

Estrada Index and Some Properties of Partially Signed Graphs

Sheila Razavi and Gholam Hossein Fath-Tabar*

Abstract

A signed graph is a graph with signed edges. Recently, a special graph consisting of signed and unsigned edges has been introduced and is called a partially signed graph. Also, some properties of particular types of these graphs have been discussed. Now, in this article, we pursue some other features of other types of partially signed graphs, such as unicyclic and bipartite partially signed graphs and investigate the reconstruction of the characteristic from their polynomial decks.

Keywords: Unicyclic partially signed graph, Bipartite partially signed graph, Index of partially signed graph.

2020 Mathematics Subject Classification: 05C22; 05C50; 05C90.

How to cite this article

S. Razavi and G.H. Fath-Tabar, Estrada index and some properties of partially signed graphs, *Math. Interdisc. Res.* 11 (2) (2026) 185-199.

1. Introduction

Suppose that $G = (V(G), E(G))$ is a simple graph with n vertices and m edges. Considering σ^* as a signature function from $E(G)$ to $\{1, -1, a\}$. Then, $G_a^* = (G, \sigma^*)$ with underlying graph G and signature function σ^* is defined as a partially signed graph[1]. $V(G_a^*)$, the vertex set of a partially signed graph, is $V(G)$, while the edge set of this graph is divided into three disjoint subsets of positive, negative and unsigned edges.

$A(G_a^*)$, the adjacency matrix of G_a^* , is $A(G_a^*) = [a_{ij}^*]$. If v_i is adjacent to v_j by a

*Corresponding author (E-mail: fathtabar@kashanu.ac.ir)

Academic Editor: Saeid Alikhani

Received 31 December 2024, Accepted 3 June 2025

DOI: 10.22052/MIR.2025.256040.1490

signed edge $a_{ij}^* = \sigma^*(v_{ij})$ and if v_i is adjacent to v_j by an unsigned edge $a_{ij}^* = a$, a is a real number, otherwise $a_{ij}^* = 0$. Briefly,

$$a_{ij}^* = \begin{cases} 1, & \text{if } e_{ij} \text{ has a positive sign,} \\ -1, & \text{if } e_{ij} \text{ has a negative sign,} \\ a, & \text{if } e_{ij} \text{ does not have any sign,} \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $A(G_a^*) = A(G_1) + aA(G_2)$, where G_1 is a subgraph of G_a^* that consists of only signed edges, and G_2 is a subgraph of G_a^* having unsigned edges. Also, $A(G_1)$ is the adjacency matrix of a signed graph with $\{-1, 0, 1\}$ components, and $A(G_2)$ is the adjacency matrix of a simple graph. The spectrum of this kind of graph is defined as $Sp(G_a^*) = (\lambda_1(G_a^*)_{m_1} \quad \lambda_2(G_a^*)_{m_2} \quad \dots \quad \lambda_k(G_a^*)_{m_k})$ where $\lambda_i(G_a^*)$ ($i = 1, 2, \dots, k$) is an eigenvalue of $A(G_a^*)$. As we know, since $A(G_a^*)$ is symmetric, the eigenvalues of G_a^* are real number.

Most concepts were defined for simple graphs, and signed graphs, such as subgraphs, connectedness, and perfect matching, will be defined for partially signed graphs [2].

A subgraph (H_a^*, δ^*) of a partially signed graph (G_a^*, σ^*) , is a graph whose vertex set and edge set are subsets, of $E(G_a^*)$ and $V(G_a^*)$, and its sign (signature) function δ^* is the restriction of the sign function σ^* to the subset of $E(G_a^*)$. In this situation, we say that (H_a^*, δ^*) is a partially subgraph of (G_a^*, σ^*) and is shown by $(H_a^*, \delta^*) \leq (G_a^*, \sigma^*)$.

The spanning subgraph of (G_a^*, σ^*) is a partially signed graph that has all the vertices that are present in (G_a^*, σ^*) . If v is a vertex of G_a^* , we write $G_a^* - v$ is a subgraph where all edges connected to v will be deleted. So $m(G_a^* - v) < m(G_a^*)$. Also, $G_a^* - e$ for the corresponding edge is a subgraph of G_a^* , which $m(G_a^* - e) < m(G_a^*)$ while $n(G_a^* - e) = n(G_a^*)$.

2. Preliminaries

Throughout this paper, the properties of partially signed graphs will be investigated. We used the standard notations for the partially signed graphs. For example, a partially signed path, a partially signed tree, or a partially signed cycle with n vertices were denoted by $(P_n)_a^*$, $(T_n)_a^*$, and $(C_n)_a^*$. For others, we have the following definition:

Definition 2.1. A connected partially signed graph with n vertices and n edges is called unicyclic partially signed graph and denoted by $(U_n)_a^*$ if it contains exactly one cycle. In other words, a unicyclic partially signed graph is a unicyclic graph which its edges have signed and unsigned edges.

