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Generalized Recurrent and ψ-Recurrent Curvature on Mixed 3-Sasakian Manifolds

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Abstract

Considering the importance of mixed 3-Sasakian manifolds which admit Einstein metrics, we introduce generalized mixed 3-ψ-recurrent manifolds on mixed 3-Sasakian structures. We prove that a non-flat generalized mixed 3-ψ-recurrent manifold is a generalized recurrent manifold with its horizontal vector fields. Also, we obtain a relation between associated 1-forms γ_i's and θ_i's for a generalized mixed 3-ψ-recurrent manifold. Moreover, we give a necessary and sufficient condition for a mixed 3-Sasakian manifold to be a generalized mixed 3-ψ-recurrent. Next, we find the Riemannian curvature representation of a mixed 3-Sasakian manifold when that is a generalized mixed 3-ψ-recurrent.

Keywords: Mixed 3-structures, Recurrent manifold, Generalized 3-ψ-recurrent manifold, Einstein manifold.

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Corrected Proof

1. Introduction

An Einstein manifold plays an important role in Riemannian geometry as well as in general relativity theory. This type of manifold holds immense significance in the mathematical formulation of general relativity, cosmology, black holes, and string theory. The critical point of the functional $f(g) = \int_M s_g \; \eta_g$ leads to an Einstein metric. Einstein's general relativity is a theory of gravitation that describes how matter and energy influence the curvature of spacetime, which we perceive as gravity. Einstein proposed that a general relativity equation takes the following form:

$$\mathcal{R}c(B, A) - \frac{1}{2} sg(B, A) + \alpha g(B, A) = \frac{8\pi G}{c^4} \mathcal{T}(B, A), \tag{1}$$

in which $\mathcal{R}c$ is the Ricci tensor, s is the Ricci scalar (trace of the Ricci), g is the metric tensor, α is the cosmological constant, G is the gravitational constant, $\mathcal{T}$ is the stress-energy tensor, and c is the speed of light [1]. If $\mathcal{T} = 0$ holds in Equation (1), then the manifold reduces to be an Einstein manifold ($\mathcal{R}c = \alpha g$). The critical value for the Einstein manifold is the cosmological constant α, which represents the vacuum energy density in cosmological models. It can be positive, zero, or negative depending on the type of spacetime. $\alpha > 0, \alpha < 0, \alpha = 0$, corresponding to a de Sitter, anti-de Sitter and flat space, respectively.

On the other hand, the Riemannian curvature tensor is a powerful tool for describing how a geometric object, such as a curve deviates from a straight line or a surface from a flat plane. It has a significant impact on both differential geometry and great significance in different parts of physics, such as the theories of general relativity and gravity, as well as the curvature of spacetime.

Historically, flat manifolds, such as Euclidean spaces with a vanishing Riemannian curvature tensor, were studied by authors that were a special case of locally symmetric manifolds. Then, as a generalization, the locally symmetric manifolds were introduced, where their curvature tensor $\mathcal{R}$ satisfies $\nabla\mathcal{R} = 0$, in which ∇ is a Levi-Civita connection [2]. For example, the locally symmetric contact metric manifold (M, ζ, μ, ψ) with dimension 3 or 5, is either Sasakian or has constant curvature 1 or locally isometric to the unit tangent sphere bundle of Euclidean space [3]. So these types of manifolds are too strong for contact manifolds. Then locally ψ-symmetric, ψ-recurrent and generalized ψ-recurrent Sasakian manifolds were investigated by many authors and proved that such manifolds are always Einstein [4–8]. Also, 3-dimensional locally generalized ψ-recurrent Sasakian manifolds are spaces of constant curvature [9]. Moreover, mixed 3-ψ-recurrent manifolds were introduced by Kazemi and Raei, $\psi_i^2(\nabla\mathcal{R}) = \gamma_i\mathcal{R}$, in [10]. They proved every 3-ψ-recurrent manifold is recurrent. Also, they showed every 3-ψ-recurrent mixed 3-Sasakian manifold is locally ψ-symmetric and locally symmetric. Recently, we introduced statistical generalized recurrent manifolds, and we proved that, despite the Riemannian manifold, a statistical generalized recurrent manifold is not statistical concircular recurrent [11]. In 1970, almost contact 3-structures and 3-Sasakian manifolds were defined by Kuo [12]. Every 3-contact metric manifold is

