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# Investigation of Electromagnetic Wave Propagation in Fractional Space in Elliptical Coordinates System and its Application in Elliptical Waveguide

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**Abstract**

Basic vector differential operators, such as gradient, divergence, Laplacian, and curl operators, are generalized and developed in elliptical coordinate systems in fractional space. The equation of Helmholtz in fractional space in an elliptical coordinate system is modified and solved in this coordinate and space for the investigation of wave propagation in the mentioned configuration. The general solutions of the scalar wave equation in fractional space in the elliptical coordinate system are obtained regarding the confluent Heun function. In terms of longitudinal electromagnetic field components in elliptical coordinate systems in fractional space, transverse electromagnetic field components are obtained. The electric and magnetic fields in elliptical coordinates systems in fractional space are developed in terms of vector potentials, and so TE and TM modes are investigated. For all cases studied, when the dimension is assumed to be an integer, the classical results are rewritten to validate the results.

Keywords: Fractional space, Elliptical coordinates system, Elliptical waveguide, Heun function, Non-linear media

2020 Mathematics Subject Classification: 78A25, 78A30, 33E30, 35J05.

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# 1. Introduction

The concept of fractional-dimensional space(FS) is used in many fields of physics to describe the effective parameters of physical systems. In the real world, space is typically Euclidean three-dimensional space, but material objects do not always move in this space. Electromagnetic wave propagation, radiation, and scattering in fractal structures can be described by replacing these finite fractal structures with D-dimensional FS. So, according to this simple value of D, the actual system can be modeled analytically. Some electromagnetic phenomena in fractal structures can be described by replacing these limited fractal structures with FS of dimension D. Therefore, given this simple value D, the real system can be modeled analytically.

Many objects in the world, such as clouds, tree branches, dust particles, etc., have irregular shapes, so Euclidean geometry is limited, leading to fractals and non-fractals. A fractal is geometrically described as a shape that can be divided into parts, where each part is a small-sized copy of the whole shape. An essential feature of fractals is that they allow complex geometries to be represented with a few parameters because all fractals are similar and repeat themselves at distinct scales. Due to the abundance of natural fractals, the laws of physics were invented for them.

For the first time, Mandelbrot classified media as either fractal or non-fractal categories [1]. To describe fractals that cannot be modeled in Euclidean space, fractional space effectively suggests an equivalent system. Therefore, using the FS, the real anisotropic enclosure structure can be replaced with an isotropic FS in which the enclosure is measured with fractional dimensions [2, 3]. However, the motion equation in a non-integer-dimensional space has been developed [4]. Electromagnetic fields, waves [5], and differential electromagnetic equations [6] in FS have been investigated. Solutions for electromagnetic wave propagation in rectangular and cylindrical waveguides in FS have been obtained [7–9]. Electromagnetic wave propagation [10], general plane wave solutions [11], solution of the cylindrical [12], and the spherical wave equation [13] in FS have been studied. A lot of research has been done about behaviors of various phenomena in FS [14–20], and some researchers have studied the fractal [21–25].

It is noted that the research conducted by the researchers has shown that elliptical waveguides have advantages compared to circular waveguides. For example, for two elliptic and circular dielectric waveguides with the same cross-section, the attenuation in the dominant mode is lower for the elliptic waveguide than for the circular waveguide. Attenuation effects and power flow expression for wave propagation in a surface wave transmission line with an elliptical cross-section show that some guide modes have less damping than the corresponding modes in a circular guide. Some modes in the elliptical waveguide have lower attenuation than the circular waveguide in similar modes[26, 27]. However, the generalization of operators of the divergence, Laplacian, curl, and gradient in rectangular, cylindrical, and spherical coordinate systems for FS has been derived. Electromagnetic fields and waves in rectangular, cylindrical, and spherical coordinate systems for

FS have been investigated. General solutions of electromagnetic fields in the mentioned coordinate system in fractional space have been calculated. Rectangular and cylindrical waveguides for FS have been studied.

The aim of this paper is to generalize and expand the range of applications and calculations of FS to the elliptical coordinates system(ECS). In this research, we expanded and developed the range of applications and calculations in ECS for FS. We generalize operators of the divergence, Laplacian, curl, and gradient in ECS for FS. We investigate electromagnetic fields, the Helmholtz equation, and solutions of electromagnetic fields in ECS for FS.

The present paper is formed of six sections, of which Section 1 is the Introduction. Section 2 deals with the calculation of the generalization of operators of divergence, Laplacian, curl, and gradient in ECS for FS. In Section 3, we obtain the solution of the wave equation in FS in ECS, and this solution inside an elliptical waveguide is also rewritten. In Section 4, we calculate transverse electromagnetic field components in terms of longitudinal electromagnetic field components in ECS in FS. General wave solution in terms of vector potentials in ECS in FS is investigated in Section 5. Finally, the conclusion is stated in Section 6.

## 2. Generalization of gradient, divergence, Laplacian, and curl operators in ECS for FS

Elliptical coordinates are indicated by $(\zeta, \vartheta, z)$ and are expressed as [28, 29]:

$$x = l\cosh\zeta\cos\vartheta, \quad y = l\sinh\zeta\sin\vartheta, \quad z = z\ ,$$

where $0 \leq \zeta < \infty$, $0 \leq \vartheta \leq 2\pi$, and $l = \sqrt{a_{xB}^2 - a_{yB}^2}$ is the semi-focal length. Also, $a_{xB}$, and $a_{yB}$ are defined as the semi-major and minor axes, that those are related to the ellipse of cross-section. The elliptic boundary is indicated by $\zeta_B = \operatorname{arctanh}(a_{yB}/a_{xB})$. Moreover, $\mathbf{e}_\zeta$, $\mathbf{e}_\vartheta$, $\mathbf{e}_z$ are unit vectors in the ECS. It is mentioned that gradient, divergence, curl, and Laplacian operators for fractals in rectangular and cylindrical coordinate systems are calculated [5]. Here, we similarly obtain these operators in ECS. The gradient for FS in ECS is developed and derived in the following form:

$$\nabla_D = \mathbf{e}_\zeta \frac{1}{h}\left[\frac{\partial}{\partial\zeta} + \frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right]$$

$$+\mathbf{e}_\vartheta \frac{1}{h}\left[\frac{\partial}{\partial\vartheta} - \frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right] + \mathbf{e}_z\left[\frac{\partial}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)}{z}\right].$$

This operator is rewritten to be more compact as:

$$\nabla_D = \nabla_E + \mathbf{S},$$

where $\nabla_E$ is the gradient operator in ECS when the dimension space is an integer,in other words: $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\nabla_E$ is defined as:

$$\nabla_E = \mathbf{e}_\zeta \frac{1}{h}\frac{\partial}{\partial \zeta} + \mathbf{e}_\vartheta \frac{1}{h}\frac{\partial}{\partial \vartheta} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

Here, $h = l\sqrt{\cosh^2\zeta - \cos^2\vartheta}$ is the scale factor, and $\mathbf{S} = \mathbf{e}_\zeta S_\zeta + \mathbf{e}_\vartheta S_\vartheta + \mathbf{e}_z S_z$ is a vector whose components are defined in the following form:

$$S_\zeta = \frac{1}{h}\left[\frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right],$$

$$S_\vartheta = \frac{1}{h}\left[-\frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right],$$

$$S_z = \left[\frac{\partial}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)}{z}\right].$$

