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Perfect Star Packings in (2,6)-Fullerene Graphs

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Abstract

A (2,6)-fullerene graph is a planar and cubic graph with hexagonal and 2-length faces. A perfect star packing is a spanning subgraph of a graph G in which each component is isomorphic to the star graph $K_{1,3}$. In this paper, we investigate which (2,6)-fullerene graphs allow such packings.

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1. Introduction

A $(k,6)$-fullerene graph is a 3-regular planar graph in which each face has either k or six edges. In [1], Došlić proved that $(k,6)$-fullerenes exist for $k=2,3,4$, and 5. Euler's formula shows that the number of faces of length k in $(k,6)$-fullerene graphs for $k=2,3,4$, and 5 equal $3,4,6$, and 12, respectively. A perfect star packing in a graph G is a spanning subgraph of G whose every component is isomorphic to the star graph $K_{1,3}$.

In [2–4], the authors obtained results about perfect star packings of $(k,6)$-fullerene graphs for $k=3,4$, and 5. Also, in [5, 6], some properties of these graphs have been examined. Došlić et al. [4] have investigated which fullerene graphs allow perfect star packings, and L. Shi proved that classical fullerene graphs with $8n+4$ $(n\in\mathbb{N})$ vertices do not have a perfect star packing [7]. Much research has been done on fullerene graphs and their properties, and this subject has always interested mathematicians and chemists [8–10]. In [11], R. Yang et al. investigated

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the structure of $(2,6)$-fullerene graphs. The results obtained about this type of fullerene graphs are very limited compared to other known types. In this paper, we investigate the structure of the $(2,6)$-fullerene graphs and check the existence of perfect star packings in these graphs.

2. Definitions and auxiliary results

This section states the important and used definitions and preliminary results. Further reading [12, 13] is recommended. A $(2,6)$-fullerene graph is a cubic and planar graph with faces of lengths 2 and 6. By Euler's formula, the number of faces of length two equals 3. A $n\times K_2$ graph is a graph with n parallel edges and two vertices. The smallest $(2,6)$-fullerene graph is $3\times K_2$, which contains three parallel edges between two vertices [11]. (See Figure 1).

Figure 1: $3\times K_2$ is the smallest (2,6)-fullerene graph.

A connected graph is called 2-extendable if every matching of size 2 can be extended to a perfect matching. All classical fullerenes are 2-extendable [14]. A graph G is called bicritical if, for all pairs of distinct vertices u and v, $G-u-v$ has perfect matching. Every $(2,6)$-fullerene graph is 1-extebdable; that is, every edge of the graph is contained in a perfect matching [11]. Also, it is proved in [11] that there is no 2-extendable and bicritical $(2,6)$-fullerene graph, and the following results are obtained.

Lemma 2.1 ([11])**.** *The* $(2,6)$*-fullerene* F *has edge-connectivity 2, where* $|V(F)| > 2$.

Lemma 2.2 ([11])**.** *Every 2-edge-cut of a* $(2,6)$*-fullerene isolates a 2-length face.*

In the next section, according to the structures introduced in [11], we will examine the existence of star packings in $(2,6)$-fullerene graphs.

3. Packing star

First, we examine the structure of $(2,6)$-fullerene graphs, as stated by the authors in[11]. As mentioned, $(2,6)$-fullerene graphs only have faces of lengths 2 and 6, and the smallest $(2,6)$-fullerene graph is $3\times K_2$. Another simplest $(2,6)$-fullerene graph is a graph with six vertices, each vertex having two adjacent vertices, see Figure 2.

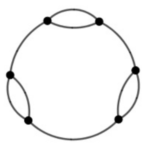

Figure 2: The graph F_6.

In the following, according to the structure of $(2,6)$-fullerene graphs, we mention some forbidden configurations precluding the existence of perfect star packings and state some simple results.

Proposition 3.1. *No vertex of* $2{\times}K_2$ *can be the central vertex of the star.*

Proof. Suppose that u is the vertex of the central star in a perfect star packing in $2{\times}K_2$. Then, the two edges of the star are adjacent to the other vertex, which is impossible. □

Corollary 3.2. $3{\times}K_2$ and F_6 do not have perfect star packing.