Corrected Proof

necessarily a 3-Sasakian manifold [13]. Every 3-Sasakian manifold is of constant curvature. In 2024, Kazemi and his collaborators defined 3-Sasakian statistical manifolds and they proved that invariant statistical submersions from 3-Sasakian statistical manifolds with vertical structure vector fields have 3-Sasakian statistical totally geodesic fibers [14, 15]. Also, mixed 3-structure manifolds and mixed 3-Sasakian manifolds were introduced by Ianus and Caldarella [16, 17]. Every mixed 3-Sasakian manifold is Einstein. So, mixed 3-Sasakian structures hold immense significance not only in the field of differential geometry but also in different parts of physics, such as supersymmetry, supergravity, string theory and geometric flow equations.

Considering the importance of mixed 3-Sasakian manifolds, which admit Einstein metrics as mentioned above, in this article, we introduce generalized mixed 3-ψ-recurrent structures on mixed 3-structure manifolds and then investigate some results of these types of manifolds.

This paper is organized as follows. In Section 2, we review some basic information about mixed 3-structures and mixed 3-Sasakian manifolds. In Section 3, we introduce generalized mixed 3-ψ-recurrent manifolds and prove that such manifolds are generalized recurrent for their horizontal vector fields. In section 4, we obtain a relation between the associated 1-forms γ_i and θ_i for a generalized mixed 3-ψ-recurrent manifold.

Corrected Proof

2. Preliminaries

Throughout this paper, we denote the set of vector fields on M by $T(M)$. In this section, we review some information about mixed 3-structure manifolds in the context of [18, 19].

A $(2m+1)$-dimensional semi-Riemannian manifold (M, ζ, μ, ψ) is said to be an almost contact manifold, if it admits a vector field ζ, a 1-form μ and a (1,1)-tensor field ψ where

$$\psi^2 B = \epsilon(-B + \mu(B)\zeta), \ \mu(\zeta) = 1, \quad \forall\ B \in T(M), \tag{2}$$

for $\epsilon = 1$, and it is said to be an almost para-contact manifold, for $\epsilon = -1$.

Definition 2.1. ([10]). A semi–Riemannian $(4m + 3)$–dimensional manifold $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ is said to be a mixed 3-structure manifold, if it admits two almost para-contact structures (ζ_i, μ_i, ψ_i), $i = 1, 2$, and an almost contact structure (ζ_3, μ_3, ψ_3), where for $i \neq j$,

$$\mu_i(\zeta_j) = 0, \quad \psi_i(\zeta_j) = \epsilon_j \zeta_k, \ \ \psi_j(\zeta_i) = -\epsilon_i \zeta_k, \quad \mu_i(\psi_j) = -\mu_j(\psi_i) = \epsilon_k \mu_k, \tag{3}$$

$$\psi_i \circ \psi_j - \epsilon_i \mu_j \otimes \zeta_i = -\psi_j \circ \psi_i + \epsilon_j \mu_i \otimes \zeta_j = \epsilon_k \psi_k, \tag{4}$$

in which (i, j, k) permutes over $\{1, 2, 3\}$.