It should be noted that three parameters ($0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate, and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. Obviously, if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then $D = 3$.
Similarly, we calculate the divergence operator in ECS for FS in the following form:

$$\nabla_D \cdot \mathbf{F} = \frac{1}{h^2}\frac{\partial}{\partial\zeta}(hF_\zeta) + \frac{1}{h^2}\frac{\partial}{\partial\vartheta}(hF_\vartheta)$$

$$+\frac{1}{2h}\left[(\alpha_1 - 1)\tanh\zeta + (\alpha_2 - 1)\coth\zeta\right]F_\zeta + \frac{1}{2h}\left[-(\alpha_1 - 1)\tan\vartheta + (\alpha_2 - 1)\cot\vartheta\right]F_\vartheta$$

$$+\left[\frac{\partial F_z}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)}{z}F_z\right].$$

This operator is rewritten to be more compact as:

$$\nabla_D \cdot \mathbf{F} = \nabla_E \cdot \mathbf{F} + \mathbf{S} \cdot \mathbf{F},$$

and $\nabla_E \cdot$ is the divergence operator in ECS when the dimension space is an integer; specifically, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\nabla_E \cdot \mathbf{F}$ is defined as:

$$\nabla_E \cdot \mathbf{F} = \frac{1}{h^2}\frac{\partial}{\partial\zeta}(hF_\zeta) + \frac{1}{h^2}\frac{\partial}{\partial\vartheta}(hF_\vartheta) + \frac{\partial F_z}{\partial z}.$$

Similarly, we calculate the Laplacian operator in ECS for FS in the following form:

$$\nabla_D^2 = \frac{1}{h^2}\left\{\frac{\partial^2}{\partial\zeta^2} + \left[(\alpha_1 - 1)\tanh\zeta + (\alpha_2 - 1)\coth\zeta\right]\frac{\partial}{\partial\zeta}\right\}$$

$$+\frac{1}{h^2}\left\{\frac{\partial^2}{\partial\vartheta^2} + \left[-(\alpha_1 - 1)\tan\vartheta + (\alpha_2 - 1)\cot\vartheta\right]\frac{\partial}{\partial\vartheta}\right\}$$

$$+\left[\frac{\partial^2}{\partial z^2}+\frac{(\alpha_3-1)}{z}\frac{\partial}{\partial z}\right].$$

This operator is rewritten to be more compact as:

$$\nabla_D^2=\nabla_E^2+2\mathbf{S}\cdot\nabla_E,$$

and $\nabla_E^2$ is the Laplacian operator in ECS when the dimension space is an integer; specifically, $\alpha_1=\alpha_2=\alpha_3=1$, and $\nabla_E^2$ is defined as:

$$\nabla_E^2=\frac{1}{h^2}\left[\frac{\partial^2}{\partial\zeta^2}+\frac{\partial^2}{\partial\vartheta^2}\right]+\frac{\partial^2}{\partial z^2}.$$

The curl operator in ECS for FS is obtained in the following form:

$$\nabla_D\times\mathbf{F}=\mathbf{e}_\zeta\left\{\frac{1}{h}\frac{\partial F_z}{\partial\vartheta}-\frac{\partial F_\vartheta}{\partial z}+\frac{1}{2h}\left[-(\alpha_1-1)\tan\vartheta+(\alpha_2-1)\cot\vartheta\right]F_z-\frac{1}{2}\frac{(\alpha_3-1)}{z}F_\vartheta\right\}$$

$$+\mathbf{e}_\vartheta\left\{-\frac{1}{h}\frac{\partial F_z}{\partial\zeta}+\frac{\partial F_\zeta}{\partial z}-\frac{1}{2h}\left[(\alpha_1-1)\tanh\zeta+(\alpha_2-1)\coth\zeta\right]F_z+\frac{1}{2}\frac{(\alpha_3-1)}{z}F_\zeta\right\}$$

$$+\mathbf{e}_z\frac{1}{h^2}\left[\frac{\partial}{\partial\zeta}(hF_\vartheta)-\frac{\partial}{\partial\vartheta}(hF_\zeta)\right]$$

$$+\mathbf{e}_z\frac{1}{2h}\left\{\left[(\alpha_1-1)\tanh\zeta+(\alpha_2-1)\coth\zeta\right]F_\vartheta-\left[-(\alpha_1-1)\tan\vartheta+(\alpha_2-1)\cot\vartheta\right]F_\zeta\right\}.$$

This operator is rewritten to be more compact as:

$$\nabla_D\times\mathbf{F}=\nabla_E\times\mathbf{F}+\mathbf{S}\times\mathbf{F},$$

and $\nabla_E$ is the curl operator in ECS when the dimension space is an integer; specifically, $\alpha_1=\alpha_2=\alpha_3=1$, and $\nabla_E\times\mathbf{F}$ is defined as:

$$\nabla_E\times\mathbf{F}=\mathbf{e}_\zeta\left\{\frac{1}{h}\frac{\partial F_z}{\partial\vartheta}-\frac{\partial F_\vartheta}{\partial z}\right\}+\mathbf{e}_\vartheta\left\{-\frac{1}{h}\frac{\partial F_z}{\partial\zeta}+\frac{\partial F_\zeta}{\partial z}\right\}+\mathbf{e}_z\frac{1}{h^2}\left\{\frac{\partial}{\partial\zeta}(hF_\vartheta)-\frac{\partial}{\partial\vartheta}(hF_\zeta)\right\}.$$

## 3. Solutions of the wave equation in FS in ECS in terms of the Heun function

It is mentioned that Maxwell's equations in the far-field region in FS are presented in the following forms [5]:

$$\nabla_D\cdot\mathbf{D}=\rho_v,\ \nabla_D\cdot\mathbf{B}=0,\ \nabla_D\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\ \nabla_D\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}.$$

It is well known that the problems with elliptical geometries need to be calculated and solved in ECS. We assume that the considered space is FS and source-free.

