviour and memory dependence, which is marked in a variety of physical, biological, and engineering phenomena [1, 2]. Generalizing the concept of an integer-order derivative to fractional orders, FDEs furnish more sophisticated and flexible mathematical propositions for the description of such phenomena [3, 4]. The origin of distributed-order fractional differential equations lies in the need to model systems where the memory and hereditary properties are not adequately described by a single fractional order. Instead, DFDEs generalize classical FDEs by integrating the fractional derivative over a range of orders, weighted by a distribution function. This approach was first systematically explored by Caputo (1969) and later developed by Bagley and Torvik (2000), enabling the mathematical representation of processes with multiple or evolving memory scales. As a result, DFDEs have found wide application in modeling anomalous diffusion, complex viscoelastic materials, and systems with distributed relaxation times, providing a flexible and powerful framework for capturing real-world dynamics that exhibit nonlocal and multi-scale behavior [5–7]. Within this half-decade, the concept of the DFDEs has vastly broadened the ability to vary the order of differentiation from fixed points to a continuous measure over some interval [8]. In this way, DFDEs facilitate modelling processes with dynamic behaviour, especially in applications, such as viscoelastic [9], signal processing [10], optimal control [11], diffusions [12], and dielectrics [13].
Analytical techniques for the examination of solutions pertaining to DFDEs have been explored, for instance, in [14, 15]. But, despite of DFDEs theoretical attractions, it is extremely complex and challenging to derive closed forms for such equations, especially when it comes to complicated systems. That is why the new development regarding numerical methods has become the tremendous areas of focus in research [12, 16]. For now, existing numerical methods for solving DFDEs include the finite difference method [16], fractional Chebyshev wavelets method [17–19], Hahn hybrid functions method [20], Chebyshev collocation method [21], shifted Legendre polynomials [22], Müntz-Legendre polynomials [23], Pell wavelet optimization method [24], and orthonormal Bernoulli polynomials method [25]. Although these methods yield remarkable accuracy, they have certain limitations. For example, finite difference methods usually have difficulties converging for strongly nonlinear systems, and conventional wavelet-based techniques might not have the flexibility needed for problems involving distributed-order derivatives. The variety of DFDEs and possible applications often require an innovative approach combining distinct mathematical properties that fractional operators possess in wavelet theory.

Wavelet-based techniques have recently garnered a lot of focused attention, especially those relying on GWs [26, 27]. The GWs are orthonormal, compactly supported, and therefore might give a very good representation of very complicated functions [28]. The combination of wavelet theory and fractional calculus has provided a very promising tool towards improving accuracy in numerical computation and efficiency when solving DFDEs [29].

This study focuses on presenting a new numerical approach for DFDE based on weighted residual techniques using the GWs. It transforms the original problem into an algebraic equation system by Riemann-Liouville fractional integral operator and Gauss-Legendre numerical integration. The method is efficient in overcoming the current drawbacks of existing methods in terms of precision and complexity. In addition, the GWs introduce a systematic approach framework for making solutions lie in the structure of wavelet-based norms. Within this framework, the GWs could be the promising option. Their high accuracy in representing functions makes them apt for the challenge of dealing with DFDEs. The GWs coincide perfectly with the weighted residual methods and thereby facilitate numerical computations. Besides, by creating more manageable systems in terms of computation, this will form further simplifications in transforming DFDEs to algebraic systems for solution purposes.

In the present study, we examine the following DFDEs:

$$\int_{\alpha}^{\beta} \mathbb{A}(q, {}_0^C D_t^q u(t))dq = \mathbb{F}(t), \tag{1}$$

with the specified initial conditions

$$u^{(i)}(0) = u_0^{(i)}, \qquad i = 0, 1, \ldots, \lceil \beta \rceil - 1, \tag{2}$$

here $\mathbb{A}$ and $\mathbb{F}$ are given functions, the parameters α and β are defined as positive real numbers, the operator ${}_0^C D_t^q$ denotes the Caputo fractional derivative of order q, and $\alpha < q < \beta$. Existence, uniqueness, and approximate solutions for the general nonlinear distributed-order fractional differential considered in [30].

The choice of the Genocchi polynomials as basis functions was made due to their specific properties that suit the above problem. The Genocchi polynomials provide a good approximation for the solution of differential equations, particularly in the context of series expansion methods. The Genocchi polynomials $G_n(x)$ are defined on the interval $[0, 1]$ via the generating function [31]:

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

The key features of the proposed polynomials can be summarized as follows [32]:

- The coefficients of the individual terms in the Genocchi polynomials are integers, which eliminates computational errors. In contrast, many classical polynomials, such as Legendre and Bernoulli polynomials, generally have non-integer coefficients. This feature highlights the advantage of the Genocchi polynomials over the aforementioned ones.