Proposition 3.3. *The necessary condition for the fullerene graph F to have a perfect star packing is that the number of its vertices is a multiple of 4.*

Grünbaum et al. proved in [15] that for every non-negative integer, $h \neq 1$, there exists a $(k,6)$-fullerene graph, (for $k = 3,4,5$), with h hexagons. In paper [11], $(2,6)$-fullerene graphs are classified. However, the authors have not answered whether there is a $(2,6)$-fullerene graph with h hexagons for every non-negative integer, $h{\neq}1$. They proved that there are $(2,6)$-fullerene graphs with f_F hexagonal faces, so that f_F depends on a parameter such as s.

Let F be a $(2,6)$-fullerene graph. A **fragment** H of F is a subgraph of F consisting of a cycle with its interior, and every inner face of H is also a face of F [11]. In [11], $(2,6)$-fullerene graphs are classified as follows.

If we denote the number of hexagons of the fragment H by f_H, then we have:

Proposition 3.4 ([11])**.** *In every $(2,6)$-fullerene graph, there is a fragment, say G_s, which $f_{G_s} = s^2 + s, s \in \mathbb{Z}$.*

In the previous proposition, s is the number of hexagonal layers. If G_0 is a cycle of length 2, then G_s, with s layers of hexagons, is shown in Figure 3.

We call the $(2,6)$-fullerene graph containing the fragment shown in Figure 3 a fullerene graph of type 1.

Proposition 3.5. *If F is a $(2,6)$-fullerene graph of type 1 with an even number of hexagonal layers, it cannot have a perfect star packing.*

Proof. Let F be a $(2,6)$-fullerene graph of type 1. For F to have a perfect star packing, the first layer should be covered, as shown in Figure 4. There are two

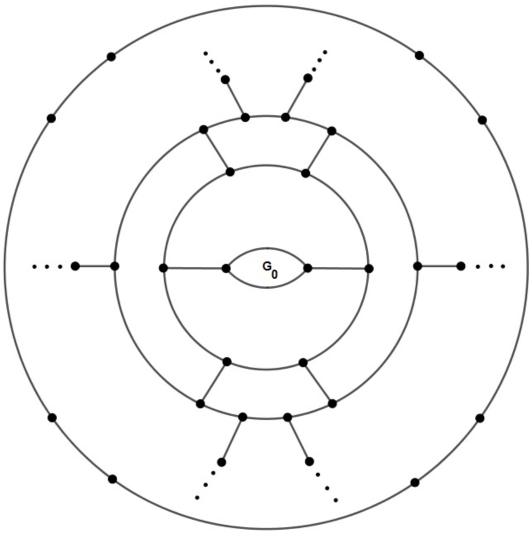

Figure 3: The fragment G_s.

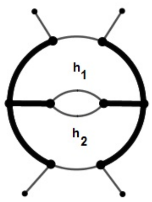

Figure 4: First layer cover.

hexagons in the first layer, four hexagons between the first and second layers, and $2s$ hexagons between the $(s-1)$th and sth layers. Also, there are 6 vertices in the first layer, 10 in the second layer, 14 in the third layer, and $4s+2$ vertices in the sth layer. The number of vertices in each layer is even. Based on the above explanation, we have Table 1 about $(2,6)$-fullerene graphs of type 1.

According to the previous explanations and the information in Table 1, if the $(2,6)$-fullerene graph of type 1 has s layers, then the number of its vertices is equal to

$$2s^2+4s+2=2\left(s^2+1\right)+4s,$$

which is an even number. This value is a multiple of four when s is odd. Because otherwise, if s is an even number, s^2+1 will be odd. On the other hand, according to Proposition 3.3, for F to have a perfect star packing, the number of its vertices must be a multiple of four, which completes the proof. □

Table 1: Information from $(2,6)$-fullerene graphs.

Layer	Number of hexagons	Vertices in each layer	Number of graph vertices
1	2	6	8
2	4	10	18
3	6	14	32
4	8	18	50
⋮	⋮	⋮	⋮
s	$2s$	$4s+2$	$2s^2+4s+2$

According to the Proposition 3.5, we have the following theorem:

Theorem 3.6. *If F is a $(2,6)$-fullerene graph of type 1 with s hexagonal layers, then*

1. *If s is odd, then F has a perfect star packing.*
2. *If s is even, then F does not have a perfect star packing.*

Proof. To find a perfect star packing in graph F, we must look for graphs with odd layers. Suppose F is such a graph. As mentioned, we have the packing in the first layer, as shown in Figure 4. The number of vertices in each layer is equal to $4s+2, s \in \mathbb{Z}$. If we discard the first layer, we will have an even number of layers. We start with the second and third layers. We are packing the vertices of these two layers as follows (see Figure 5): We are packing the vertices of the fourth and