Definition 2.2. A signed graph is called balanced if the product of the signs of edges around every cycle is positive. It means that, if a partially signed graph

has an even number of negative edges, it will be called balanced; otherwise, it is unbalanced. Also a similar definition is used for a partially signed cycle. This kind of graph with an even number of negative edges is said to be balanced, otherwise it is unbalanced.

The weight of a partially signed graph is $\omega(G_a^*) = \prod_{C_a^* \in G_a^*} \sigma(C_a^*) a^i$, where i is the number of unsigned edges of each cycle and $\sigma(C_a^*)$ is the sign of cycles. By this definition, the identity of a partially unicyclic balanced graph is a^i . A partially signed cycle has a positive weight if it contains an even number of negative edges and an even number of unsigned edges. Therefore, its weight would be $\omega(C_a^*) = a^i$. Now, we want to define the characteristic polynomial of a partially signed graph G_a^* of order n . If G_a^* is a partially signed graph of order n , the characteristic polynomial of this graph is defined as:

$$\Phi_a(G_a^*, \lambda) = \det(\lambda I - A) = \prod_{i=1}^n (\lambda - \lambda_i), \tag{1}$$

and also

$$\Phi_a(G_a^*, \lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1} \lambda + a_n. \tag{2}$$

As you can see the characteristic polynomial of partially signed graph depends on "a".

By using the Coates formula, the characteristic polynomial of a partially signed graph has the following coefficient generalization [3].

$$(-1)^i a_i = \sum_{H_a^*} (-1)^{r(H_a^*)} 2^{s(H_a^*)} \prod(H_a^*), \tag{3}$$

where H_a^* is a partially signed subgraph of G_a^* , spanned over i vertices, whose components are edges or cycles (of length at least 3); $r(H_a^*)$ is the number of components of H_a^* , $s(H_a^*)$ is the number of cycles in H_a^* , and

$$\prod(H_a^*) = \prod_{h \in E(H_a^*)} (w(h))^{f(h, H_a^*)}. \tag{4}$$

Here, $E(H_a^*)$ is the set of edges of H_a^* , $w(h)$ is the weight of the edge h ,

$$f(h; H_a^*) = \begin{cases} 1, & \text{if } h \text{ is contained in any circuit of } H_a^*, \\ 2, & \text{otherwise.} \end{cases} \tag{5}$$

With $i = n$, we can obtain a simple formula calculating the determinant of an arbitrary square matrix A_a considered as a generalized adjacency matrix of graph G_a^*

$$\det(A_a) = (-1)^n \sum_{h \in H} (-1)^{r(H_a^*)} 2^{s(H_a^*)} \prod(H_a^*). \tag{6}$$

So, by the mentioned formula, we can calculate the coefficients of the characteristic polynomial of a partially signed graph. Therefore, we have the following theorem:

Theorem 2.3. Suppose that G_a^* is the partially signed graph of order n vertices. The characteristic polynomial of a partially signed graph G_a^* can easily derive the following generalization $\Phi(G_a^*, \lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n$ and their coefficients are

$$(-1)^i a_i = \sum_{H_a^*} (-1)^{r(H_a^*)} 2^{s(H_a^*)} \sigma(H_c^*) a^{f(h, H_a^*)}.$$

Where H_a^* is a partially signed subgraph of G_a^* , spanned over i vertices, whose components are edges or cycles (of length at least 3), $r(H_a^*)$ is the number of components of H_a^* and $s(H_a^*)$ is the number of cycles. Also H_c^* is a signed subgraph of H containing only its cycles and

$$f(h, H_a^*) = \begin{cases} 1, & \text{if } h \text{ is contained in any circuit of } H_a^*, \\ 2, & \text{otherwise.} \end{cases} \tag{7}$$

By the mentioned formula, we could easily get the constant terms in the characteristic polynomial of some partially signed graphs, such as paths, trees and cycles.

Now we want to investigate the constant terms in the characteristic polynomials of some partially signed graphs.

Definition 2.4. Suppose (G_a^*, σ^*) is a partially signed graph. By reversing the sign of each edge from $U^* \subseteq V^*$ joining to a vertex in $V(G_a^*) \setminus U^*$ even the sign of a , we have switched the equivalence of G_a^* which is denoted by $(G_a^*)_{se}$. Clearly, the switching equivalence is an equivalence relation that has the same characteristic polynomial and preserves the eigenvalue.

For example, the following partially signed graph has a unique characteristic polynomial for different switching equivalences and different vertex sets.