$(M, \zeta_i, \mu_i, \psi_i, g)_{i\in\{1,2,3\}}$ is said to be a metric mixed 3-structure manifold, if there exist a semi-Riemannian metric g on M such that

$$g(\psi_i B, \psi_i A) = \epsilon_i[g(B, A) - \tau_i \mu_i(B, A)], \quad \forall\ A, B \in T(M), \tag{5}$$

in which $\tau_i = g(\zeta_i, \zeta_i) = \pm 1$. From Equation (5) we have

$$g(\psi_i B, A) = -g(B, \psi_i A). \tag{6}$$

Definition 2.2. ([18]). A semi-Riemannian manifold that admits two metric para-Sasakian structures $(\zeta_i, \mu_i, \psi_i, g)$, $i = 1, 2$, and a metric Sasakian structure $(\zeta_3, \mu_3, \psi_3, g)$, is said to be a mixed 3-Sasakian manifold if

$$(\nabla_E \psi_i)A = \epsilon_i[g(E, A)\zeta_i - \tau_i \mu_i(A)E], \tag{7}$$

for all $A, E \in T(M)$, $i \in \{1, 2, 3\}$, and $\tau_1 = \tau_2 = -1 = -\tau_3$.

Let $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ be a $(4m+3)$-dimensional mixed 3-structure Sasakian manifold. Then the following relations hold:

$$\nabla_E \zeta_i = -\tau_i \psi_i E, \tag{8}$$

$$\mu_i(B) = \tau_i g(B, \zeta_i), \tag{9}$$

$$(\nabla_E \mu_i)(A) = g(E, \psi_i A), \tag{10}$$

$$\mathcal{R}(B, A)\zeta_i = \epsilon_i [\mu_i(A)B - \mu_i(B)A], \tag{11}$$

$$\mu_i(\mathcal{R}(B, A)Z) = \tau_i \epsilon_i [\mu_i(B)g(A, Z) - \mu_i(A)g(B, Z)]. \tag{12}$$

If the Ricci tensor of a manifold satisfies $\mathcal{R}c(B, A) = \alpha g(B, A)$, for a scalar function α, then it is said to be an Einstein manifold.

Lemma 2.3. *([20]).(Contract Bianchi identity) The covariant derivatives of the Ricci and scalar curvatures satisfy $div(\mathcal{R}c) = \frac{1}{2}\nabla s$, where div is the divergence operator,*

$$div(\mathcal{R}c) = \sum_{j=1}^{4m+3} g(\nabla_{e_j}\mathcal{R}c, e_j).$$

3. Generalized ψ-recurrent mixed 3-structures

In this section, we introduce generalized mixed 3-ψ-recurrent manifolds.

Definition 3.1. We say a metric mixed 3-structure almost contact manifold $(M, \zeta_i, \mu_i, \psi_i, g)$ for $i \in \{1, 2, 3\}$ is generalized mixed 3-ψ-recurrent, if

$$\psi_i^2(\nabla_E \mathcal{R})(B, A, Z) = \gamma_i(E)\mathcal{R}(B, A)Z + \theta_i(E)[g(A, Z)B - g(B, Z)A], \tag{13}$$

Corrected Proof

for all $A, B, E, Z \in T(M)$. Here, γ_i's and θ_i's are nowhere vanishing unique 1-forms such that

$$\gamma(E) = g(E, \rho_1), \qquad \theta(E) = g(E, \rho_2), \tag{14}$$

in which ρ_1 and ρ_2 are vector fields associated with 1-forms γ and θ, respectively. In particular, if $\gamma_i = \theta_i = 0$, then the manifold is locally mixed 3-ψ-symmetric.

Lemma 3.2. *([10]). Let $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ be a metric mixed 3-structure manifold. Then*

$$\psi_i^2 \circ \psi_j^2 = -\epsilon_k[\epsilon_i\psi_i^2 + \mu_j \otimes \zeta_j] = -\epsilon_k[\epsilon_j\psi_j^2 + \mu_i \otimes \zeta_i],$$

where $-\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ and (i, j, k) is an even permutation of $\{1, 2, 3\}$.

Theorem 3.3. *Let $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ be a non-flat generalized mixed 3-ψ-recurrent manifold. Then relations*

$$\gamma_j(E) = \epsilon_k\gamma_i(E), \qquad and \qquad \theta_j(E) = \epsilon_k\theta_i(E),$$

hold for all horizontal vector fields A, B, Z, that is, for all vector fields A, B, Z which are orthogonal to ζ_i, and $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$.