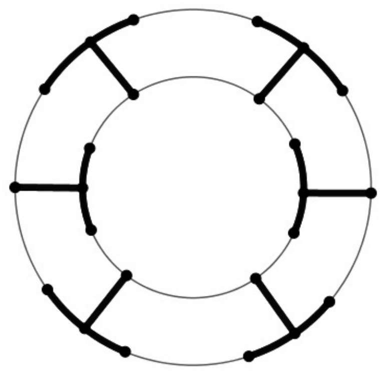

Figure 5: Packing of second and third layers.

fifth layers as follows: In the fifth layer, we have two pairs of consecutive centers of a star's vertices. In fact, in each pair of layers, we have two pairs of consecutive

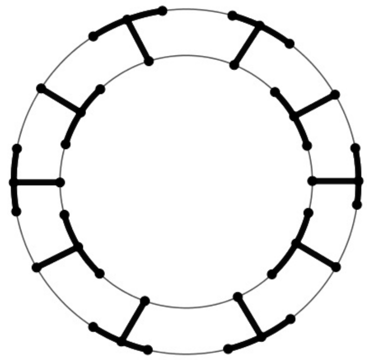

Figure 6: Packing of the fourth and fifth layers.

central star vertices in the outer layer, which are located on both sides of the axis shown in Figure 6 (dashed lines). Other vertices of the center of the star are equally divided among the vertices of the layers. By continuing this process in the last two layers, considering that according to Proposition 3.1, the vertices of $2 \times K_2$ cannot be the central vertex of the star, we consider each of the two vertices adjacent to $2 \times K_2$ as two consecutive central vertices. Finally, we will have a perfect star packing in F. □

Figure 7 shows an example of a perfect star packing in a $(2, 6)$-fullerene graph of type 1 with 72 vertices.

Proposition 3.7 ([11]). *In every $(2, 6)$-fullerene graph, there is a fragment, say C_s, which $f_{C_s} = s, s \in \mathbb{N}$.*

If C_0 is $3 \times K_2$, then $f_{C_0} = 0$. Figure 8 shows the fragments C_1 and C_s. We call the $(2, 6)$-fullerene graph containing the fragment C_s a fullerene graph of type 2.

Theorem 3.8. *If F be a $(2, 6)$-fullerene graph of type 2 with s hexagonal faces in the first layer, then*

1. *If s is odd, then F has a perfect star packing.*
2. *If s is even, then F does not have a perfect star packing.*

Proof. Clearly, the fullerene graph containing the fragment C_1 has a perfect star packing. Now, we consider the $(2, 6)$-fullerene graph F, including the fragment C_3. A perfect star packing for this graph is shown in Figure 9. Now, we examine the graph F containing the fragment C_s. This graph has s hexagonal faces between two cycles $2 \times K_2$. We consider two paths P_1 and P_2 from vertex v_1 to v_2 and $v_1{}'$ to $v_2{}'$, respectively, as follows:

$$P_1 : v_1 x_1 x_2 \cdots x_{2s-1} v_2 \quad , \quad P_2 : v_1{}' x_1{}' x_2{}' \cdots x_{2s-1}{}' v_2{}'.$$

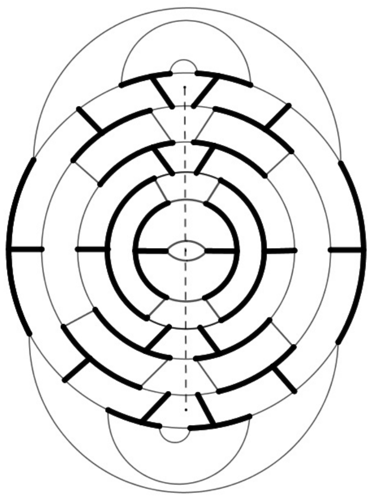

Figure 7: A perfect star packing in a $(2, 6)$-fullerene graph of type 1.