Example 2.5. Suppose (G_a^*, σ^*) is the following partially signed graph in Figure 1 with its characteristic polynomial $\phi_a(G_a^*, \lambda) = \lambda^4 - (2a^2 + 2)\lambda^2 + 2a^2\lambda + a^2$. Now, we consider $U_1^* = \{v_2\}$, $U_2^* = \{v_1v_2\}$, $U_3^* = \{v_3\}$, $U_4^* = \{v_4\}$.

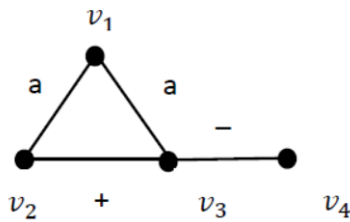


Figure 1: Partially signed graph G_a^* .

By changing the sign of unsigned and signed edges corresponding to a joining vertex in $(U_n)_a^*$, we observe that the polynomial of its characteristic has not changed. As you can see, for a various vertex sets switching equivalence is unique. Therefore, the characteristic polynomials of G_a^* and $(G_a^*)_{se}$ for each subset of the vertex set are equal.

Lemma 2.6. *Let n vertices partially unicyclic signed graph $(U_n)_a^*$, with unique partially signed cycle C_a^* , length $g = g((U_n)_a^*)$ and $s = \sigma(C_a^*)$. If U^* and $V(C_a^*)$ are the number of $(U_n)_a^*$ and C_a^* respectively, we have:*

1. *If n and g are odd then*

$$a_0((U_n)_a^*) = \begin{cases} 2(-1)^{\frac{1}{2}(n-g)+1}a^{2i+j}, & \text{if } U^* - V(C_a^*) \text{ has a perfect matching} \\ & \text{and cycle has some unsigned edges,} \\ 2(-1)^{\frac{1}{2}(n-g)}a^{2i+j}s, & \text{if } U^* - V(C_a^*) \text{ has a perfect matching} \\ & \text{and cycle has no any unsigned edges,} \\ 0, & \text{otherwise,} \end{cases}$$

where i is the number of unsigned edges in $U^* - V(C_a^*)$, and j is the number of unsigned edges in C_a^* .

2. *If n is odd and g is even, then $a_0((U_n)_a^*) = 0$.*

3. *If n is even and g is odd then*

$$a_0((U_n)_a^*) = \begin{cases} (-1)^{\frac{n}{2}}a^{2i}, & \text{if } U^* \text{ has a perfect matching,} \\ 0, & \text{otherwise,} \end{cases}$$

where i is the number of unsigned edges in $U^* - V(C_a^*)$.

4. *If n and g are even then*

$$a_0((U_n)_a^*) = \begin{cases} (-1)^{\frac{n}{2}}a^{2i}(1 - sa^j)^{\frac{g}{2}}, & \text{if } U^* \text{ and } U^* - V(C_a^*) \text{ has a perfect matching,} \\ (-1)^{\frac{n}{2}}a^{2j}, & \text{if } U^* \text{ has a perfect matching but } U^* - V(C_a^*) \\ & \text{does not have any perfect matching,} \\ 0, & \text{if } U^* \text{ does not have any perfect matching,} \end{cases}$$

where i is the number of unsigned edges in $U^* - V(C_a^*)$ and j is the number of unsigned edges in C_a^* .

Proof. The proof of this theorem is similar to the unicyclic signed graph theorem with a little change. By using the switching equivalence properties for any balanced unicyclic partially signed graph and its underlying partially signed graph, this theorem will be proven. The proof follows [4, Lemma 2.2] and [5, Theorem 2.4]. \square

Notation 2.7. If $(U_n)_a^*$ is unicyclic, then $a_0((U_n)_a^*) \neq 0$ only if $(U_n)_a^*$ has a perfect matching.

Example 2.8. Consider the following three Figures 2 to 4. As you can see, three different figures with 6 vertices and 4 vertices of cycle have different constant coefficients.

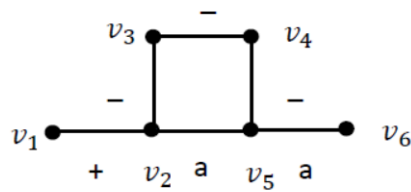


Figure 2: Partially unicyclic signed graph of order 6, $U^* - V(C_a^*)$ does not have any perfect matchings.

In Figure 2, $(U_n)_a^*$ has a perfect matching, while $U^* - V(C_a^*)$ does not have any perfect matching, the constant coefficient is $(-1)^{\frac{n}{2}} a^{2j} = -a^2$.

Figure 3 does not have any matching. So, its constant coefficient is zero.