Corrected Proof

Proof. From (13), it follows

$$\psi_j^2(\nabla_E\mathcal{R})(B, A, Z) = \gamma_j(E)\mathcal{R}(B, A)Z + \theta_j(E)[g(A, Z)B - g(B, Z)A]. \tag{15}$$

By applying ψ_i^2 on both sides of (15), using Lemma 3.2 and Equation (2), we obtain

$$-\epsilon_k\left[\epsilon_i\psi_i^2\left(\nabla_E\mathcal{R}\right)(B, A, Z) + \mu_j((\nabla_E\mathcal{R})(B, A, Z))\zeta_j\right]$$
$$= \epsilon_i\gamma_j(E)\left[-\mathcal{R}(B, A)Z + \mu_i\left(\mathcal{R}(B, A)Z\right)\zeta_i\right]$$
$$+\epsilon_i\theta_j(E)\left[g(B, Z)A - g(A, Z)B\right].$$

So from (13) and direct computation, we get:

$$[\epsilon_i\gamma_j(E) - \epsilon_k\epsilon_i\gamma_i(E)]\,\mathcal{R}(B, A)Z$$
$$+[\epsilon_i\theta_j(E) - \epsilon_k\epsilon_i\theta_i(E)]\,[g(A, Z)B - g(B, Z)A]$$
$$= \epsilon_k\mu_j\left((\nabla_E\mathcal{R})(B, A, Z)\right)\zeta_j + \epsilon_i\gamma_j(E)\,\mu_i\left(\mathcal{R}(B, A)Z\right)\zeta_i. \tag{16}$$

By applying μ_k on both sides of (16), we obtain:

$$[\epsilon_i\gamma_j(E) - \epsilon_k\epsilon_i\gamma_i(E)]\,\mu_k(\mathcal{R}(B, A)Z) = 0, \tag{17}$$

for all horizontal vector fields A, B. Since M is non-flat, Equation (17) implies that

$$\gamma_j(E) = \epsilon_k \gamma_i(E). \tag{18}$$

By replacing (18) in (16) and applying ψ_j and then ψ_k on both sides of (16) and considering Equation (4), we obtain:

$$[\epsilon_i \theta_j(E) - \epsilon_k \epsilon_i \theta_i(E)] [\epsilon_i g(B, Z) \psi_i(A) - \epsilon_i g(\mathcal{A}, Z) \psi_i(B)] = 0. \tag{19}$$

By applying ψ_i on (19) and considering Equation (2), we get:

$$[\epsilon_i \theta_j(E) - \epsilon_k \epsilon_i \theta_i(E)] [g(A, Z)B - g(B, Z) A] = 0. \tag{20}$$

Since M is generalized mixed 3-ψ-recurrent, so Equation (20) implies that $g(A, Z)B - g(B, Z) A \neq 0$. Therefore $\theta_j(E) = \epsilon_k \theta_i(E)$. □

Theorem 3.4. *A non-flat generalized mixed 3-ψ-recurrent manifold is a generalized recurrent manifold for all horizontal vector fields A, B, Z.*

Proof. Let M be a generalized mixed 3-ψ-recurrent manifold. It follows from Theorem 3.3,

$$\gamma(E) = \gamma_1(E) = \gamma_2(E) = -\gamma_3(E), \qquad \theta(E) = \theta_1(E) = \theta_2(E) = -\theta_3(E).$$