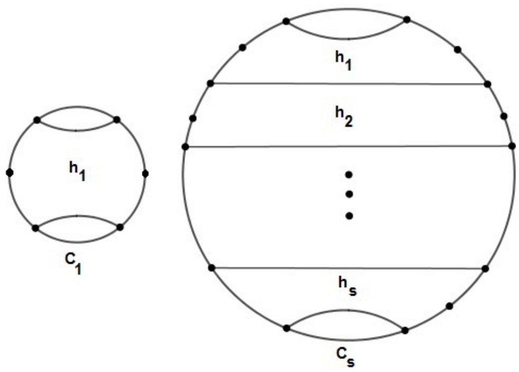

Figure 8: The fragments C_1 and C_s.

These paths are shown in Figure 10. The number of vertices in each of these two paths is equal to $2s+1$. The total number of vertices in this layer (first layer) equals $4s+2$. From the second layer onwards, four vertices are reduced at each step. After that, two vertices are added in the last layer, which has six vertices. Therefore, in a $(2, 6)$-fullerene graph of type 2, if we have s hexagons in the first layer, then the number of vertices is equal to

$$|V| = 2 + 6 + 10 + \cdots + (4s + 2) = 2s^2 + 4s + 2.$$

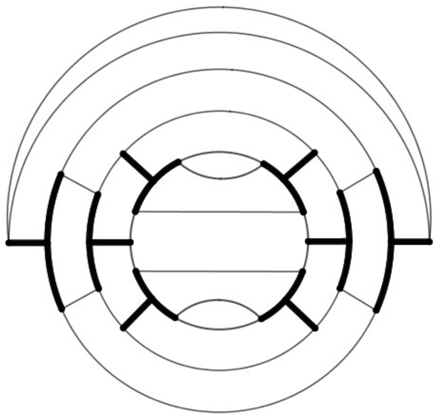

Figure 9: A perfect star packing in a $(2, 6)$-fullerene graphs of type 2 containing the fragment C_3.

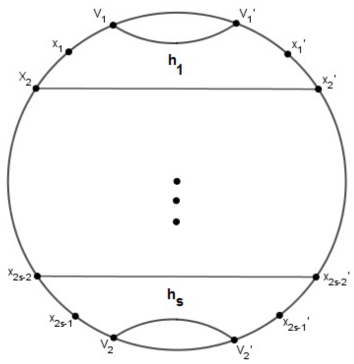

Figure 10: The paths P_1 and P_2.

The value of $2s^2 + 4s + 2$ is a multiple of four if $s^2 + 1$ is an even number, which means that s must be an odd number. Therefore, if s is an even number, then the number of vertices of F is not a multiple of four, and in this case, F does not have a perfect star packing.

Now, suppose that the number of hexagons in the first layer of F is odd. In the last layer, we have a packing, as illustrated in Figure 11. An even number of layers remains. We start from the first and second layers. In the first layer, we have $4s+2$ vertices. The number of vertices between u_1 and u_2 and v_1 and v_2 equals to $2s-5$, which is odd. Therefore, the first and second layers can be packed as depicted in Figure 12. By covering the remaining pairs of layers similarly, we

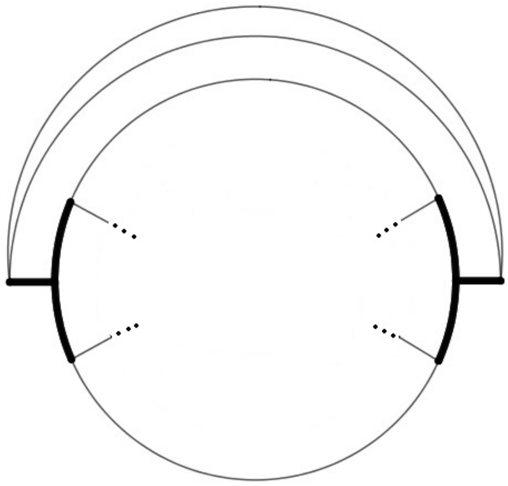

Figure 11: Packing in the last layer.

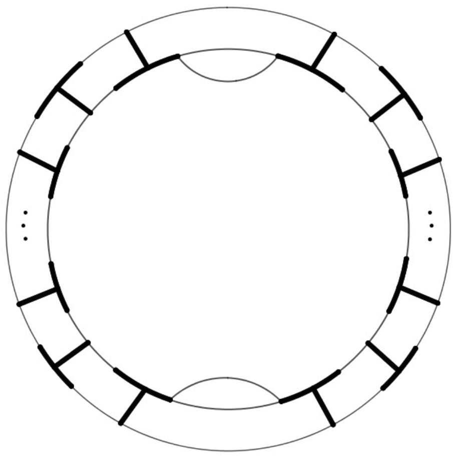

Figure 12: Packing in the first and second layers.

obtain a perfect packing,which completes the proof. □

Next, we study two other cases of $(2,6)$-fullerene graphs and investigate the existence of perfect star packing in these graphs.