- The Genocchi polynomials contain fewer terms compared to other well-known polynomials. For instance, the Genocchi polynomial $G_6(x)$ consists of four terms, whereas the Bernoulli polynomial $B_6(x)$ includes five terms. Moreover, the shifted Chebyshev polynomial $T_6(x)$ and the shifted Legendre polynomial $L_6(x)$ both contain seven terms. As a result, when used for function approximation, the Genocchi polynomials require less computational time than Bernoulli, shifted Chebyshev, and shifted Legendre polynomials.

The study is organized in these sections: Section 2 presents a short overview of fractional operators and the GWs. The Riemann-Liouville fractional integral operator articulated in Section 3. Proposed numerical method with steps for implementation has been provided in Section 4. Analysis of errors is given in Section 5 to justify the reliability of the method. Applying this method through illustrative examples is Section 6, while Section 7 ends with discussions on the findings and future directions of the research.

2. Preliminaries

In this section, we introduce the essential concepts including fractional operators, definitions, properties, and a formulation of the GWs and their properties in solving DFDEs. This would be necessary for continuing deeper into the numerical method and its properties in later sections.

2.1 Fractional operators

Definition 2.1. The Riemann-Liouville fractional integral operator of order $q \geq 0$ is defined as [33, 34]:

$$ {}_0^R I_t^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-z)^{q-1} u(z) dz, \qquad t > 0. \tag{3}$$

Definition 2.2. The Caputo fractional derivative operator of order q is defined as [34, 35]:

$$ {}_0^C D_t^q u(t) = \tfrac{1}{\Gamma(n-q)} \int_0^t (t-z)^{n-q-1} u^{(n)}(z) dz, \qquad n-1 < q \leq n,$$

where $n \in \mathbb{N}$, and $t > 0$.

Proposition 2.3. *The characteristics of the Caputo fractional derivative and R-LFIO are expressed as follows [36]:*

1. ${}_0^C D_t^q {}_0^R I_t^q u(t) = u(t)$,
2. ${}_0^R I_t^q {}_0^C D_t^q u(t) = u(t) - \sum_{i=0}^{n-1} u^{(i)}(0) \frac{t^i}{i!}$,
3. ${}_0^C D_t^q u(t) = {}_0^R I_t^{n-q} {}_0^C D_t^n u(t)$,

4. ${}_0^C D_t^q t^\beta = \begin{cases} 0, & \beta \in N_0,\ \beta < \lceil q \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-q)} t^{\beta-q}, & \textit{otherwise}, \end{cases}$

5. ${}_0^R I_t^q t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+q)} t^{\beta+q}$,

6. ${}_0^C D_t^q \ell = 0$,

where ℓ is real constant, the ceiling function $\lceil q \rceil$ denotes the smallest integer greater than or equal to q, $N_0 = \{0, 1, 2, \cdots\}$ and $n - 1 < q \le n$.

Definition 2.4. The definition of the Caputo distributed-order fractional derivative is as follows [18]:

$$ {}_0^C D_t^{\rho(q)} u(t) = \int_0^1 \rho(q) {}_0^C D_t^q u(t) dq, \tag{4}$$

where $\rho(q) > 0, q \in (0, 1)$, and $\int_0^1 \rho(q)dq < \infty$.

2.2 Genocchi wavelets and their properties

Now, we revisit the definition of the GWs on the interval $[0, 1)$ as [28]:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\mathsf{G}}_m(2^{k-1}t - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\hat{n} = n - 1, n = 1, 2, \ldots, 2^{k-1}, \qquad m = 1, 2, \ldots, M, \qquad \hat{m} = 2^{k-1}M,$$

with

$$\tilde{\mathsf{G}}_m(2^{k-1}t - \hat{n}) = \begin{cases} 1, & m = 1, \\ \frac{1}{\sqrt{\frac{2(-1)^m (m!)^2}{(2m)!} g_{2m}}} \mathsf{G}_m(2^{k-1}t - \hat{n}), & m > 1. \end{cases}$$

Here, $\mathsf{G}_m(t)$ is the Genocchi polynomials of order m as [28]:

$$\mathsf{G}_m(t) = \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right) g_{m-i} t^i. \tag{5}$$

where $g_i = 2B_i - 2^{i+1}B_i$ is the Genocchi numbers and B_i is the well-known Bernoulli number. From the properties of the Genocchi polynomials, the following relation is established [28]:

$$\int_0^1 \mathsf{G}_m(t)\mathsf{G}_n(t)dt = \frac{2(-1)^m m!n!}{(m+n)!} g_{m+n}, \qquad m, n \ge 1.$$

3. Riemann-Liouville fractional integral operator

The primary aim of this section is to present the exact formula for the R-LFIO of the GWs in terms of the regularized beta function (RBF). The RBF is defined as [29]:

$$\mathbb{I}(t;\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t s^{\alpha-1}(1-s)^{\beta-1}ds.$$