Therefore, the Helmholtz equation for source-free and lossless media is presented as follows [5]:

$$\nabla_D^2 \mathbf{E} + \beta^2 \mathbf{E} = 0,$$
$$\nabla_D^2 \mathbf{H} + \beta^2 \mathbf{H} = 0,$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and time dependency is assumed as: $\exp(i\omega t)$. In elliptical coordinates, a general solution for the electric field $\mathbf{E}(\zeta,\vartheta,z)$ is written in the following form:

$$\mathbf{E}(\zeta,\vartheta,z) = \mathbf{e}_\zeta E_\zeta(\zeta,\vartheta,z) + \mathbf{e}_\vartheta E_\vartheta(\zeta,\vartheta,z) + \mathbf{e}_z E_z(\zeta,\vartheta,z).$$

So, substituting $\mathbf{E}(\zeta,\vartheta,z)$ we can obtain:

$$\nabla_D^2[\mathbf{e}_\zeta E_\zeta + \mathbf{e}_\vartheta E_\vartheta + \mathbf{e}_z E_z] + \beta^2[\mathbf{e}_\zeta E_\zeta + \mathbf{e}_\vartheta E_\vartheta + \mathbf{e}_z E_z] = 0.$$

It is clear that:

$$\nabla_D^2[\mathbf{e}_\zeta E_\zeta] \neq \mathbf{e}_\zeta \nabla_D^2[E_\zeta],$$
$$\nabla_D^2[\mathbf{e}_\vartheta E_\vartheta] \neq \mathbf{e}_\vartheta \nabla_D^2[E_\vartheta],$$
$$\nabla_D^2[\mathbf{e}_z E_z] = \mathbf{e}_z \nabla_D^2[E_z].$$

Therefore, the solution of the above equation cannot be easily solved, but we can convert it into a scaler coupled partial differential equation. However, for simplicity, the solution of the wave expressed in ECS must apply to the following equation:

$$\nabla_D^2 \psi(\zeta,\vartheta,z) + \beta^2 \psi(\zeta,\vartheta,z) = 0,$$

where, $\psi(\zeta,\vartheta,z)$ is defined as a scalar function such as a component of a vector potential, or component of electric field, or component of magnetic field. Substituting $\psi(\zeta,\vartheta,z)$ we can obtain:

$$\frac{1}{h^2}\left\{\frac{\partial^2\psi(\zeta,\vartheta,z)}{\partial\zeta^2} + [(\alpha_1-1)\tanh\zeta + (\alpha_2-1)\coth\zeta]\,\psi(\zeta,\vartheta,z)\right\}$$
$$+\frac{1}{h^2}\left\{\frac{\partial^2\psi(\zeta,\vartheta,z)}{\partial\vartheta^2} + \left[-(\alpha_1-1)\tan\vartheta + \frac{1}{2}(\alpha_2-1)\cot\vartheta\right]\psi(\zeta,\vartheta,z)\right\}$$
$$+\left[\frac{\partial^2\psi(\zeta,\vartheta,z)}{\partial z^2} + \frac{(\alpha_3-1)}{z}\psi(\zeta,\vartheta,z)\right] + \beta^2\psi(\zeta,\vartheta,z) = 0,$$

or in compact form as follows:

$$\nabla_E^2\psi(\zeta,\vartheta,z) + 2\mathbf{S}\cdot\nabla_E\psi(\zeta,\vartheta,z) + \beta^2\psi(\zeta,\vartheta,z) = 0.$$

Using the method of separation of variables, we can separate the above equation. For this purpose, we assume:

$$\psi(\zeta,\vartheta,z) = \psi_\zeta(\zeta)\psi_\vartheta(\vartheta)\psi_z(z).$$

Therefore, three ordinary differential equations are derived as follows:

$$\frac{d^2\psi_\zeta(\zeta)}{d\zeta^2}+[(\alpha_1-1)\tanh\zeta+(\alpha_2-1)\coth\zeta]\frac{d\psi_\zeta(\zeta)}{d\zeta}+\kappa^2 l^2\cosh^2\zeta\psi_\zeta(\zeta)-b\psi_\zeta(\zeta)=0,$$

$$\frac{d^2\psi_\vartheta(\vartheta)}{d\vartheta^2}+[-(\alpha_1-1)\tan\vartheta+(\alpha_2-1)\cot\vartheta]\frac{d\psi_\vartheta(\vartheta)}{d\vartheta}-\kappa^2 l^2\cos^2\vartheta\psi_\vartheta(\vartheta)+b\psi_\vartheta(\vartheta)=0,$$

$$\frac{d^2\psi_z(z)}{dz^2}+\frac{(\alpha_3-1)}{z}\frac{d\psi_z(z)}{dz}+\beta_z^2\psi_z(z)=0,$$

where $b$ is a separation constant and $\beta^2-\beta_z^2=\kappa^2$. In addition, the first two equations are transformed into the following equations by replacing and changing the appropriate variable:

$$\psi_\zeta''(\zeta)+\frac{1}{2}[\frac{\alpha_2}{X_\zeta}-\frac{\alpha_1}{(1-X_\zeta)}]\psi_\zeta'(\zeta)-\frac{1}{4}[\frac{-\kappa_2 l^2+b}{X_\zeta}+\frac{b}{(1-X_\zeta)}]\psi_\zeta(\zeta)=0,$$

$$\psi_\vartheta''(\vartheta)+\frac{1}{2}[\frac{\alpha_2}{X_\vartheta}-\frac{\alpha_1}{(1-X_\vartheta)}]\psi_\vartheta'(\vartheta)+\frac{1}{4}[\frac{-\kappa_2 l^2+b}{X_\vartheta}+\frac{b}{(1-X_\vartheta)}]\psi_\vartheta(\vartheta)=0.$$

The solutions of the above three differential equations can be written in the following forms:

$$\psi_\zeta(\zeta)=B_1 HeunC(\alpha_\zeta,\beta_\zeta,\gamma_\zeta,\delta_\zeta,\vartheta_{;}X_\zeta)+B_1X_\zeta^{-\beta_\zeta}HeunC(\alpha_\zeta,-\beta_\zeta,\gamma_\zeta,\delta_\zeta,\vartheta_\zeta;X_\zeta),$$

$$\psi_\vartheta(\vartheta)=B_3 HeunC(\alpha_\vartheta,\beta_\vartheta,\gamma,\delta_\vartheta,\vartheta_\vartheta;X_\vartheta)+B_4X_\vartheta^{-\beta_\vartheta}HeunC(\alpha_\vartheta,-\beta_\vartheta,\gamma,\delta_\vartheta,\vartheta_{;}X_\vartheta),$$

$$\psi_z(z)=z^\lambda[B_5J_\lambda(\beta_z z)+B_6Y_\lambda(\beta_z z)]\quad,\quad\lambda=1-\frac{\alpha_3}{2},$$

where $HeunC$ is defined as the solution of the confluent Heun function which is explained in the next subsection in summary. In the case where the dimension of space is an integer; specifically, $\alpha_1=\alpha_2=\alpha_3=1$, the classical results are obtained as:

$$\nabla_E^2\psi(\zeta,\vartheta,z)+\beta^2\psi(\zeta,\vartheta,z)=0,$$

which separates into three equations:

$$\frac{d^2\psi_\zeta(\zeta)}{d\zeta^2}+\kappa^2l^2\cosh^2\zeta\psi_\zeta(\zeta)-b\psi_\zeta(\zeta)=0,$$

$$\frac{d^2\psi_\vartheta(\vartheta)}{d\vartheta^2}-\kappa^2l^2\cos^2\vartheta\psi_\vartheta(\vartheta)+b\psi_\vartheta(\vartheta)=0,$$

$$\frac{d^2\psi_z(z)}{dz^2}+\beta_z^2\psi_z(z)=0.$$

The solutions of the first two equations are presented as Mathieu and modified Mathieu functions [28, 29].