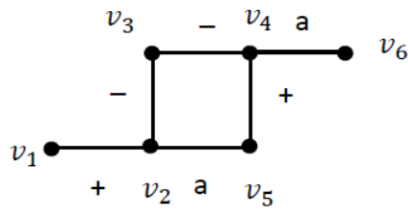


Figure 3: Partially unicyclic signed graph of order 6 without any perfect matchings.

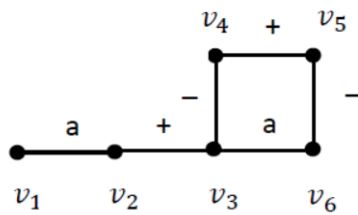


Figure 4: $(U_n)_a^*$ and $U^* - V(C_a^*)$ have two different perfect matchings.

In Figure 4, $(U_n)_a^*$ and $U^* - V(C_a^*)$ have two different perfect matchings. Therefore, $a_0 = (-1)^{\frac{n}{2}} a^{2i} (1 - sa^j)^{\frac{n}{2}} = -a^2(1 - a)^2$.

Definition 2.9. A bipartite graph, denoted by $G = (U, V, E)$, is a graph two disjoint, independent vertex sets U and V , where every edge connects a vertex in U to one in V .

If a bipartite graph has both signed and unsigned edges, then it is called a bipartite partially signed graph, and it will be shown $(B_n)_a^* = G_a^*(U^*, V^*, E)$. For a bipartite partially signed graph, similar to [5], we get the following theorem:

Theorem 2.10. *The spectrum of a bipartite partially signed graph is symmetric with respect to the origin.*

Proof. We know that every bipartite partially signed graph has the following adjacency matrix

$$A = \begin{pmatrix} 0 & (B_n)_a^{*T} \\ (B_n)_a^* & 0 \end{pmatrix}.$$

Let μ be an eigenvalue of A , and let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the corresponding eigenvector.

If $x' = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$, we obtain

$$Ax' = \begin{pmatrix} 0 & (B_n)_a^{*T} \\ (B_n)_a^* & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} -(B_n)_a^{*T} x_2 \\ (B_n)_a^* x_1 \end{pmatrix} = \begin{pmatrix} -\mu x_1 \\ x_2 \end{pmatrix} = -\mu x',$$

and the proof is completed. □

Note that the converse of the previous theorem is not true.

Example 2.11. Figure 5 has spectra symmetric with respect to the origin, while it is not a bipartite partially signed graph.

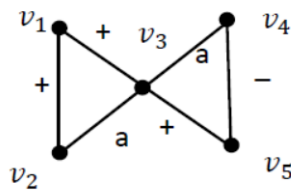


Figure 5: A partially signed graph that has a spectre symmetric with respect to the origin.

The spectra of this non-bipartite partially signed graph are symmetric with respect to the origin. It has one negative and two unsigned edges. Its eigenvalues are $0, 1, -1, (2a^2 + 3)^{\frac{1}{2}}, -(2a^2 + 3)^{\frac{1}{2}}$. However, this graph is not bipartite, and its eigenvalues are symmetric.

Corollary 2.12. *The spectrum of a unicyclic partially signed graph is symmetric if it is bipartite.*

Proof. In order to show that it is bipartite, we can prove only the converse. Suppose $(U_n)_a^*$ is not bipartite partially signed graph. So, it has an odd cycle and $g(U_n)_a^*$ is odd, and $a_{n-g}((U_n)_a^*) \neq 0$. If $n \geq 4$, in particular $a_{n-4}((U_n)_a^*) \neq 0$. So the spectrum of $(U_n)_a^*$ can not be symmetric with respect to the origin because its characteristic polynomial is neither odd nor even, and it is contradictory to the assumption. \square

The previous example shows a fact which is declared in the following corollary:

Corollary 2.13. *If the eigenvalues of a partially signed graph G_a^* are symmetric with respect to the origin, then the numbers of balanced and unbalanced partially signed cycles are equal in G_a^* .*

Proof. We know that $a_{n-g}(G_a^*)$ in the corresponding characteristic polynomial is zero, and it is enough to prove this statement. \square

In the previous example, you can see the balanced and unbalanced cycles in G_a^* . Also, we have $a_{5-3}(G_a^*) = 0$.

$$a_0 = 0, \quad a_1 = 2(a^2 + 3), \quad a_2 = 0, \quad a_3 = -(2a^2 - 4), \quad a_4 = 0, \quad a_5 = 1.$$

Theorem 2.14. *Let G_a^* be a bipartite partially signed graph with partially signed cycles. Then the odd coefficient of its characteristic polynomial, $a_{2i+1}(G_a^*)$, is equal to zero.*

Proof. An invariant of G_a^* is determined from $\phi(G_a^*)$. \square

Some results of this section have been derived from simple and signed graphs.