Lemma 3.1. *Let γ and c be positive real numbers. Then*

$${}_0^R I_t^q(t^\gamma\mu_c(t))=\frac{\Gamma(\gamma+1)t^{\gamma+q}}{\Gamma(\gamma+1+q)}[1-\mathbb{I}(\tfrac{c}{t};\gamma+1,q)]\mu_c(t),$$

where μ_c is defined by [29]

$$\mu_c(t)=\begin{cases}1, & if \quad t\geq c,\\ 0, & otherwise.\end{cases} \tag{6}$$

Proof. We have $\mu_c(t)=0$, for $t<c$. So ${}_0^R I_t^q(t^\gamma\mu_c(t))$ is identically zero on the interval $[0,c)$. At this point, it is reasonable to presume that $t\geq c$. Using the definition of R-LFIO (3), their properties, and Equation (6), we obtain

$$\begin{aligned}
{}_0^R I_t^q(t^\gamma\mu_c(t)) &= \frac{1}{\Gamma(q)}\int_0^t s^\gamma\mu_c(s)(t-s)^{q-1}ds\\
&= \frac{1}{\Gamma(q)}\int_0^t s^\gamma(t-s)^{q-1}ds-\frac{1}{\Gamma(q)}\int_0^c s^\gamma(t-s)^{q-1}ds\\
&= {}_0^R I_t^q(t^\gamma)-\frac{1}{\Gamma(q)}\int_0^c t^\gamma(\frac{s}{t})^\gamma t^{q-1}(1-\frac{s}{t})^{q-1}td\frac{s}{t}\\
&= {}_0^R I_t^q(t^\gamma)-\frac{t^{\gamma+q}}{\Gamma(q)}\int_0^{\frac{c}{t}} y^\gamma(1-y)^{q-1}dy\\
&= {}_0^R I_t^q(t^\gamma)-\frac{t^{\gamma+q}}{\Gamma(q)}\frac{\Gamma(\gamma+1)\Gamma(q)}{\Gamma(\gamma+q+1)}\mathbb{I}(\frac{c}{t};\gamma+1,q)\\
&= \frac{\Gamma(\gamma+1)t^{\gamma+q}}{\Gamma(\gamma+1+q)}[1-\mathbb{I}(\frac{c}{t};\gamma+1,q)].
\end{aligned}$$

□

Theorem 3.2. *The R-LFIO of GWs ($\Omega(t,q)$) for $q>0$ is obtained as:*

$${}_0^R I_t^q(\Psi(t))=\Omega(t,q), \tag{7}$$

where

$$\Omega(t,q)=$$
$$[{}_0^R I_t^q(\psi_{1,1}(t)),\ldots,{}_0^R I_t^q(\psi_{1,M}(t)),{}_0^R I_t^q(\psi_{2,1}(t)),\ldots,{}_0^R I_t^q(\psi_{2,M}(t)),\ldots,{}_0^R I_t^q(\psi_{2^{k-1},1}(t)),\ldots,{}_0^R I_t^q(\psi_{2^{k-1},M}(t))]^T.$$

That, for the case where $m = 1$, we have:

$$ {}_{0}^{R}I_t^q(\psi_{n,1}(t)) = \begin{cases} 0, & 0 \leq t < \frac{\hat{n}}{2^{k-1}}, \\ \frac{2^{\frac{k-1}{2}} t^q}{\Gamma(q+1)} [1 - \mathbb{I}(\frac{c_1}{t}; 1, q)], & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ \frac{2^{\frac{k-1}{2}} t^q}{\Gamma(q+1)} [\mathbb{I}(\frac{c_2}{t}; 1, q) - \mathbb{I}(\frac{c_1}{t}; 1, q)], & \frac{\hat{n}+1}{2^{k-1}} \leq t < 1. \end{cases} \tag{8} $$

For $m > 1$, we get:

$$ {}_{0}^{R}I_t^q(\psi_{n,m}(t)) = \begin{cases} 0, & 0 \leq t < \frac{\hat{n}}{2^{k-1}}, \\ \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m (m!)^2 g_{2m}}} \sum_{s=0}^{m} \sum_{j=0}^{s} \left(\begin{array}{c} m \\ s \end{array} \right) \left(\begin{array}{c} s \\ j \end{array} \right) (-1)^{s-j} g_{m-s} (\hat{n})^{s-j} 2^{(k-1)j} & \\ \times \frac{\Gamma(j+1) t^{j+q}}{\Gamma(j+q+1)} [1 - \mathbb{I}(\frac{c_1}{t}; j+1, q)], & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m (m!)^2 g_{2m}}} \sum_{s=0}^{m} \sum_{j=0}^{s} \left(\begin{array}{c} m \\ s \end{array} \right) \left(\begin{array}{c} s \\ j \end{array} \right) (-1)^{s-j} g_{m-s} (\hat{n})^{s-j} 2^{(k-1)j} & \\ \times \frac{\Gamma(j+1) t^{j+q}}{\Gamma(j+q+1)} [\mathbb{I}(\frac{c_2}{t}; j+1, q) - \mathbb{I}(\frac{c_1}{t}; j+1, q)], & \frac{\hat{n}+1}{2^{k-1}} \leq t < 1, \end{cases} \tag{9} $$