## 3.1 Confluent Heun equation

Solutions of Heun's differential equation are widely applied across various fields of the natural sciences, particularly in physics [30]. The confluent Heun's differential equation is written in the following form [31, 32]:

$$\frac{d^2Y(X)}{dX^2} + \left(\alpha_h + \frac{\beta_h+1}{X} + \frac{\gamma_h+1}{X-1}\right)\frac{dY(X)}{dX} + \left(\frac{\mu_h}{X} + \frac{\nu_h+1}{X-1}\right)Y(X) = 0.$$

The solutions of the confluent Heun's differential equation denoted by

$$HeunC(\alpha_h, \beta_h, \gamma_h, \delta_h, \vartheta_h; X),$$

and:

$$\delta_h = \mu_h + \nu_h - \frac{\alpha_h}{2}(\beta_h + \gamma_h + 2), \quad \eta_h = \frac{\alpha_h}{2}(\beta_h + 1) - \mu - \frac{1}{2}(\beta_h + \gamma_h + \beta_h\gamma_h).$$

This solution is defined by the convergent Taylor series expansion around $X = 0$, and normalization:

$$HeunC(\alpha_h, \beta_h, \gamma_h, \delta_h, \eta_h; 0) = 1,$$

$$\frac{d}{dX}HeunC(\alpha_h, \beta_h, \gamma_h, \delta_h, \eta_h; 0) = \frac{1}{2}\frac{\beta_h(-\alpha_h + \gamma_h + 1) - \alpha_h + \gamma_h + 2\eta_h}{\beta_h + 1}.$$

The general solution is defined as:

$$Y(X) = H_1 HeunC(\alpha_h, \beta_h, \gamma_h, \delta_h, \eta_h; X) + H_2 X^{-\beta_h} HeunC(\alpha_h, -\beta_h, \gamma_h, \delta_h, \eta_h; X).$$

Note that the problem does not need to be very complicated to work with these equations. We encounter Mathieu functions if we consider two-dimensional problems with elliptical shapes. Let us use $x = \ell\cosh\zeta\cos\vartheta,\ y = \ell\sinh\zeta\sin\vartheta$. Then the Helmholtz equation can be written as:

$$\frac{\partial^2\psi(\zeta,\vartheta)}{\partial\zeta^2} + \frac{\partial^2\psi(\zeta,\vartheta)}{\partial\vartheta^2}\psi(\zeta,\vartheta,z) + \ell^2k^2(\cosh^2\zeta - \cos^2\vartheta)\psi(\zeta,\vartheta) = 0,$$

which separates into two equations:

$$\begin{aligned}
\frac{d^2\Theta(\vartheta)}{d\vartheta^2} + (b - 2q\cos 2\vartheta)\Theta(\vartheta) &= 0,\\
\frac{d^2\Lambda(\zeta)}{d\zeta^2} + (b - 2q\cosh 2\zeta)\Lambda(\zeta) &= 0.
\end{aligned}$$

The solutions to these two equations can be represented as Mathieu and modified Mathieu functions and $\psi(\zeta,\vartheta) = \Theta(\vartheta)\Lambda(\zeta)$.

# 4. Transverse electromagnetic field components in terms of longitudinal electromagnetic field components in ECS in FS

We assume time dependence is $e^{i\omega t}$ and wave propagation is along the z-axis. In addition, it is important to mention that all differential operators presented [5, 6]. The generalized Maxwell's equations for a source-free region are written in the following forms:

$$\nabla_D \times \mathbf{E} = -i\omega\mu\mathbf{H},$$

$$\nabla_D \times \mathbf{H} = i\omega\varepsilon\mathbf{E}.$$

Furthermore, the fields can be written in the form of transverse and longitudinal components as:

$$\mathbf{E}(\zeta,\vartheta,z) = \mathbf{E}_T(\zeta,\vartheta,z) + \mathbf{e}_z E_z(\zeta,\vartheta,z),$$

$$\mathbf{H}(\zeta,\vartheta,z) = \mathbf{H}_T(\zeta,\vartheta,z) + \mathbf{e}_z H_z(\zeta,\vartheta,z),$$

where:

$$\mathbf{E}_T(\zeta,\vartheta,z) = \mathbf{e}_\zeta E_\zeta(\zeta,\vartheta,z) + \mathbf{e}_\vartheta E_\vartheta(\zeta,\vartheta,z),$$

$$\mathbf{H}_T(\zeta,\vartheta,z) = \mathbf{e}_\zeta H_\zeta(\zeta,\vartheta,z) + \mathbf{e}_\vartheta H_\vartheta(\zeta,\vartheta,z).$$

Therefore, we obtain the transverse field components in terms of longitudinal fields components in ECS in FS as:

$$E_\zeta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\omega\mu\frac{\partial H_z}{\partial\vartheta} + \omega\mu\left[-\frac{1}{2}(\alpha_1-1)\tan\vartheta + \frac{1}{2}(\alpha_2-1)\cot\vartheta\right]H_z \right.$$
$$\left. + \beta_z\frac{\partial E_z}{\partial\zeta} + \beta_z\left[\frac{1}{2}(\alpha_1-1)\tanh\zeta + \frac{1}{2}(\alpha_2-1)\coth\zeta\right]E_z\right\},$$

$$E_\vartheta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{-\omega\mu\frac{\partial H_z}{\partial\zeta} - \omega\mu\left[\frac{1}{2}(\alpha_1-1)\tanh\zeta + \frac{1}{2}(\alpha_2-1)\coth\zeta\right]H_z \right.$$
$$\left. + \beta_z\frac{\partial E_z}{\partial\vartheta} + \beta_z\left[-\frac{1}{2}(\alpha_1-1)\tan\vartheta + \frac{1}{2}(\alpha_2-1)\cot\vartheta\right]E_z\right\},$$

$$H_\zeta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\beta_z\frac{\partial H_z}{\partial\zeta} + \beta_z\left[\frac{1}{2}(\alpha_1-1)\tanh\zeta + \frac{1}{2}(\alpha_2-1)\coth\zeta\right]H_z \right.$$
$$\left. - \omega\varepsilon\frac{\partial E_z}{\partial\vartheta} - \omega\varepsilon\left[-\frac{1}{2}(\alpha_1-1)\tan\vartheta + \frac{1}{2}(\alpha_2-1)\cot\vartheta\right]E_z\right\},$$

$$H_\vartheta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\beta_z\frac{\partial H_z}{\partial\vartheta} + \beta_z\left[-\frac{1}{2}(\alpha_1-1)\tan\vartheta + \frac{1}{2}(\alpha_2-1)\cot\vartheta\right]H_z \right.$$
$$\left. + \omega\varepsilon\frac{\partial E_z}{\partial\zeta} + \omega\varepsilon\left[\frac{1}{2}(\alpha_1-1)\tanh\zeta + \frac{1}{2}(\alpha_2-1)\coth\zeta\right]E_z\right\}.$$

In the limit of integer dimensional space $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the components of transverse fields in ECS can be obtained as the classical forms:

$$E_\zeta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left[\omega\mu\frac{\partial H_z}{\partial\vartheta} + \beta_z\frac{\partial E_z}{\partial\zeta}\right],$$

$$E_\vartheta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left[-\omega\mu\frac{\partial H_z}{\partial\zeta} + \beta_z\frac{\partial E_z}{\partial\vartheta}\right],$$

$$H_\zeta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left[\beta_z\frac{\partial H_z}{\partial\zeta} - \omega\varepsilon\frac{\partial E_z}{\partial\vartheta}\right],$$

$$H_\vartheta = -\frac{i}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left[\beta_z\frac{\partial H_z}{\partial\vartheta} + \omega\varepsilon\frac{\partial E_z}{\partial\zeta}\right].$$