Proposition 3.9 ([11]). *In every $(2,6)$-fullerene graph, there is a fragment, say K_s, which $f_{K_s} = 2s^2 + 3s, s \in \mathbb{Z}$.*

Suppose G' and G'' are two fragments, as in Figure 3. There are $4s+2$ vertices in the layer s of each of these fragments. Suppose $e_i = u_iv_i$, where i is an odd number, see Figure 13.

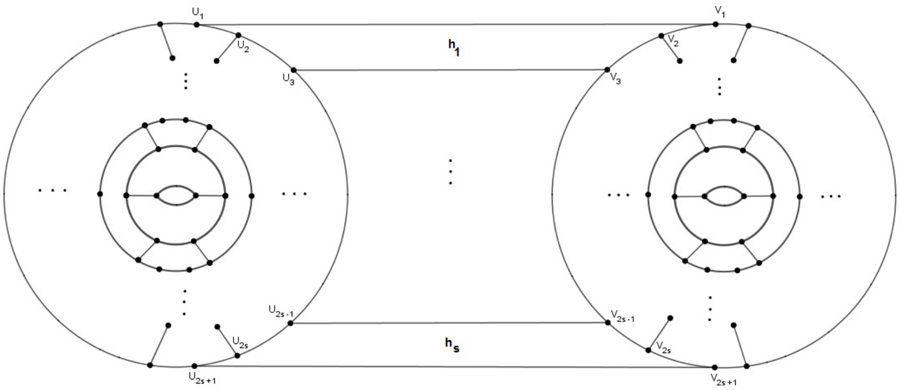

Figure 13: The initial fragment of $(2,6)$-fullerene graph type 3.

We have the number of s hexagons, h_1,h_2,$\cdots$,h_s; we call the created fragment K_s, and the $(2,6)$-fullerene graph including this fragment is called a $(2,6)$-fullerene graph of type 3. Figure 14 shows a $(2,6)$-fullerene graph of type 3.

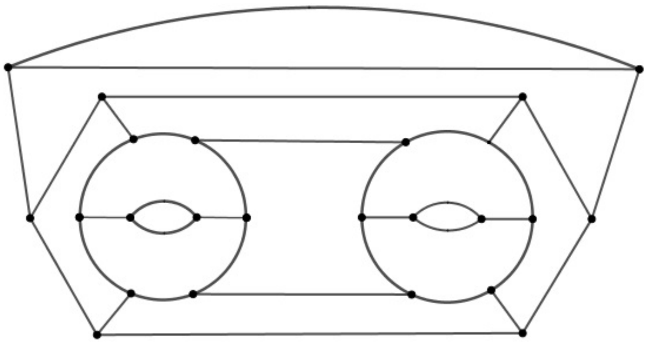

Figure 14: A $(2,6)$-fullerene graph type 3.

Now, we try to create a perfect star packing for this graph. As mentioned, a $(2,6)$-fullerene graph of type 3 consists of an initial fragment including two fragments G' and G'' and edges e_i=u_iv_i.

Theorem 3.10. *If F be a $(2,6)$-fullerene graph of type 3 with s hexagonal layers in the fragments G' and G'', then*

1. *If s is odd, then F has a perfect star packing.*

2. *If s is even, then F does not have a perfect star packing.*

Proof. If the number of hexagons in the fragments G' and G'' is an odd number, such as s, then s layers are formed around the initial fragment. The first layer has $4s+2$ vertices. In the second layer, we will have $4s-2$ vertices. Similarly, in the sth layer, the vertices of a hexagon, form six vertices. Finally, two vertices are added to them. Therefore, the number of vertices of a $(2,6)$-fullerene graph of type 3 that has s layers in the initial fragment is equal to

$$|V(F)| = 2\left(2s^2+4s+2\right)+2s^2+4s+2 = 6s^2+12s+6 = 4k+2\left(s^2+1\right), k \in \mathbb{Z}.$$

The value is a multiple of four when s is an odd number. Therefore, if s is even, F does not have a perfect star packing. Now, suppose s is an odd number. In this case, we show that F has a perfect star packing. For this purpose, we create a packing for fragments G' and G'' as described in the packing of $(2,6)$-fullerene graphs of type 1. By doing this, the initial fragment of graph F is covered. Now, we consider the vertices as the external layers. We start from the outermost layer. For this layer, we must have the following packing as shown in Figure 15.