Lemma 2.15. *The number of signed and unsigned edges and the number of vertices of G_a^* can be determined from its $\phi(G_a^*)$.*

Proof. If q is the number of unsigned edges and r, s are the numbers of signed edges, then the number of unsigned edges is $-2q$ times of a^2 . Because

$$m = -2(qa^2 + r + s),$$

and the other half is the numbers of signed edges. The order of G_a^* , $m(G_a^*) = n+1$, where "n" is the degree of the characteristic polynomial. \square

The following lemma is available in a similar article [4].

Lemma 2.16. For $n > 2$, $\phi(G_a^*)$ can determine the collection of vertex degrees G_a^* .

We also had these two theorems are obtained from the Schwenk-Like formula for a directed and signed graph G and G^σ .

The first formula uses the Schwenk-Like formula for a directed graph.

Theorem 2.17. ([6]). Let v be a vertex of the directed graph G . Then

$$\phi(G, \lambda) = \lambda\phi(G-v, \lambda) - \sum_u \phi(G-v-u, \lambda) + \sum_w \phi(G-v-w, \lambda) - 2 \sum_c (-1)^{\sigma(G)} \phi(G-C, \lambda). \tag{8}$$

Where the first summation is taken over all vertices u adjacent to v in G , the second summation is taken over all vertices w are in the neighborhood of v such that (v, w) or (w, v) is an arc in G , and the third summation is taken over all cycles containing even number of arcs and the vertex v .

Also, the second formula is taken from the Schwenk-Like formula for a signed graph.

Theorem 2.18. ([4]). Let u be a vertex of a signed graph G^σ . Then

$$\phi_{G^\sigma}(\lambda) = \lambda\phi_{G^\sigma-u}(\lambda) - \sum_{u \sim v} \phi_{G^\sigma-u-v}(\lambda) - 2 \sum_{C^\sigma \in G^\sigma_u} \sigma(C^\sigma) \phi_{G^\sigma-C^\sigma}(\lambda). \tag{9}$$

Where G^σ_u denotes the following set of signed cycles pressing through u , and $G^\sigma - C^\sigma$ denotes the signed graph obtained from G^σ by deleting C^σ . We assume that $\phi_{G^\sigma-C^\sigma}(\lambda) = 1$ if $G^\sigma - C^\sigma$ has no vertices.

If u has degree 1 (u is a pendant vertex) and v is its neighbor, Equation (9) is reduced to

$$\phi_{G^\sigma}(\lambda) = \lambda\phi_{G^\sigma-u}(\lambda) - \sum_{u \sim v} \phi_{G^\sigma-u-v}(\lambda). \tag{10}$$

For a symmetric matrix, we have the following Schwenk-like formula for a weighted graph:

Theorem 2.19. ([7]). Let G be a directed graph, and v be a fixed vertex of G . Then

$$\phi(G) = (\lambda - a_{uv})\phi(G - V) - \sum a_{uv}^2 \phi(G - u - v) - 2 \sum_{C \in C_v} w_A(C) \phi(G - v(C)),$$

where uv is an arc, C_v is a set of all undirected cycles of G length ≥ 3 passing through v , while $w_A(C) = \prod_{ij \in e(C)} a_{ij}$.

Now we have a similar formula for partially signed graphs:

Theorem 2.20. Suppose that $\phi_{G_a^*}(\lambda) = \phi(G_a^*, \lambda)$ is a partially signed graph with vertex u . Therefore,

$$G_a^*(\lambda) = \lambda\phi_{G_a^*-u}(\lambda) - a^2 \sum_{u \sim v} \phi_{G_a^*-u-v}(\lambda) - \sum_{u \sim w} \phi_{G_a^*-u-w}(\lambda) - 2 \sum_{C^* \in C_u^*} w(C^*)\phi_{G_a^*-C^*}(\lambda). \quad (11)$$

Where v is the neighbor vertex of u with an unsigned edge, and w is the neighbor vertex of u with signed edges, C_u^* is a partially signed cycle pressing through u , and $G_a^* - C^*$ is a partially signed graph of G_a^* by omitting C^* . Also $\phi_{G_a^*-C^*}(\lambda) = 1$, if $G_a^* - C^*$ has no vertices.