## 4.1 TE mode: Transverse electric mode in ECS in FS

For transverse electric mode $E_z = 0$ and $H_z \neq 0$,therefore by solving following wave equation

$$\nabla_D^2 H_z + \beta_z^2 H_z = \nabla_E^2 H_z + 2\mathbf{S}\cdot\nabla_E H_z + \beta_z^2 H_z = 0,$$

the solution can be obtained, and transverse field components can be calculated by:

$$E_\zeta = -\frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial H_z}{\partial\vartheta} + \left[-\frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right]H_z\right\},$$

$$E_\vartheta = \frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial H_z}{\partial\zeta} + \left[\frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right]H_z\right\},$$

$$H_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial H_z}{\partial\zeta} + \left[\frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right]H_z\right\},$$

$$H_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial H_z}{\partial\vartheta} + \left[-\frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right]H_z\right\}.$$

After calculating the electric and magnetic field components, unknown constant coefficients are determined using appropriate boundary conditions.

## 4.2 TM mode: Transverse magnetic mode in ECS in FS

For transverse magnetic mode $H_z = 0$ and $E_z \neq 0$, by solving the following wave equation

$$\nabla_D^2 E_z + \beta_z^2 E_z = \nabla_E^2 E_z + 2\mathbf{S}\cdot\nabla_E E_z + \beta_z^2 E_z = 0,$$

the solution can be derived, and the transverse field components can be computed by:

$$E_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial E_z}{\partial\zeta} + \left[\frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right]E_z\right\},$$

$$E_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial E_z}{\partial\vartheta} + \left[-\frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right]E_z\right\},$$

$$H_\zeta = \frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial E_z}{\partial\vartheta} + \left[-\frac{1}{2}(\alpha_1 - 1)\tan\vartheta + \frac{1}{2}(\alpha_2 - 1)\cot\vartheta\right]E_z\right\},$$

$$H_\vartheta = -\frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h}\left\{\frac{\partial E_z}{\partial\zeta} + \left[\frac{1}{2}(\alpha_1 - 1)\tanh\zeta + \frac{1}{2}(\alpha_2 - 1)\coth\zeta\right]E_z\right\}.$$

After calculating the electric and magnetic field components, unknown constant coefficients are determined using appropriate boundary conditions.

## 4.3 Elliptical waveguide

### 4.3.1 TE waves inside the elliptical waveguide in FS

Here we consider an elliptical waveguide with the elliptic boundary defined by $\zeta = \zeta_0$. To simplify, we assumed fractionality is along the z-axis. The waveguide is filled by a material with permeability $\mu$ and permittivity $\varepsilon$.

For TE modes, considering a single parameter for a non integer dimension, $H_z$ must satisfy in [11]:

$$\left[\frac{1}{h^2}\left(\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial\vartheta^2}\right) + \frac{\partial^2}{\partial z^2} + \frac{D-3}{z}\frac{\partial}{\partial z} + \beta^2\right]H_z = 0.$$

Solution of the above equation is in the following form:

$$H_z(\zeta,\vartheta,z) = [C_m Ce_m(\zeta,q)ce_m(\vartheta,q) + S_m Se_m(\zeta,q)se_m(\vartheta,q)]z^\lambda[H_{\lambda 1}H_\lambda^{(1)}(\beta_z z) + H_{\lambda 2}H_\lambda^{(2)}(\beta_z z)],$$

where $ce_m(\vartheta,q)$, $se_m(\vartheta,q)$, and $Ce_m(\zeta,q)$, $Se_m(\zeta,q)$ are the even and odd solutions of the angular and radial Mathieu equation of the first kind [28, 29]. $H_\lambda^{(1)}(\beta_z z)$ and $H_\lambda^{(2)}(\beta_z z)$ are the Hankel functions of the first and second kind of order $\lambda$. We consider:

$$H_z(\zeta,\vartheta,z) = A_{m\lambda}Ce_m(\zeta,q)ce_m(\vartheta,q)z^\lambda H_\lambda^{(2)}(\beta_z z),$$

or:

$$H_z(\zeta,\vartheta,z) = A_{m\lambda}Se_m(\zeta,q)se_m(\vartheta,q)z^\lambda H_\lambda^{(2)}(\beta_z z),$$

and using the boundary condition, it is obtained $Ce'_m(\zeta_0,q) = 0$ or $Se'_m(\zeta_0,q) = 0$. Therefore, transverse fields components are presented as:

$$E_\zeta = -\frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h}A_{m\lambda}Ce_m(\zeta,q)ce'_m(\vartheta,q)z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$E_\vartheta = \frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta,q) ce_m(\vartheta,q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$H_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta,q) ce_m(\vartheta,q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$H_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce_m(\zeta,q) ce'_m(\vartheta,q) z^\lambda H_\lambda^{(2)}(\beta_z z).$$

For a limit $D = 3$ [28]

$$H_z(\zeta,\vartheta,z) = A_{m\lambda} Ce_m(\zeta,q) ce_m(\vartheta,q) z^{\frac{1}{2}} H_{\frac{1}{2}}^{(2)}(\beta_z z),$$

and using $H_{\frac{1}{2}}^{(2)}(\beta_z z) = i\sqrt{\frac{2}{\pi\beta_z z}} e^{-i\beta_z z}$ [33], we obtain:

$$H_z(\zeta,\vartheta,z) = A'_{m\lambda} Ce_m(\zeta,q) ce_m(\vartheta,q) e^{-i\beta_z z},$$

$$E_\zeta = -\frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h} A'_{m\lambda} Ce_m(\zeta,q) ce'_m(\vartheta,q) e^{-i\beta_z z},$$

$$E_\vartheta = \frac{i\omega\mu}{(\omega^2\varepsilon\mu - \beta_z^2)h} A'_{m\lambda} Ce'_m(\zeta,q) ce_m(\vartheta,q) e^{-i\beta_z z},$$

$$H_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A'_{mn} Ce'_m(\zeta,q) ce_m(\vartheta,q) e^{-i\beta_z z},$$

$$H_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A'_{m\lambda} Ce_m(\zeta,q) ce'_m(\vartheta,q) e^{-i\beta_z z}.$$

#### 4.3.2 TM waves inside the elliptical waveguide in fractional dimensional space

For TM modes, considering a single parameter for a non integer dimension, $E_z$ must satisfy the following equation as proposed for rectangular coordinates in [11]:

$$\left[\frac{1}{h^2}\left(\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial\vartheta^2}\right) + \frac{\partial^2}{\partial z^2} + \frac{D-3}{z}\frac{\partial}{\partial z} + \beta^2\right] E_z = 0.$$