By virtue of Equations (2) and (13), we get

$$\begin{aligned} -(\nabla_E \mathcal{R})(B, A, Z) + \mu_i((\nabla_E \mathcal{R})(B, A, Z)) \zeta_i &= \gamma(E) \mathcal{R}(B, A) Z \\ +\theta(E) [g(A, Z) B - g(B, Z) A], & \end{aligned} \tag{21}$$

for $i = 1, 2, 3$. By applying μ_j and μ_i on both sides of (21), we obtain:

$$\mu_j((\nabla_E \mathcal{R})(B, A, Z)) = -\gamma(E) \mu_j(\mathcal{R}(B, A) Z), \tag{22}$$

and

$$\gamma(E) \mu_i(\mathcal{R}(B, A) Z) = 0, \tag{23}$$

respectively. By virtue of (21), Equations (22) and (23) show that

$$(\nabla_E \mathcal{R})(B, A, Z) = -\gamma(E) \mathcal{R}(B, A) Z - \theta(E) [g(A, Z) B - g(B, Z) A].$$

Therefore, M is a generalized recurrent manifold. □

Corrected Proof

4. Generalized 3-ψ-recurrent Sasakian manifolds

Let $(M, \zeta_i, \mu_i, \psi_i, g)_{i\in\{1,2,3\}}$ be a $(4m+3)$-dimensional generalized mixed 3-ψ-recurrent almost contact manifold. Then, by virtue of (2), Equation (13) yields

$$\begin{aligned}(\nabla_E \mathcal{R})(B,A,Z) &= \mu_i((\nabla_E \mathcal{R})(B,A,Z))\zeta_i \\ &- \gamma(E)\mathcal{R}(B,A)Z - \theta(E)[g(A,Z)B - g(B,Z)A]. \quad (24)\end{aligned}$$

So, from (24)

$$\begin{aligned}g\left((\nabla_E \mathcal{R})(B,A,Z),F\right) &= \mu_i((\nabla_E \mathcal{R})(B,A,Z))\mu_i(F) - \gamma(E)g(\mathcal{R}(B,A)Z,F) \\ &- \theta(E)[g(A,Z)g(B,F) - g(B,Z)g(A,F)]. \quad (25)\end{aligned}$$

Lemma 4.1. *Let $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ be a generalized mixed 3-ψ-recurrent mixed 3-Sasakian manifold. Then for any vector field E, we have:*

$$\tau_i\epsilon_i\gamma(E) + \theta(E) = \{\tau_i\mu_i(\rho_1) + \epsilon_i\mu_i(\rho_2)\}\mu_i(E). \quad (26)$$

Proof. From (24) and the Bianchi identity, we obtain:

$$\begin{aligned}\gamma(E)\mu_i(\mathcal{R}(B,A)Z) &+ \gamma(B)\mu_i(\mathcal{R}(A,E)Z) + \gamma(A)\mu_i(\mathcal{R}(E,B)Z) \\ &+ \theta(E)[g(A,Z)\mu_i(B) - g(B,Z)\mu_i(A)] \\ &+ \theta(B)[g(E,Z)\mu_i(A) - g(A,Z)\mu_i(E)] \\ &+ \theta(A)[g(B,Z)\mu_i(E) - g(E,Z)\mu_i(B)] = 0.\end{aligned}$$

By virtue of (12), we get:

$$\begin{aligned}&\{\tau_i\epsilon_i\gamma(E) + \theta(E)\}[g(A,Z)\mu_i(B) - g(B,Z)\mu_i(A)] \\ + &\{\tau_i\epsilon_i\gamma(B) + \theta(B)\}[g(E,Z)\mu_i(A) - g(A,Z)\mu_i(E)] \\ + &\{\tau_i\epsilon_i\gamma(A) + \theta(A)\}[g(B,Z)\mu_i(E) - g(E,Z)\mu_i(B)] = 0. \quad (27)\end{aligned}$$

By putting $A = Z = \{e_i\}$ in (27), where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold, and taking summation over i, $1 \le i \le 4m+3$, we obtain:

$$\{\tau_i\epsilon_i\gamma(E) + \theta(E)\}\mu_i(B) = \{\tau_i\epsilon_i\gamma(B) + \theta(B)\}\mu_i(E), \quad \forall\ B, E \in T(M). \quad (28)$$

By replacing B by ζ_i in (28), we get (26). □

Corrected Proof

Theorem 4.2. *([10]). A $(4m+3)$-dimensional manifold with mixed 3-Sasakian structures is an Einstein manifold.*

Theorem 4.3. *Let $(M, \zeta_i, \mu_i, \psi_i)_{i\in\{1,2,3\}}$ be a generalized mixed 3-ψ-recurrent Sasakian manifold. Then*

$$\gamma(E) = -\alpha\{4m+2\}\theta(E), \tag{29}$$

where α is a scalar function.