where $c_1 = \frac{n-1}{2^{k-1}}$, and $c_2 = \frac{n}{2^{k-1}}$.

Proof. For $m = 1$, the GWs can be rewritten as:

$$ \psi_{n,1}(t) = 2^{\frac{k-1}{2}} (t^0 \mu_{c_1}(t) - t^0 \mu_{c_2}(t)). \tag{10} $$

From Equation (10), Proposition 2.3, and Lemma 3.1, we yield

$$ \begin{aligned} {}_{0}^{R}I_t^q(\psi_{n,1}(t)) &= 2^{\frac{k-1}{2}} [{}_{0}^{R}I_t^q(t^0 \mu_{c_1}(t)) - {}_{0}^{R}I_t^q(t^0 \mu_{c_2}(t))] \\ &= \frac{2^{\frac{k-1}{2}} t^q}{\Gamma(q+1)} [(1 - \mathbb{I}(\frac{c_1}{t}; 1, q)) \mu_{c_1}(t) - (1 - \mathbb{I}(\frac{c_2}{t}; 1, q)) \mu_{c_2}(t)]. \end{aligned} \tag{11} $$

Thus, we conclude the relation (8).
For $m > 1$, the GWs can be rewritten as:

$$ \begin{aligned} \psi_{n,m}(t) &= \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m (m!)^2 g_{2m}}} (\mathsf{G}_m(2^{k-1}t - \hat{n}) \mu_{c_1}(t) - \mathsf{G}_m(2^{k-1}t - \hat{n}) \mu_{c_2}(t)) \\ &= \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m (m!)^2 g_{2m}}} \sum_{s=0}^{m} \sum_{j=0}^{s} \left(\begin{array}{c} m \\ s \end{array} \right) \left(\begin{array}{c} s \\ j \end{array} \right) (-1)^{s-j} g_{m-s} (\hat{n})^{s-j} 2^{(k-1)j} [t^j \mu_{c_1}(t) - t^j \mu_{c_2}(t)]. \end{aligned} \tag{12} $$

Considering Equation (12), Proposition 2.3, and Lemma 3.1, we get:

$$
\begin{aligned}
{}_0^R I_t^q(\psi_{n,m}(t)) &= \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m(m!)^2 g_{2m}}}\sum_{s=0}^{m}\sum_{j=0}^{s}\left(\begin{array}{c} m \\ s \end{array}\right)\left(\begin{array}{c} s \\ j \end{array}\right)(-1)^{s-j} g_{m-s}(\hat{n})^{s-j}2^{(k-1)j} \\
\times & \quad [{}_0^R I_t^q(t^j\mu_{c_1}(t)) - {}_0^R I_t^q(t^j\mu_{c_2}(t))] \\
= & \quad \sqrt{\frac{2^{k-1}(2m)!}{2(-1)^m(m!)^2 g_{2m}}}\sum_{s=0}^{m}\sum_{j=0}^{s}\left(\begin{array}{c} m \\ s \end{array}\right)\left(\begin{array}{c} s \\ j \end{array}\right)(-1)^{s-j} g_{m-s}(\hat{n})^{s-j}2^{(k-1)j} \\
\times & \quad \frac{\Gamma(j+1)t^{j+q}}{\Gamma(j+q+1)}[(1-\mathbb{I}(\frac{c_1}{t};j+1,q))\mu_{c_1}(t) - (1-\mathbb{I}(\frac{c_2}{t};j+1,q))\mu_{c_2}(t)]. \qquad (13)
\end{aligned}
$$

Finally, we conclude the relation (9).

□

4. Explanation of the proposed method

This section looks into the methodology suggested for solving DFDEs. The approach using GWs with weighted residual methods would transform the initial problem into a system of algebraic equations.