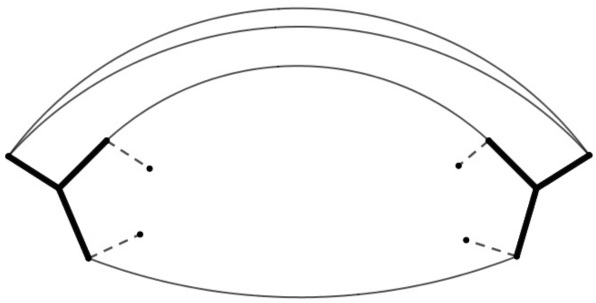

Figure 15: Packing of the outermost layer.

An even number of layers remains. We cover the vertices of each pair of layers as shown in Figure 16. Therefore, we have a perfect star packing. □

Figure 17 shows a perfect star packing in a $(2,6)$-fullerene graph of type 3 with three layers.

In [11], graph F is constructed in the following form; we call this graph $(2,6)$-fullerene graphs of type 4. First, we consider the graph F_6, which has three cycles of length two, including uvu, $u_1v_1u_1$, and $u_2v_2u_2$, see Figure 18.
We create paths P_1 and P_3 on two parallel edges uv as follow (see Figure 19):

$$P_1 : ux_{11}x_{12}x_{13}\cdots x_{1,2s}v \quad , \quad P_3 : ux_{31}x_{32}x_{33}\cdots x_{3,2s}v.$$

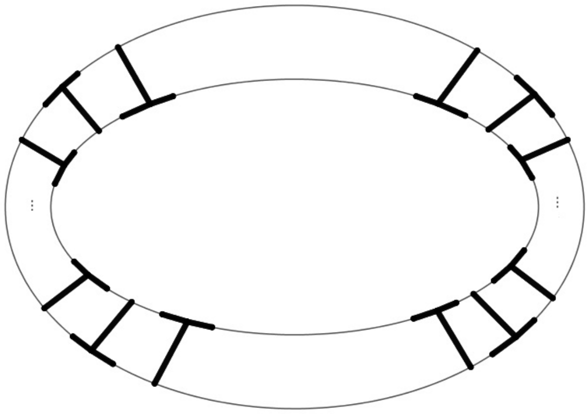

Figure 16: Packing of each pair of layers.

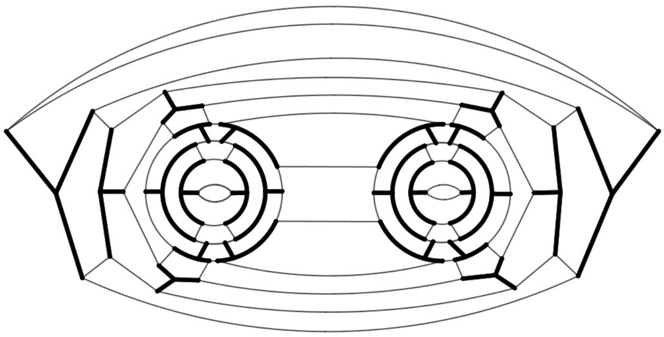

Figure 17: A perfect star packing in a $(2, 6)$-fullerene graph of type 3.

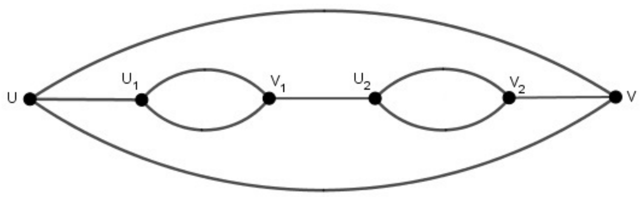

Figure 18: The graph F_6.