If u is a pendant vertex of degree one and v is its neighbor with signed edges, Equation (11) is reduced to

$$\phi_{G_a^*}(\lambda) = \lambda\phi_{G_a^*-u}(\lambda) - \phi_{G_a^*-u-v}(\lambda). \quad (12)$$

Also, if u is a pendant vertex of degree one and v is its neighbor with an unsigned edge, Equation (11) is reduced to

$$\phi_{G_a^*}(\lambda) = \lambda\phi_{G_a^*-u} - a^2\phi_{G_a^*-u-v}(\lambda). \quad (13)$$

Proof. It is completely similar to Theorem 3.2 of [7]. \square

In order to write the characteristic polynomial by edge-deleting, the following theorem will be declared

Theorem 2.21. (i) Let $e = uv$ be an unsigned edge in G_a^* . Then

$$\phi_{G_a^*}(\lambda) = \phi_{G_a^*-e}(\lambda) - a^2\phi_{G_a^*-u-v}(\lambda) - 2 \sum_{C^* \in C_u^*} w(C^*)\phi_{G_a^*-C^*}(\lambda). \quad (14)$$

(ii) Let $e = uv$ be a signed edge in G^* . Then

$$\phi_{G_a^*}(\lambda) = \phi_{G_a^*-e}(\lambda) - \phi_{G_a^*-u-v}(\lambda) - 2 \sum_{C^* \in C_u^*} w(C^*)\phi_{G_a^*-C^*}(\lambda). \quad (15)$$

Where the summation is taken over all partially signed cycles C^* containing e .

We also have the following theorem:

Theorem 2.22. Suppose G_a^* and H_a^* are two partially signed graphs and $G_a^*uvH_a^*$ is a partially signed graph of adjoining two G_a^* and H_a^* by an edge uv which $u \in V_{G_a^*}$ and $v \in V_{H_a^*}$. Then

(i) If uv is a signed bridge, then

$$\phi(G_a^*uvH_a^*) = \phi(G_a^*)\phi(H_a^*) - \phi(G_a^* - u)\phi(H_a^* - v). \quad (16)$$

(ii) If uv is an unsigned bridge, then

$$\phi(G_a^*uvH_a^*) = \phi(G_a^*)\phi(H_a^*) - a^2\phi(G_a^* - u)\phi(H_a^* - v). \quad (17)$$

3. The Estrada index in a partially signed graph

In chemical graph theory, the Estrada index is a topological index of protein folding, the number of possible conformations that a protein can adopt is enormous. For instance, a protein of length 100 can take a total of about conformation [8–10] characterization of 3D molecular structure.

Definition 3.1. A walk of length k in a partially signed graph G_a^* from v_0 to v_k is a sequence of the form $v_0e_1v_1e_2v_2 \dots v_{k-1}e_kv_k$ of vertices and edges such that for any $i = 0, 1, 2, \dots, k - 1$ and $v_{i-1}v_i = e_i$. $w_k(ij)$ is the number of all walks of length k starting from v_i and ending at v_j .

Theorem 3.2. ([11]). Let A_a be the adjacency matrix of a partially signed graph G_a^* and v_i and v_j be two vertices of G_a^* . Then $[A_a^k]_{ij} = w_k(ij)$.

Theorem 3.3. ([3]). Let A_a be the adjacency matrix of a graph G_a^* . Then the k -spectral moment of G_a^* is equal to $\text{tr}(A_a^k) = \sum_{i=1}^k w_k(ii)$.

Let G_a^* be a partially signed graph of order n . Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues. The Estrada index of G_a^* is defined as the trace of the adjacency matrix of the exponential and was proposed by Estrada in 2000 for the first time. It means $EE(G_a^*) = EE(G, \sigma^*) = \sum_{i=1}^n e^{\lambda_i}$, λ_i 's are the eigenvalues of G_a^* [12]. For a non-negative integer k , let denote $M_k(G_a^*) = \sum_{i=1}^n \lambda_i^k$ the k -spectral moment of G_a^* . From the Taylor expansion of e^{λ_i} , $EE(G_a^*)$ can be written as:

$$EE(G_a^*) = \sum_{k=0}^{\infty} \frac{M_k(G_a^*)}{k!}. \tag{18}$$

It is well known that $M_k(G_a^*)$ is equal to the difference of the number of positive, negative, and unsigned closed walks of length k in G_a^* . In particular we have,

$$M_0(G_a^*) = n, \quad M_1(G_a^*) = 0, \quad M_2(G_a^*) = 2(qa^2 + r + s), \quad M_3(G_a^*) = 6 \sum_{k=1}^n a^{i_k} (-1)^{j_k},$$

where i_k is the number of unsigned edges in the k -triangle and j_k is the number of negative edges in each k -triangle.

Calculating the Estrada index of a partially signed star of order n is a good example.