For solving problem (1)-(2), we expand ${}_0^C D_t^{\beta} u(t)$ in the terms of GWs as

$$ {}_0^C D_t^{\beta} u(t) \simeq C^T \Psi(t) = {}_0^C D_t^{\beta} u^{\star}(t). \qquad (14)$$

Using Proposition 2.3, and Equations (7) and (14), we have

$$ u(t) \simeq C^T \Omega(t,\beta) + \sum_{i=0}^{\lceil\beta\rceil-1} \frac{t^i}{i!} u_0^{(i)} = u^{\star}(t). \qquad (15)$$

From Equation (15), we achieve

$$ {}_0^C D_t^{q} u(t) \simeq C^T \Omega(t,\beta-q) + \sum_{i=0}^{\lceil\beta\rceil-1} \frac{{}_0^C D_t^{q} t^i}{i!} u_0^{(i)} = {}_0^C D_t^{q} u^{\star}(t). \qquad (16)$$

Substituting Equation (16) in Equation (1), we get

$$ R(t,C) = \int_{\alpha}^{\beta} \mathbb{A}(q, {}_0^C D_t^{q} u^{\star}(t))dq - \mathbb{F}(t). \qquad (17)$$

Through the application of the Gauss-Legendre numerical integration technique for the assessment of the integral delineated in Equation (17), we derive

$$ R(t,C) = \frac{\beta-\alpha}{2}\sum_{j=1}^{\tilde{n}} \omega_j \mathbb{A}(\frac{\beta-\alpha}{2}\xi_j + \frac{\beta+\alpha}{2}, {}_0^C D_t^{\frac{\beta-\alpha}{2}\xi_j+\frac{\beta+\alpha}{2}} u^{\star}(t)) - \mathbb{F}(t). \qquad (18)$$

where $\{\omega_j\}_{j=1}^{\tilde{n}}$ and $\{\xi_j\}_{j=1}^{\tilde{n}}$ are weights and Legendre-Gauss quadrature nodes, respectively, and $\tilde{n}$ is the number of weights and nodes.
By employing the weighted residual method by weighting function $\delta(t-t_i)$, we have

$$\mathbb{R}(t_i, C) = \int_0^1 \delta(t-t_i) R(t_i, C) = 0, \qquad i = 1, 2, \ldots, 2^{k-1}M, \tag{19}$$

where t_i represents the roots of the shifted Legendre polynomials $P_{\hat{m}}$. The above relation gives $2^{k-1}M$ equations corresponding to $2^{k-1}M$ unknowns, which can be resolved for the unknown vector C through the application of Newton's iterative method.

5. Error analysis

In this section, we intend to discuss the convergence characteristics for the application of the proposed numerical methods in solving the problem of fractional differential equations with distributed order. Some theoretical foundations will assist in defining accuracy for our approach. Attention to the best approximation in Sobolev norms within the context of GWs will lead to the extraction of significant results that describe the reliability and effectiveness of the method obtained through definitions and theorems.

Definition 5.1. The Sobolev norm of integer order $\tau \geq 0$ within the interval (α, β) is formally delineated by [37]

$$\|u\|_{\mathcal{H}^\tau(\alpha,\beta)} = \left(\sum_{j=0}^{\tau} \int_\alpha^\beta |u^{(j)}(t)|^2 \, dt \right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\tau} \|u^{(j)}(t)\|^2_{L^2(\alpha,\beta)} \right)^{\frac{1}{2}}, \tag{20}$$

where $u^{(j)}$ signifies the distributional derivative of the function u of the j-th order.

Theorem 5.2. *Let us consider the scenario where u is an element of the function space $\mathcal{H}^\tau(\alpha,\beta)$, with the condition that $\tau \geq 0$, and let u^* represent the best approximation of u within the set $\Psi(t)$; thus, [38]*

$$\|u - u^*\|_{L^2(\alpha,\beta)} \leq cM^{-\tau}(2^{k-1})^{-\tau} \|u^{(\tau)}\|_{L^2(\alpha,\beta)}, \tag{21}$$

and for $s \geq 1$

$$\|u - u^*\|_{\mathcal{H}^s(\alpha,\beta)} \leq cM^{2s-\frac{1}{2}-\tau}(2^{k-1})^{s-\tau} \|u^{(\tau)}\|_{L^2(\alpha,\beta)}. \tag{22}$$

Theorem 5.3. *Let us consider the case where $u \in \mathcal{H}^\tau(\alpha,\beta)$, with the condition that $\tau \geq 0$, $M > s$, and $0 < q \leq 1$. Then*

$$\|{}_0^C D_t^q u - {}_0^C D_t^q u^*\|_{L^2(\alpha,\beta)} \leq \frac{(\beta^{1-q} - \alpha^{1-q}) cM^{2s-\frac{1}{2}-\tau}(2^{k-1})^{s-\tau}}{\Gamma(2-q)} \|u^{(\tau)}\|_{L^2(\alpha,\beta)}, \tag{23}$$

where $1 \leq s < \tau$.

Proof. The proof involves using the application of the characteristics of fractional derivatives in conjunction with the convolution inequality.