The solution of the above equation is in the following form:

$$E_z(\zeta,\vartheta,z) = [C_m Ce_m(\zeta,q) ce_m(\vartheta,q) + S_m Se_m(\zeta,q) se_m(\vartheta,q)] z^\lambda [H_{\lambda 1} H_\lambda^{(1)}(\beta_z z) + H_{\lambda 2} H_\lambda^{(2)}(\beta_z z)].$$

Similar to the TE case we consider:

$$E_z(\zeta,\vartheta,z) = A_{m\lambda} Ce_m(\zeta,q) ce_m(\vartheta,q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

or:

$$E_z(\zeta,\vartheta,z) = A_{m\lambda} Se_m(\zeta,q) se_m(\vartheta,q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

and using boundary condition, it is obtained $Ce_m(\zeta_0, q) = 0$ or $Se_m(\zeta_0, q) = 0$. Therefore, transverse field components are presented as:

$$E_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta, q) ce_m(\vartheta, q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$E_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce_m(\zeta, q) ce'_m(\vartheta, q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$H_\zeta = \frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce_m(\zeta, q) ce'_m(\vartheta, q) z^\lambda H_\lambda^{(2)}(\beta_z z),$$

$$H_\vartheta = -\frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta, q) ce_m(\vartheta, q) z^\lambda H_\lambda^{(2)}(\beta_z z).$$

For a limit $D = 3$ [28]

$$E_z(\zeta, \vartheta, z) = A_{m\lambda} Ce_m(\zeta, q) ce_m(\vartheta, q) z^{\frac{1}{2}} H_{\frac{1}{2}}^{(2)}(\beta_z z),$$

and using $H_{\frac{1}{2}}^{(2)}(\beta_z z) = i\sqrt{\frac{2}{\pi\beta_z z}} e^{-i\beta_z z}$ [33], we obtain:

$$E_z(\zeta, \vartheta, z) = A'_{m\lambda} Ce_m(\zeta, q) ce_m(\vartheta, q) e^{-i\beta_z z},$$

$$E_\zeta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta, q) ce_m(\vartheta, q) e^{-i\beta_z z},$$

$$E_\vartheta = -\frac{i\beta_z}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce_m(\zeta, q) ce'_m(\vartheta, q) e^{-i\beta_z z},$$

$$H_\zeta = \frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce_m(\zeta, q) ce'_m(\vartheta, q) e^{-i\beta_z z},$$

$$H_\vartheta = -\frac{i\omega\varepsilon}{(\omega^2\varepsilon\mu - \beta_z^2)h} A_{m\lambda} Ce'_m(\zeta, q) ce_m(\vartheta, q) e^{-i\beta_z z}.$$



## 5. General wave solution in terms of vector potentials in ECS in FS

The vector potentials **A** and **F** satisfy the following Helmholtz equation in a source-free region [8]:

$$\left[\nabla_D^2 + \beta^2\right] \begin{bmatrix} \mathbf{A} \\ \mathbf{F} \end{bmatrix} = 0,$$

where:

$$\mathbf{A}(\zeta, \vartheta, z) = \mathbf{e}_\zeta A_\zeta(\zeta, \vartheta, z) + \mathbf{e}_\vartheta A_\vartheta(\zeta, \vartheta, z) + \mathbf{e}_z A_z(\zeta, \vartheta, z),$$

$$\mathbf{F}(\zeta, \vartheta, z) = \mathbf{e}_\zeta F_\zeta(\zeta, \vartheta, z) + \mathbf{e}_\vartheta F_\vartheta(\zeta, \vartheta, z) + \mathbf{e}_z F_z(\zeta, \vartheta, z).$$

The electric and magnetic fields in FS can be expressed as [8]:

$$\mathbf{E} = -i\omega\mathbf{A} - \frac{i}{\omega\mu\varepsilon}\nabla_D(\nabla_D\cdot\mathbf{A}) - \frac{1}{\varepsilon}\nabla_D\times\mathbf{F},$$

$$\mathbf{H} = -i\omega\mathbf{F} - \frac{i}{\omega\mu\varepsilon}\nabla_D(\nabla_D\cdot\mathbf{F}) + \frac{1}{\mu}\nabla_D\times\mathbf{A}.$$

Therefore, the generalized form for **E** and **H** for fractals in ECS can be calculated as:

$$\begin{aligned}
\mathbf{E} = \mathbf{e}_\zeta &\left\{-i\omega A_\zeta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta} + S_\zeta)(\nabla\cdot\mathbf{A} + \mathbf{S}\cdot\mathbf{A})\right.\\
&\left.-\frac{1}{\varepsilon}\left[\frac{1}{h}\frac{\partial F_z}{\partial\vartheta} - \frac{\partial F_\vartheta}{\partial z} + S_\vartheta F_z - S_z F_\vartheta\right]\right\}\\
+\mathbf{e}_\vartheta &\left\{-i\omega A_\vartheta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta} + S_\vartheta)(\nabla\cdot\mathbf{A} + \mathbf{S}\cdot\mathbf{A})\right.\\
&\left.-\frac{1}{\varepsilon}\left[-\frac{1}{h}\frac{\partial F_z}{\partial\zeta} + \frac{\partial F_\zeta}{\partial z} - S_\zeta F_z + S_z F_\zeta\right]\right\}\\
+\mathbf{e}_z &\left\{-i\omega A_z - \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z} + S_z)(\nabla\cdot\mathbf{A} + \mathbf{S}\cdot\mathbf{A})\right.\\
&\left.-\frac{1}{\varepsilon}\left[\frac{1}{h^2}(\frac{\partial}{\partial\zeta}hF_\vartheta - \frac{\partial}{\partial\zeta}hF_\zeta) + S_\zeta F_\vartheta - S_\vartheta F_\zeta\right]\right\},\\
\mathbf{H} = \mathbf{e}_\zeta &\left\{-i\omega F_\zeta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta} + S_\zeta)(\nabla\cdot\mathbf{F} + \mathbf{S}\cdot\mathbf{F})\right.\\
&\left.+\frac{1}{\mu}\left[\frac{1}{h}\frac{\partial A_z}{\partial\vartheta} - \frac{\partial A_\vartheta}{\partial z} + S_\vartheta A_z - S_z A_\vartheta\right]\right\}\\
+\mathbf{e}_\vartheta &\left\{-i\omega F_\vartheta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta} + S_\vartheta)(\nabla\cdot\mathbf{F} + \mathbf{S}\cdot\mathbf{F})\right.\\
&\left.+\frac{1}{\mu}\left[-\frac{1}{h}\frac{\partial A_z}{\partial\zeta} + \frac{\partial A_\zeta}{\partial z} - S_\zeta A_z + S_z A_\zeta\right]\right\}\\
+\mathbf{e}_z &\left\{-i\omega F_z - \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z} + S_z)(\nabla\cdot\mathbf{F} + \mathbf{S}\cdot\mathbf{F})\right.\\
&\left.+\frac{1}{\mu}\left[\frac{1}{h^2}(\frac{\partial}{\partial\zeta}hA_\vartheta - \frac{\partial}{\partial\zeta}hA_\zeta) + S_\zeta A_\vartheta - S_\vartheta A_\zeta\right]\right\}.
\end{aligned}$$