Example 3.4. Suppose that partially signed star of order n , $(S_n)_a^*$, with q number of unsigned edges and r, s number of signed edges has the following spectrum:

$$\text{spec}((S_n)_a^*) = \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 1 & n - 2 & 1 \end{pmatrix},$$

where $\lambda_1 = \sqrt{qa^2 + r + s}$ and $\lambda_2 = -\sqrt{qa^2 + r + s}$. The Estrada index of $(S_n)_a^*$ is

$$EE((S_n)_a^*) = \sum_{i=1}^n e^{\lambda_i} = n - 2 + e^{\sqrt{qa^2 + r + s}} + e^{-\sqrt{qa^2 + r + s}} = n - 2 + 2 \cosh(\sqrt{qa^2 + r + s}). \tag{19}$$

Finding upper and lower bounds for Estrada index based on the number of vertices and edges of partially signed graph is our goal.

Theorem 3.5. *Let G_a^* be a partially signed graph of order n and size m . Then*

$$EE(G_a^*) \geq n + (aq^2 + r + s) + \sum_{k=2}^{\infty} \frac{\binom{2k}{n}^{2k}}{(2k)!}, \tag{20}$$

with equality if and only if G_a^* is an edgeless graph of order n .

Proof. Since $EE(G_a^*) = \sum_{k=0}^{\infty} \frac{M_k(G_a^*)}{k!}$ and $EE(G_a^*) \geq \sum_{k=0}^{\infty} \frac{M_{2k}(G_a^*)}{2k!}$, we have

$$EE(G_a^*) \geq \sum_{k=0}^{\infty} \frac{M_{2k}(G_a^*)}{2k!} = \frac{n}{0!} + \frac{0}{1!} + \frac{2(aq^2 + r + s)}{2!} + \sum_{k=3}^{\infty} \frac{M_{2k}(G_a^*)}{2k!}.$$

The equality will hold if and only if G_a^* is bipartite.

For $k \geq 2$, $M_{2k}(G_a^*) > \lambda_{\max}^{2k}$ with equality if and only if G_a^* is an edgeless partially signed graph, but we know that $\lambda_{\max}^{2k} \geq \left(\frac{2m}{n}\right)^{2k}$ then $M_{2k}(G_a^*) > \lambda_{\max}^{2k} \geq \left(\frac{2m}{n}\right)^{2k}$ and the equality holds if and only if G_a^* is edgeless graph. So, $EE(G_a^*) \geq \sum_{k=0}^{\infty} \frac{M_{2k}(G_a^*)}{2k!} = \frac{n}{0!} + \frac{2(aq^2 + r + s)}{2!}$. \square

Similar to the results given in [8], we have the following theorem:

Theorem 3.6. *Let G_a^* be a partially signed graph with n vertices, m signed edges and q unsigned edges. Then,*

$$e_{\min} \leq EE(G_a^*) \leq e_{\max},$$

with equality in left and right holds, if and only if G_a^* equals k_2 and k_n , respectively.

Proof. First, we find the extremum of the function $EE(G_a^*) = \sum_{i=1}^n e^{\lambda_i}$ with respect to the two conditions $\sum_{i=1}^n \lambda_i = 0$ and $\sum_{i=1}^n \lambda_i^2 = 2(qa^2 + r + s)$. So, by the method of Lagrange multipliers with two constraints, we consider the function $f(\lambda_1, \lambda_2) = \alpha g(\lambda_1, \lambda_2) + \beta h(\lambda_1, \lambda_2)$ such that $g(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$, $h(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2$, $\alpha \neq 0$ and $\beta \neq 0$. Therefore, we have

$$e^{\lambda_i} = \alpha + 2\beta\lambda_i, \quad i = 1, 2.$$

It is clear that for n_1 and n_2 such that $n_1 + n_2 = n$, this equation has at most two solutions λ_1 and λ_2 . Thus

$$n_1\lambda_1 + n_2\lambda_2 = 0, \quad n_1\lambda_1^2 + n_2\lambda_2^2 = 2(qa^2 + r + s).$$

But from the first equation, we find that $\lambda_2 = -\frac{n_1}{n_2}\lambda_1$.

$$n_1\lambda_1^2 + n_2\left(\frac{-n_1}{n_2}\right)^2\lambda_1^2 = 2(qa^2 + r + s).$$

Therefore, we have

$$\begin{aligned} \lambda_1^2\left(n_1 + \left(\frac{n_1^2}{n_2^2}n_2\right)\right) &= 2(qa^2 + r + s), \\ \lambda_1^2 &= \frac{2(qa^2 + r + s)n_2}{n_1(n_1 + n_2)}, \quad \lambda_2 = -\frac{n_1}{n_2}\lambda_1, \\ \lambda_1 &= \sqrt{\frac{2(qa^2 + r + s)n_2}{nn_1}}, \quad \lambda_2 = -\sqrt{\frac{2n_1(qa^2 + r + s)}{nn_2}}. \end{aligned}$$