By employing case 3 of Proposition 2.3 and Ref. [39], we have

$$\|\mathcal{F} * \mathcal{G}\|_p \leq \|\mathcal{F}\|_1 \|\mathcal{G}\|_p.$$

So the following conclusion is reached:

$$\begin{aligned}
\|{}_0^C D_t^q u - {}_0^C D_t^q u^*\|^2_{L^2(\alpha,\beta)} &= \left\|{}_0^R I_t^{1-q}({}_0^C D_t u - {}_0^C D_t u^*)\right\|^2_{L^2(\alpha,\beta)} \\
&= \left\|\frac{1}{t^q \Gamma(1-q)} * ({}_0^C D_t u - {}_0^C D_t u^*)\right\|^2_{L^2(\alpha,\beta)} \\
&\leq \left(\frac{\beta^{1-q} - \alpha^{1-q}}{\Gamma(2-q)}\right)^2 \|{}_0^C D_t u - {}_0^C D_t u^*\|^2_{L^2(\alpha,\beta)} \\
&\leq \left(\frac{\beta^{1-q} - \alpha^{1-q}}{\Gamma(2-q)}\right)^2 \|u - u^*\|^2_{\mathcal{H}^s(\alpha,\beta)}.
\end{aligned}$$

From the implications of Theorem 5.2, the following result is derived:

$$\|{}_0^C D_t^q u - {}_0^C D_t^q u^*\|^2_{L^2(\alpha,\beta)} \leq \frac{(\beta^{1-q} - \alpha^{1-q})^2 c^2 M^{4s-1-2\tau}(2^{k-1})^{2s-2\tau}}{(\Gamma(2-q))^2} \|u^{(\tau)}\|^2_{L^2(\alpha,\beta)}.$$

By extracting the square root, the required result is obtained. □

At this juncture, we will conduct a thorough examination of the error bounds associated with the proposed methodology.

Theorem 5.4. *Let $u \in H^\tau(\alpha, \beta)$ with $\tau \geq 0$, $M > s$, and the operator $\mathbb{A}$ defined in (14) being Lipschitz continuous with the Lipschitz constant η. Then, the present error estimate of the method E is given as follows:*

$$\|E\|_{L^2(\alpha,\beta)} \leq \eta \frac{(\beta - \alpha)(\beta^{1-q} - \alpha^{1-q}) c M^{2s-\frac{1}{2}-\tau}(2^{k-1})^{s-\tau}}{\Gamma(2-q)} \|u^{(\tau)}\|_{L^2(\alpha,\beta)}, \quad (24)$$

where $1 \leq s \leq \tau$.

Proof. We define

$$\|E\|_{L^2(\alpha,\beta)} = \left\|\int_\alpha^\beta \mathbb{A}(q, {}_0^C D_t^q u(t)) dq - \mathbb{F}(t)\right\|_{L^2(\alpha,\beta)}.$$

From Equation (1), we have

$$\|E\|_{L^2(\alpha,\beta)} = \left\|\int_\alpha^\beta \mathbb{A}(q, {}_0^C D_t^q u(t)) dq - \int_\alpha^\beta \mathbb{A}(q, {}_0^C D_t^q u^*(t)) dq\right\|_{L^2(\alpha,\beta)}.$$

Given that $\mathbb{A}$ adheres to a Lipschitz condition characterized by the constant η, we derive

$$\|E\|_{L^2(\alpha,\beta)} \leq \eta \int_{\alpha}^{\beta} \|{}_0^C D_t^q u(t) - {}_0^C D_t^q u^*(t)\|_{L^2(\alpha,\beta)}\, dt.$$

From Theorem 5.3, we get

$$\|E\|_{L^2(\alpha,\beta)} \leq \eta \frac{(\beta-\alpha)(\beta^{1-q}-\alpha^{1-q})cM^{2s-\frac{1}{2}-\tau}(2^{k-1})^{s-\tau}}{\Gamma(2-q)} \|u^{(\tau)}\|_{L^2(\alpha,\beta)}.$$

This concludes the proof. □

6. Numerical examples

Here, four numerical examples are provided to show the validity and applicability of the presented scheme in Section 4.

Example 6.1. Consider the following DFDEs:

$$\int_0^2 \frac{\Gamma(6-q)}{120} {}_0^C D_t^q u(t) dq = \frac{t^5 - t^3}{Lnt}, \tag{25}$$

with the initial conditions

$$u(0) = u'(0) = 0. \tag{26}$$

The exact solution is $u(t) = t^5$.

We compare the obtained results of our method with the methods in [15](for step length in Adams solver of 0.0015625 and different values of step length in trapezium rule (ϵ)), and [40], which shows the superiority and accuracy of the proposed method in Table 1. This clearly indicates the efficiency of the proposed method in reducing absolute errors. In addition, it indicates that the method can give extremely accurate solutions with fewer required levels set in the number of nodes. Also, the absolute errors, numerical and exact solution for $k = 2, M = 8$ are demonstrated in Figure 1. This demonstrates the method's strong capability in capturing the underlying dynamics of the represented DFDE, highlighting its robustness and reliability for potential real-world applications.