In the limit of integer-dimensional space $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the generalized form of **E** and **H** for fractals in ECS can be obtained as the classical general solutions:

$$\mathbf{E} = \mathbf{e}_\zeta\left\{-i\omega A_\zeta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta})(\nabla\cdot\mathbf{A}) \; -\frac{1}{\varepsilon}[\frac{1}{h}\frac{\partial F_z}{\partial\vartheta} - \frac{\partial F_\vartheta}{\partial z}]\right\}$$

$$+\mathbf{e}_\vartheta \left\{-i\omega A_\vartheta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta})(\nabla\cdot\mathbf{A}) \; -\frac{1}{\varepsilon}[-\frac{1}{h}\frac{\partial F_z}{\partial\zeta} + \frac{\partial F_\zeta}{\partial z}]\right\}$$

$$+\mathbf{e}_z \left\{-i\omega A_z - \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z})(\nabla\cdot\mathbf{A}) \; -\frac{1}{\varepsilon}[\frac{1}{h^2}(\frac{\partial}{\partial\zeta}hF_\vartheta - \frac{\partial}{\partial\zeta}hF_\zeta)]\right\},$$

$$\mathbf{H} = \mathbf{e}_\zeta \left\{-i\omega F_\zeta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta})(\nabla\cdot\mathbf{F}) \; +\frac{1}{\mu}[\frac{1}{h}\frac{\partial A_z}{\partial\vartheta} - \frac{\partial A_\vartheta}{\partial z}]\right\}$$

$$+\mathbf{e}_\vartheta \left\{-i\omega F_\vartheta - \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta})(\nabla\cdot\mathbf{F}) \; +\frac{1}{\mu}[-\frac{1}{h}\frac{\partial A_z}{\partial\zeta} + \frac{\partial A_\zeta}{\partial z}]\right\}$$

$$+\mathbf{e}_z \left\{-i\omega F_z - \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z})(\nabla\cdot\mathbf{F}) \; +\frac{1}{\mu}[\frac{1}{h^2}(\frac{\partial}{\partial\zeta}hA_\vartheta - \frac{\partial}{\partial\zeta}hA_\zeta)]\right\}.$$

### 5.1 TE mode: Transverse electric mode

In this subsection, the **E** and **H** of $TE$ mode in an elliptical waveguide filled with FS are obtained, and in the limit case, are converted to their classical forms. For transverse electric mode:

$$\mathbf{A} = 0, \quad \mathbf{F} = \mathbf{e}_z F_z(\zeta, \vartheta, z).$$

Using the general solution for **E** and **H**, we obtain:

$$\mathbf{E} = -\mathbf{e}_\zeta \frac{1}{\varepsilon}[\frac{1}{h}\frac{\partial F_z}{\partial\vartheta} + S_\vartheta F_z - S_z F_\vartheta] - \mathbf{e}_\vartheta \frac{1}{\varepsilon}[-\frac{1}{h}\frac{\partial F_z}{\partial\zeta} - S_\zeta F_z - S_z F_\zeta],$$

$$\mathbf{H} = -\mathbf{e}_\zeta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta} + S_\zeta)(\frac{\partial F_z}{\partial z} + S_z F_z) - \mathbf{e}_\vartheta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta} + S_\vartheta)(\frac{\partial F_z}{\partial z} + S_z F_z)$$

$$- \mathbf{e}_z[i\omega F_z + \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z} + S_z)(\frac{\partial F_z}{\partial z} + S_z F_z)].$$

In the limit of integer-dimensional space $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the classical results are obtained in the following forms:

$$\mathbf{E} = -\mathbf{e}_\zeta \frac{1}{\varepsilon}[\frac{1}{h}\frac{\partial F_z}{\partial\vartheta}] + \mathbf{e}_\vartheta \frac{1}{\varepsilon}[\frac{1}{h}\frac{\partial F_z}{\partial\zeta}],$$

$$\mathbf{H} = -\mathbf{e}_\zeta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta})(\frac{\partial F_z}{\partial z}) - \mathbf{e}_\vartheta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta})(\frac{\partial F_z}{\partial z}) - \mathbf{e}_z[i\omega F_z + \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z})(\frac{\partial F_z}{\partial z})].$$

### 5.2 TM mode: Transverse magnetic mode

In this subsection, the **E** and **H** of $TM$ mode in an elliptical waveguide filled with FS are obtained, and in the limit case are converted to their classical forms. For transverse magnetic mode, we have:

$$\mathbf{F} = 0 \quad , \quad \mathbf{A} = \mathbf{e}_z A_z(\zeta, \vartheta, z).$$

Using the general solution for **E** and **H**, we obtain:

$$\mathbf{E} = -\mathbf{e}_\zeta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta} + S_\zeta)(\frac{\partial A_z}{\partial z} + S_z A_z) - \mathbf{e}_\vartheta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta} + S_\vartheta)(\frac{\partial A_z}{\partial z} + S_z A_z)$$

$$-\mathbf{e}_z[i\omega A_z + \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z} + S_z)(\frac{\partial A_z}{\partial z} + S_z A_z)],$$

$$\mathbf{H} = \mathbf{e}_\zeta \frac{1}{\mu}[\frac{1}{h}\frac{\partial A_z}{\partial\vartheta} + S_\vartheta A_z - S_z A_\vartheta] + \mathbf{e}_\vartheta \frac{1}{\mu}[-\frac{1}{h}\frac{\partial A_z}{\partial\zeta} - S_\zeta A_z - S_z A_\zeta].$$

In the limit of integer-dimensional space $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the classical results are obtained in the following forms:

$$\mathbf{E} = -\mathbf{e}_\zeta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\zeta})(\frac{\partial A_z}{\partial z}) - \mathbf{e}_\vartheta \frac{i}{\omega\mu\varepsilon}(\frac{1}{h}\frac{\partial}{\partial\vartheta})(\frac{\partial A_z}{\partial z}) - \mathbf{e}_z[i\omega F_z + \frac{i}{\omega\mu\varepsilon}(\frac{\partial}{\partial z})(\frac{\partial A_z}{\partial z})],$$

$$\mathbf{H} = \mathbf{e}_\zeta \frac{1}{\mu}[\frac{1}{h}\frac{\partial A_z}{\partial\vartheta}] - \mathbf{e}_\vartheta \frac{1}{\mu}[\frac{1}{h}\frac{\partial A_z}{\partial\zeta}].$$

# 6. Conclusion

In this work, we calculated generalized vector differential operators, such as gradient, divergence, Laplacian and curl operators in the ECS in FS. By solving Helmholtz equation in the ECS in FS, wave propagation in this system was investigated. The general solution of the scalar wave equation in fractional space in ECS was obtained in terms of the confluent Heun function. The electric and magnetic fields in an elliptical coordinate system in fractional space were developed using the vector potentials, and so TE and TM modes were investigated. Transverse electromagnetic field components in terms of longitudinal electromagnetic field components in ECS in FS were calculated. In all the cases examined, when the dimension is taken to be an integer, the classical results were rewritten to verify the findings.

**Conflicts of Interest.** The authors declare that they have no conflicts of interest regarding the publication of this article.

## References

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, New York, 1983.