In fact, we have created $2s$ vertices in each of the uv parallel edges. Also, we replaced the edge v_1u_2 with path P_2 as follows:

$$P_2 : v_1x_{21}x_{22}x_{23}\cdots x_{2,2s}u_2.$$

Therefore, we have created $2s$ new vertices on the edge v_1u_2. Now suppose for $i = 1, 2$ we have

$$x_{i2}x_{i+1,1}, x_{i4}x_{i+1,3}, \dots\ ,\ x_{i,2n}x_{i+1,2n-1} \in E\,(F)\,.$$

So, the number of $s+1$ hexagons between P_i and P_{i+1} are created. $(h_1, h_2, \dots\ ,\ h_{s+1}$ Or $h_1{}', h_2{}', \dots\ ,\ h_{s+1}{}')$. Figure 20 shows a $(2, 6)$-fullerene graph of type 4. We

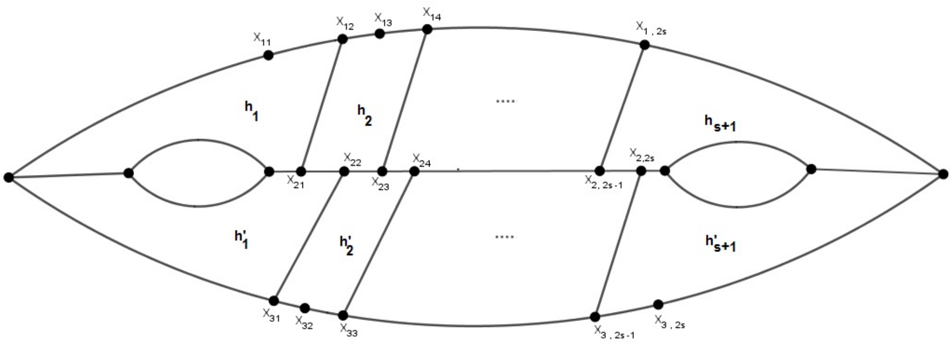

Figure 19: The paths P_1 and P_2.

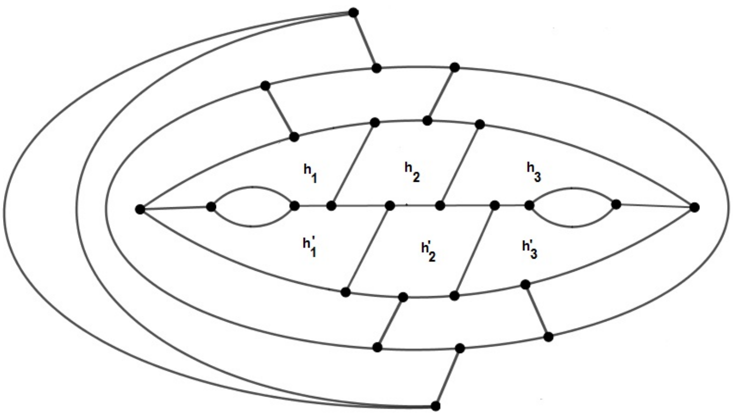

Figure 20: A $(2, 6)$-fullerene graph of type 4.

have placed $2s$ vertices in each path $P_i, i = 1, 2$. Therefore, we have $6s+6$ vertices in the first layer. If we add the four vertices in 2-cycles $u_1v_1u_1$ and $u_2v_2u_2$ to this number, we will have $6s{+}6$ vertices. In the next layers, we will have $4s-2$, $4s-6$, and so on vertices respectively, and in the last layer, we have six vertices. Therefore, the number of vertices in these layers is equal to

$$6+10+\cdots+(4s-2)=2s^2-2.$$

Finally, two vertices are added to these numbers. Therefore, the total number of vertices F is equal to

$$6s+6+2s^2-2+2=2s^2+6s+6, \quad s\in\mathbb{N}.$$

However, we have

$$2s^2+6s+6=4\,(s+1)+2(s^2+s+1).$$

For the above value to be a multiple of four, s^2+s must be an odd number, which is impossible. The following theorem can be concluded from the above:

Theorem 3.11. *If F is a $(2,6)$-fullerene graph of type 4, then F does not have a perfect star packing.*

4. Open problems

There are many open problems related to perfect star packings in $(2,6)$-fullerene graphs. Here we explore some of them in the last part of this article.

1. Characterize $(2,6)$-fullerene graphs that allow perfect star packings.
2. Find the number of perfect star packings in $(2,6)$-fullerene graphs.

   A perfect pseudo matching of a graph is a spanning subgraph such that each component is K_2 or $K_{1,3}$. Based on this definition, we state the third problem.
3. Find the perfect pseudo matching in $(2,6)$-fullerene graphs.

Conflicts of Interest. The author declare that he has no conflicts of interest refarding the publication of this article.

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