Example 6.2. Consider the following DFDEs:

$$\int_{0.2}^{1.5} \Gamma(3-q) {}_0^C D_t^q u(t) dq = 2\frac{t^{1.8} - t^{0.5}}{Lnt}, \tag{27}$$

with the initial conditions

$$u(0) = u'(0) = 0. \tag{28}$$

The exact solution is $u(t) = t^2$.

In Table 2, the absolute errors obtained by the mentioned method for $k = 2$ and

Table 1: The comparison of absolute error in $u(0.5)$ for Example 6.1.

$Ref.$[15]	$Absolute\ errors$	CPU
$\epsilon = 1$	2.48×10^{-3}	–
$\epsilon = 0.5$	6.39×10^{-4}	–
$\epsilon = 0.25$	1.59×10^{-4}	–
$\epsilon = 0.75$	3.73×10^{-5}	–
$Ref.$[40]		
$K = 2$	2.48×10^{-3}	–
$K = 4$	6.36×10^{-4}	–
$K = 8$	1.47×10^{-4}	–
$K = 16$	1.71×10^{-5}	–
$Proposed\ method$		
$k = 2, M = 5$	1.47×10^{-10}	0.047
$k = 2, M = 8$	9.23×10^{-16}	0.219

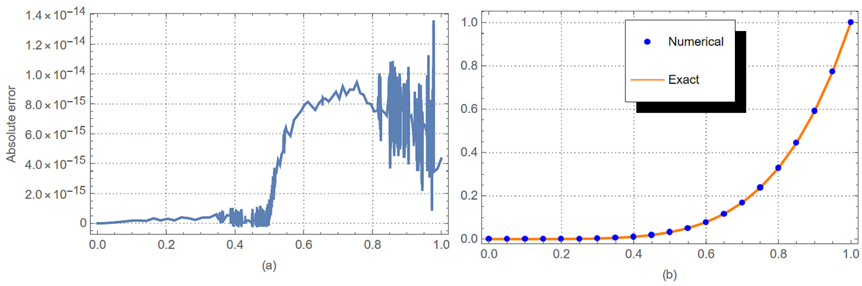

Figure 1: (a): absolute error, (b): numerical and exact solution of the presented method for $k = 2, M = 8$ for Example 6.1.

different values M are reported. As the value of M increases, errors significantly decrease, confirming that the solution converges towards the exact value. It also demonstrates the computational efficiency of the method, achieving accurate results with relatively low computational cost. In addition, Figure 2 shows the absolute error, numerical and exact solution of the presented method for $k = 2, M = 5$.

Example 6.3. Consider the following DFDEs:

$$\int_0^2 (\Gamma(4-q){}_0^C D_t^q u(t))^2 dq = 18\frac{t^6 - t^2}{Lnt}, \tag{29}$$

with the initial conditions

$$u(0) = u'(0) = 0. \tag{30}$$

Table 2: The absolute error for $k = 2$ and different values M for Example 6.2.

t	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0.1	1.26×10^{-3}	8.07×10^{-6}	7.45×10^{-5}	8.82×10^{-6}
0.2	4.61×10^{-4}	3.15×10^{-4}	1.41×10^{-4}	5.03×10^{-5}
0.3	1.79×10^{-3}	4.70×10^{-4}	9.04×10^{-5}	1.48×10^{-4}
0.4	1.35×10^{-3}	1.19×10^{-3}	7.05×10^{-4}	5.25×10^{-4}
0.5	3.65×10^{-3}	1.31×10^{-3}	1.39×10^{-3}	1.23×10^{-3}
0.9	5.80×10^{-3}	2.50×10^{-3}	2.18×10^{-3}	2.42×10^{-3}

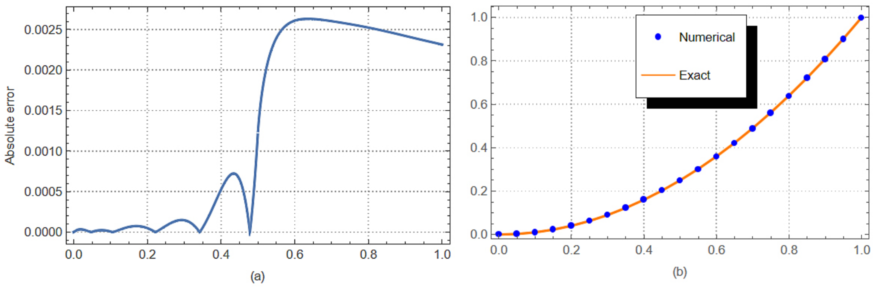

Figure 2: (a): absolute error, (b): numerical and exact solution of the presented method for $k = 2, M = 5$ for Example 6.2.