Proof. Since M is a mixed 3-structure Sasakian manifold, from Theorem 4.2, it follows

$$\mathcal{R}c(A,Z) = \alpha g(A,Z), \tag{30}$$

in which $\mathcal{R}c$ is the Ricci tensor of M. Taking contraction over A and Z of Equation (30), we get $s = \alpha(4m+3)$, where s is the scalar curvature of M. In account of Lemma 2.3, by contracting the Bianchi identity, we get $\frac{1}{2}\nabla s = \frac{1}{4m+3}\nabla s$. Hence, for $m \geq 0$, s is constant. Therefore, α is constant, and from (30), we conclude

$$(\nabla_E \mathcal{R}c)(A,Z) = \alpha(\nabla_E g)(A,Z) = 0, \ \forall \ A, E, Z \in T(M). \tag{31}$$

Since M is a generalized 3-ψ-recurrent manifold, in view of Theorem 3.4, we have

$$(\nabla_E \mathcal{R})(B,A,Z) = -\gamma(E)\mathcal{R}(B,A)Z - \theta(E)\left[g(A,Z)B - g(B,Z)A\right]. \tag{32}$$

Also, by contracting (32), we get

$$(\nabla_E \mathcal{R}c)(A,Z) = -\gamma(E)\mathcal{R}c(A,Z) - (4m+2)\theta(E)g(A,Z). \tag{33}$$

From (30), (31), and (33) we obtain (29). □

Lemma 4.4. *([5]). The covariant derivative of curvature tensor has the following symmetry for any vector fields A, B, E, F, Z.*

$$g\left((\nabla_E \mathcal{R})(B,A,Z), F\right) = -g\left((\nabla_E \mathcal{R})(B,A,F), Z\right). \tag{34}$$

Theorem 4.5. *A mixed 3-Sasakian manifold $(M, \zeta_i, \mu_i, \psi_i, g)_{i\in\{1,2,3\}}$ is generalized mixed 3-ψ-recurrent if and only if the relation*

$$\begin{aligned}(\nabla_E \mathcal{R})(B,A,Z) &= \{\tau_i\epsilon_i\left[g(A,\psi_i E)g(B,Z) - g(B,\psi_i E)g(A,Z)\right] \\ &- g(\mathcal{R}(B,A)\psi_i E, Z)\}\zeta_i - \gamma(E)\mathcal{R}(B,A)Z \\ &- \theta(E)[g(A,Z)B - g(B,Z)A], \end{aligned} \tag{35}$$

holds for all horizontal vector fields $A, B, Z \in T(M)$.

Corrected Proof

Proof. Let M be a generalized mixed 3-ψ-recurrent Sasakian manifold. From the covariant derivative of the curvature tensor, it follows:

$$\begin{aligned}(\nabla_E \mathcal{R})(B, A, \zeta_i) &= \nabla_E \mathcal{R}(B, A)\zeta_i - \mathcal{R}(\nabla_E B, A)\zeta_i \\ &- \mathcal{R}(B, \nabla_E A)\zeta_i - \mathcal{R}(B, A)\nabla_E \zeta_i. \end{aligned} \tag{36}$$

By using Equations (8), (10), and (11) in (36), we get:

$$(\nabla_E \mathcal{R})(B, A, \zeta_i) = \epsilon_i [g(E, \psi_i A)B - g(E, \psi_i B)A] + \tau_i \mathcal{R}(B, A)\psi_i E. \tag{37}$$