[2] M. Sadallah and S. I. Muslih, Solution of the equations of motion for Einstein's field in fractional dimensional space-time, *Int. J. Theor. Phys.* **48** (2009) #3312, https://doi.org/10.1007/s10773-009-0172-6.

[3] F. H. Stillinger, Axiomatic basis for spaces with noninteger dimension, *J. Math. Phys.* **18** (1977) 1224 – 1234, https://doi.org/10.1063/1.523395.

[4] C. Palmer and P. N. Stavrinou, Equations of motion in a non-integer-dimensional space, *J. Phys. A: Math. Gen.* **37** (2004) #6987, https://doi.org/10.1088/0305-4470/37/27/009.

[5] M. Zubair, M. J. Mughal and Q. A. Naqvi, *Electromagnetic Fields and Waves in Fractional Dimensional Space*, Springer, Berlin, 2012.

[6] M. Zubair, M. J. Mughal, Q. A. Naqvi and A. A. Rizvi, Differential electromagnetic equations in fractional space, *Prog. Electromagn. Res.* **114** (2011) 255 – 269, https://doi.org/10.2528/PIER11011403.

[7] S. Khan and M. J. Mughal, General solution for TEM, TE and TM waves in fractional dimensional space and its application in rectangular waveguide filled with fractional space, *J. Electromagn. Waves Appl.* **27** (2013) 2298 – 2307, https://doi.org/10.1080/09205071.2013.840543.

[8] S. Khan, A. Noor and M. J. Mughal, General solution for waveguide modes in fractional space, *Prog. Electromagn. Res.* **33** (2013) 105 – 120, https://doi.org/10.2528/PIERM13062807.

[9] S. Khan, F. M. A. Khan, Gulalai and A. Noor, General solution for electromagnetic wave propagation in cylindrical waveguide filled with fractional space, *Waves Random Complex Media* **33** (2023) 49 – 61, https://doi.org/10.1080/17455030.2021.1874076.

[10] M. Zubair, M. J. Mughal and Q. A. Naqvi, On electromagnetic wave propagation in fractional space, *Nonlinear Anal. Real World Appl.* **12** (2011) 2844 – 2850,https://doi.org/10.1016/j.nonrwa.2011.04.010.

[11] M. Zubair, M. J. Mughal and Q. A. Naqvi, The wave equation and general plane wave solutions in fractional space, *Prog. Electromagn. Res. Lett.* **19** (2010) 137 – 146, https://doi.org/10.2528/PIERL10102103.

[12] M. Zubair, M. J. Mughal and Q. A. Naqvi, An exact solution of the cylindrical wave equation for electromagnetic field in fractional dimensional space, *Prog. Electromagn. Res.* **114** (2011) 443 – 455, https://doi.org/10.2528/PIER11021508.

[13] M. Zubair, M. J. Mughal and Q. A. Naqvi, An exact solution of the spherical wave equation in d-dimensional fractional space, *J. Electromagn. Waves Appl.* **25** (2011) 1481 – 1491, https://doi.org/10.1163/156939311796351605.

[14] V. E. Tarasov, Electromagnetic fields on fractals, *Mod. Phys. Lett. A* **21** (2006) 1587 – 1600, https://doi.org/10.1142/S0217732306020974.

[15] M. Ostoja-Starzewski, Electromagnetism on anisotropic fractal media, *Z. Angew. Math. Phys.* **64** (2013) 381 – 390, https://doi.org/10.1007/s00033-012-0230-z.

[16] A. M. Attiya, Reflection and transmission of electromagnetic wave due to a quasi-fractional-space slab, *Prog. Electromagn. Res.* **24** (2011) 119 – 128, https://doi.org/10.2528/PIERL11051105.

[17] H. Asad, M. Zubair and M. J. Mughal, Reflection and transmission at dielectric-fractal interface, *Prog. Electromagn. Res.* **125** (2012) 543 – 558, https://doi.org/10.2528/PIER12012402.

[18] M. Omar and M. J. Mughal, Behavior of electromagnetic waves at dielectric fractal-fractal interface in fractional spaces, *Prog. Electromagn. Res.* **28** (2013) 229 – 244, https://doi.org/10.2528/PIERM12121903.

[19] D. Baleanu, A. K. Golmankhaneh and A. K. Golmankhaneh, On electromagnetic field in fractional space, *Nonlinear Anal. Real World Appl.* **11** (2010) 288 – 292, https://doi.org/10.1016/j.nonrwa.2008.10.058.

[20] S. I. Muslih and D. Baleanu, Fractional multipoles in fractional space, *Nonlinear Anal. Real World Appl.* **8** (2007) 198 – 203, https://doi.org/10.1016/j.nonrwa.2005.07.001.

[21] A. Shreevastava, P. S. C. Rao and G. S. McGrath, Emergent self-similarity and scaling properties of fractal intra-urban heat islets for diverse global cities, *Phys. Rev. E* **100** (2019) #032142, https://doi.org/10.1103/PhysRevE.100.032142.

[22] M. Kompaniets and K. J. Wiese, Fractal dimension of critical curves in the O(n)-symmetric $\phi^4$ model and crossover exponent at 6-loop order: Loop-erased random walks, self-avoiding walks, Ising, XY, and Heisenberg models, *Phys. Rev. E* **101** (2020) #012104, https://doi.org/10.1103/PhysRevE.101.012104.

[23] J. Zierenberg, N. Fricke, M. Marenz, F. P. Spitzner, V. Blavatska and W. Janke, Percolation thresholds and fractal dimensions for square and cubic lattices with long-range correlated defects, *Phys. Rev. E* **96** (2017) #062125, https://doi.org/10.1103/PhysRevE.96.062125.

[24] O. Praud and H. L. Swinney, Fractal dimension and unscreened angles measured for radial viscous fingering, *Phys. Rev. E* **72** (2005) #011406, https://doi.org/10.1103/PhysRevE.72.011406.

[25] J. R. Westernacher-Schneider, Fractal dimension of turbulent black holes, *Phys. Rev. D* **96** (2017) #104054, https://doi.org/10.1103/PhysRevD.96.104054.

[26] C. Yeh, Elliptical dielectric waveguides, *J. Appl. Phys.* **33** (1962) 3235 – 3243, https://doi.org/10.1063/1.1931144.

[27] S. R. Rengarajan and J. E. Lewis, Dielectric loaded elliptical waveguides, *IEEE Trans. Microw. Theory Tech.* **28** (1980) 1085 – 1089, https://doi.org/10.1109/TMTT.1980.1130229.

[28] N. W. McLachlan, *Theory and Application of Mathieu Functions*, Dover, New York, 1964.

[29] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw Hill, New York, 1953.

[30] M. Hortaçsu, Heun functions and some of their applications in physics, *Adv. High Energy Phys.* **2018** (2018) #8621573, https://doi.org/10.1155/2018/8621573

[31] A. Ronveaux, *Heun's Differential Equations*, Oxford University Press, Oxford, UK, 1995.

[32] S. Y. Slavyanov and W. Lay, *Special Functions: A Unified Theory Based on Singularities*, Oxford University Press, Oxford, UK, 2000.

[33] C. A. Balanis, *Advanced Engineering Electromagnetics*, Wiley, New York, 1989.



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