The exact solution is $u(t) = t^3$.
In Table 3, we compare the results obtained using our method with those presented in Ref. [15](for a step length of 0.0015625 in the Adams solver and various step lengths ϵ in the trapezium rule). Furthermore, Figure 3 illustrates the absolute error, numerical and exact solution obtained using the proposed method for $k = 1$ and $M = 5$.

Example 6.4. Consider the following DFDEs:

$$\int_0^2 e^{-q}\Gamma(6-q){}_0^C D_t^q u(t)dq = 120\frac{t^3(t^2 - e^{-t})}{1 + Lnt}, \tag{31}$$

with the initial conditions

$$u(0) = u'(0) = 0. \tag{32}$$

The exact solution is $u(t) = t^5$.
Figure 4 shows the absolute error and the numerical solution obtained using the proposed method for $k = M = 1$.

Table 3: The comparison of absolute error in $u(0.5)$ for Example 6.3.

Ref.[15]	*Absolute errors*	*CPU*
$\epsilon = 1$	2.58×10^{-2}	–
$\epsilon = 0.5$	8.95×10^{-3}	–
$\epsilon = 0.25$	2.49×10^{-3}	–
$\epsilon = 0.125$	6.37×10^{-4}	–
Proposed method		
$k = 1, M = 2$	2.84×10^{-7}	0.015
$k = 1, M = 3$	3.93×10^{-11}	0.047
$k = 1, M = 4$	1.63×10^{-14}	0.062
$k = 1, M = 5$	8.87×10^{-14}	0.062

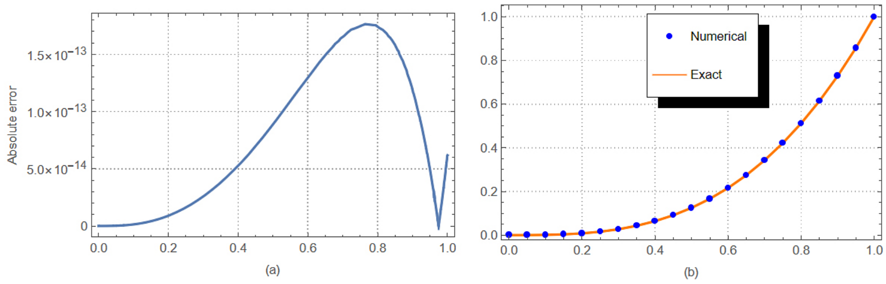

Figure 3: (a): absolute error, (b): numerical and exact solution of the presented method for $k = 1, M = 5$ for Example 6.3.

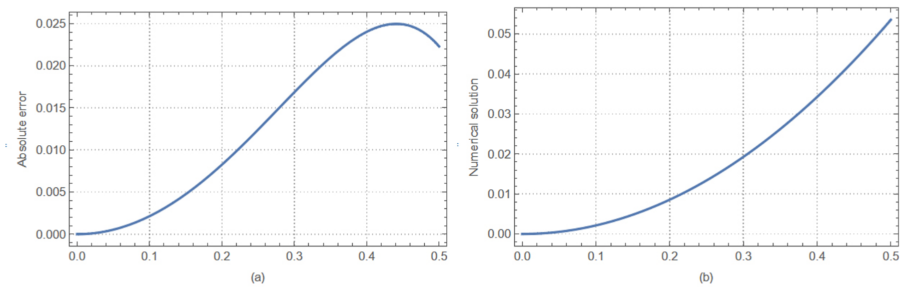

Figure 4: (a): absolute error, (b): numerical solution of the presented method for $k = M = 1$ for Example 6.4.

7. Conclusions

This paper presents a method based on the GWs for solving DFDEs. By deriving the exact formula for the Riemann-Liouville fractional integral operator of the GWs using the regularized beta function, the study introduces an effective numerical approach. The integration of GWs with Gauss-Legendre numerical techniques transforms the DFDEs into algebraic equations, facilitating precise and efficient computation. The numerical experiments conducted validate the method's accuracy and computational efficiency, outperforming existing techniques. The results demonstrate that this approach not only resolves the challenges associated with solving DFDEs but also offers a structured framework for extending wavelet-based solutions to more complex systems. We envisage the following directions for future research:

- The proposed method can be extended to solve various types of problems, including fractional partial differential equations, fractional pantograph equations, and others.
- Conducting a stability analysis of the proposed scheme for the numerical approximation of the current problem is engaging direction for future investigation.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgment. Authors are very grateful to the reviewers for carefully reading the paper and for his(her) comments and suggestions which have improved the paper.

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