By virtue of (24), (34), and (37), we obtain the relation (35). Conversely, if Equation (35) holds for a generalized mixed 3-ψ-recurrent Sasakian manifold, then by applying ψ_i^2 on both sides of (35) and keeping in mind that A, B, E, Z are orthogonal to ζ_i, we get (13), and this completes the proof. □

Theorem 4.6. *If the relation*

$$\psi_i^2(\nabla_E \mathcal{R})(B, A, \zeta_i) = \gamma(E)\mathcal{R}(B, A)\zeta_i + \theta(E)[g(A, \zeta_i)B - g(B, \zeta_i)A], \tag{38}$$

holds for all horizontal vector fields $A, B \in T(M)$ in a mixed 3-structure Sasakian manifold $(M, \zeta_i, \mu_i, \psi_i, g)_{i\in\{1,2,3\}}$, then

$$\begin{aligned}\mathcal{R}(B, A)E &= g(B, E)A - g(A, E)B + 2g(\psi_i A, E)\psi_i B \\ &- 2g(\psi_i B, E)A. \end{aligned} \tag{39}$$

Proof. Theorem 4.5 and Equations (11) and (12) imply

$$(\nabla_E \mathcal{R})(B, A, \zeta_i) = 0,$$

holds for all horizontal vector fields $A, B, E \in T(M)$. Hence, by virtue of (37), we get:

$$\mathcal{R}(B, A)\psi_i E = \tau_i \epsilon_i \{g(\psi_i B, E)A - g(\psi_i A, E)B\}, \tag{40}$$

holds for all horizontal vector fields $A, B, E \in T(M)$. Suppose that A, B, Z are vector fields such that for a fixed point p of M^{4m+3}, $(\nabla B)_p = (\nabla A)_p = (\nabla Z)_p = 0$. By the Ricci identity for ψ_i, we have:

$$-(\mathcal{R}(B, A)\psi_i)E = (\nabla_B \nabla_A \psi_i)E - (\nabla_A \nabla_B \psi_i)E.$$

At the point p, it holds

$$-\mathcal{R}(B, A)(\psi_i E) + \psi_i \mathcal{R}(B, A)E = \nabla_B((\nabla_A \psi_i)E) - \nabla_A((\nabla_B \psi_i)E).$$

So, from (7), we have:

$$- \quad \mathcal{R}(B, A)(\psi_i E) + \psi_i \mathcal{R}(B, A)E$$

Corrected Proof

$$= \epsilon_i \left[\nabla_B \left(g\left(A,E\right)\zeta_i - \tau_i\mu_i\left(E\right)A\right) - \nabla_A\left(g\left(B,E\right)\zeta_i - \tau_i\mu_i\left(E\right)B\right)\right]$$

$$= \epsilon_i \left\{g\left(A,E\right)\nabla_B\zeta_i - \tau_i\left(\nabla_B\mu_i\right)\left(E\right)A - g\left(B,E\right)\nabla_A\zeta_i + \tau_i\left(\nabla_A\mu_i\right)\left(E\right)B\right\}.$$

Hence, by virtue of (8) and (10), we get:

$$\begin{aligned}\mathcal{R}\left(B,A\right)\left(\psi_i E\right) &= \tau_i\epsilon_i\{g\left(A,E\right)\psi_i B - g\left(B,E\right)\psi_i A - g\left(\psi_i B,E\right)A \\ &+ g\left(\psi_i A,E\right)B\} + \psi_i\mathcal{R}\left(B,A\right)E. \qquad (41)\end{aligned}$$

From (40) and (41), we have:

$$\begin{aligned}\psi_i\mathcal{R}\left(B,A\right)E &= \{g\left(B,E\right)\psi_i A - g\left(A,E\right)\psi_i B + 2g\left(\psi_i B,E\right)A \\ &- 2g\left(\psi_i A,E\right)B\}. \qquad (42)\end{aligned}$$

Applying ψ_i on both sides of (42) and using Equation (2) it follows (39) for any vector fields A, B that are orthogonal to ζ_i. □

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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Corrected Proof

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