This means that $f(\lambda_1, \lambda_2, \dots, \lambda_n) = n_1e^{\lambda_1} + n_2e^{\lambda_2} = n_1e^{\lambda_1} + (n - n_1)e^{\lambda_2}$. Now, by substituting 1 to $n - 1$ instead of n_1 , we have

$$e_{\max} = (n - 1)e^{-\sqrt{\frac{2(qa^2+r+s)}{n(n-1)}}} + e^{\sqrt{\frac{2(qa^2+r+s)(n-1)}{n}}},$$

and

$$e_{\min} = (n - 1)e^{\sqrt{\frac{2(qa^2+r+s)}{n(n-1)}}} + e^{-\sqrt{\frac{2(qa^2+r+s)(n-1)}{n}}}.$$

□

4. Conclusion

In this article, we defined partially signed graphs. We also discussed some special features, including calculating the last coefficient of the characteristic polynomial of two important types of partially signed graphs, namely unicyclic partially signed graphs and bipartite partially signed graphs. In another article we had discussed the initial definition and examining some other features of these graphs. Rewriting characteristic polynomials, whether of vertex removal or edge removal, was another important feature of this article. We also examined the energy and Estrada index as an important and key strategy in the category of features of this type of graph, and calculated the upper and lower bounds for this index. The question that arises is whether all the features of the Estrada index of simple graphs will be inherited by this group of graphs? For example, the Estrada index of a star is greater than the Estrada index of a tree, and this index is also greater than the Estrada index of a path. Can such an inequality be written for the Estrada index of the partially signed graphs of trees, stars, and paths? This means, can we express the following inequality?

$$EE((P_n)_a^*) \leq EE((T_n)_a^*) \leq EE((S_n)_a^*).$$

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

- [1] E. Ghasemian and G. H. Fath-Tabar, On signed graphs with two distinct eigenvalues, *Filomat* **31** (2017) 6393 – 6400, <https://doi.org/10.2298/FIL1720393G>.
- [2] R. Balakrishnan and K. Ranganathan, *A textbook of graph theory*, Springer, New York, NY, 2012.
- [3] D. M. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs. Theory and Application*, Huthig Pub Ltd, 1994.
- [4] S. K. Simić and Z. Stanić, Polynomial reconstruction of signed graphs, *Linear Algebra Appl.* **501** (2016) 390 – 408, <https://doi.org/10.1016/j.laa.2016.03.036>.
- [5] S. K. Simić and Z. Stanić, The polynomial reconstruction of unicyclic graphs is unique, *Linear Multilinear Algebra* **55** (2007) 35 – 43, <https://doi.org/10.1080/03081080500426133>.
- [6] C. Adiga, B. R. Rakshith and W. So, On the mixed adjacency matrix of a mixed graph, *Linear Algebra Appl.* **495** (2016) 223 – 241, <https://doi.org/10.1016/j.laa.2016.01.033>.
- [7] F. Belardo, E. M. Li Marzi and S. K. Simić, Combinatorial approach for computing the characteristic polynomial of a matrix, *Linear Algebra Appl.* **433** (2010) 1513 – 1523, <https://doi.org/10.1016/j.laa.2010.05.010>.
- [8] M. Arabzadeh, G. H. Fath-Tabar, H. Rasoli and A. Tehranian, Estrada and L-Estrada indices of a graph and their relationship with the number of spanning trees, *MATCH Commun. Math. Comput. Chem* **90** (2023) 787 – 798, <https://doi.org/10.46793/match.90-3.787A>.
- [9] G. H. Fath-Tabar and A. R. Ashrafi, New upper bounds for Estrada index of bipartite graphs, *Linear Algebra Appl.* **435** (2011) 2607 – 2611, <https://doi.org/10.1016/j.laa.2011.01.034>.
- [10] M. A. Iranmanesh and R. Nejati, The minimum Estrada index of spiro compounds with k quadrangles, *Iranian J. Math. Chem.* **10** (2019) 241 – 249, <https://doi.org/10.22052/IJMC.2019.149094.1392>.
- [11] N. Biggs, *Algebraic Graph Theory*, Cambridge university press, 1974.

- [12] T. Shamsheer, S. Pirzada and M. A. Bhat, On the Estrada index of unicyclic and bicyclic signed graphs, arXiv preprint arXiv:2309.13252 (2023).

Sheila Razavi
Department of Pure Mathematics,
Faculty of Mathematical Sciences, University of Kashan,
Kashan, 87317-53153 Iran
e-mail: Sheila.razavi@yahoo.com

Gholam Hossein Fath-Tabar
Department of Pure Mathematics,
Faculty of Mathematical Sciences, University of Kashan,
Kashan, 87317-53153 Iran
e-mail: Fathtabar@kashanu.ac.ir