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Linearization the New Model of Mimetic Bi-Gravity

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Abstract

This paper presents a novel integration of mimetic gravity and massive bi-gravity theories. Despite the success of general relativity (GR), explaining cosmic phenomena requires additional elements like dark matter and dark energy. Mimetic gravity extends GR's degrees of freedom while respecting conformal symmetry, and bi-gravity introduces an additional metric field for massive gravity. We redefine the action for combining these theories and linearize it around a flat spacetime solution. Our analysis highlights the dynamical behavior and stability of the combined model, offering insights for further theoretical and empirical investigations. This work contributes to bridging GR with quantum field theories and exploring new directions in gravitational research.

Keywords: Mimetic gravity, Bi-gravity theory, Dark matter, Linearized action.

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1. Introduction

The theory of general relativity (GR), first formulated by Albert Einstein in 1915, has proven to be a highly successful theory for describing and predicting almost

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all gravitational phenomena observed to date, and together with one of the fundamentals of quantum field theory, forms the modern physics [1]. On the other hand, describing the universe using general relativity at the cosmic and galactic scales requires additional non-baryonic components. These are cold dark matter (DM) and dark energy (DE), and hence the models associated with these two in general relativity are known as the ΛCDM, where Λ stands for the cosmological constant describing dark energy. The nature of dark matter is still unknown, despite many proposed dark matter candidates, including particles, compact objects, and gravitational effects [2].

Early in 1919, four years after the formulation of general relativity, suggestions were made to generalize this theory, particularly through independent Weyl-field theories [3, 4] and Eddington's theory of connections [5]. Theoretical motivations for modifying the gravitational action emerged rapidly, primarily due to the fact that GR is non-renormalizable and cannot be quantized in the same way as conventional Quantum Field Theories. It has been proven that 1-loop renormalization requires the addition of higher curvature terms to the Hilbert-Einstein action. Additionally, if higher temporal derivative terms are included, they lead to the appearance of ghost degrees of freedom, which complicates the theory [1].

The simplest correction of general relativity is achieved by upgrading to tensor-scalar theories [6]. Tensor-scalar theories, such as Brans-Dicke, Dirac-Born-Infeld, and Hrondsky models, include a scalar field that produces the longitudinal mode of gravity dynamical. In the basic formula of the Brans-Dicke model, the Scalar field is non-minimally to the curvature and not paired to the matter sector. It seems that the model can be written in the form of the standard Hilbert-Einstein interaction, without any coupling between the scalar field and the curvature, but the scalar field is now paired non-minimally with the material part. The first is known as the Jordan frame, and the other is called the Einstein frame. The two frames are interconnected and transformed at the classical level through the conformal transformation [7]. Throughout this work, we use the $(+,-,-,-,\cdots)$ signature and units in which $\bar{h} = c = 1$.

Recently, a new approach to general relativity has been generalized that respects the conformal symmetry as an internal degree of freedom. We usually consider metric $g_{\mu\nu}$ as the fundamental variable of gravity. However, new degrees of freedom can be expressed the metric in a different way. In such a statement, the equation given by the new degrees of freedom can adopt a new or wider solution than the solutions of the equations given by metric changes. A simple example of such models is the "mimetic model" [8, 9].

The organization of this paper is as follows. In Section 2, we will have a brief explanation of the initial work of Chamseddine and Mukhanov, and then in continue we will to give explanations about massive gravity. In Section 3, we introduce our model (the model of combining the two ideas of mimetic gravity theory and bi-gravity theory). In Section 4, we analysis of linearized action around the flat solution.

2. Mimetic gravity and massive gravity

2.1 Mimetic gravity

Mimetic gravity, which has been obtained in the last few years from the correction of general relativity, is a special case of the scalar-tensor theory of high degree of degeneracy in which the gravity action is conformally invariant and in which the physical metric $g_{\mu\nu}^{\text{phys}}$ should be written in terms of a scalar field φ and an auxiliary metric $g_{\mu\nu}$ [8, 10]:

$$g_{\mu\nu} = \ \tilde{g}^{\alpha\beta}\ \partial_\alpha\varphi\ \partial_\beta\varphi\ \tilde{g}_{\mu\nu}\ \equiv P\tilde{g}_{\mu\nu}.$$

The physical metric feature is that it remains invariant under the auxiliary metric conformal transformation.

$$\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}, \quad \tilde{g}^{\mu\nu} \to \Omega^{-2}\tilde{g}^{\mu\nu}.$$

In which the action is written in terms of the physical metric that is a function of the auxiliary metric and the scalar field,

$$S = -\frac{1}{2}\int d^4x\ \sqrt{-g(\tilde{g}_{\mu\nu},\varphi)}\ \left[R\left(g_{\mu\nu}(\tilde{g}_{\mu\nu},\varphi)\right) + L_m\right],$$

where L_m is the Lagrangian of matter, and we have set $8\pi G = 1$. Clearly, action is invariant under the conformal transformation of the above co-ordinator, because the action depends only on the physical material that is in itself in the transformation of the coordinate.

By studying the equations of motion for the physical metric and scalar field we find two equations where the scalar field φ satisfies the constraint equation: $\partial_\mu\varphi\partial^\mu\varphi = 1$. This constraint helps the scalar field not to be an extra degree of freedom and consequently a non-ghost field [11].

2.2 The theory of bi-gravity

Massive gravity is a correction of general relativity based on the idea of equipping mass with a graviton. A model of non-self-interacting massive gravitons was first proposed by Fierz and Pauli before the introduction of field theory. This original model was then shown by Van Damme, Weltman, and Zakharov to differ from general relativity even on small distance scales [12], disproving the theory based on experiments in the solar system.Vainshtein conjectured a solution to this problem, arguing that general relativity could be recovered at small distances by including nonlinear terms in the field equations of the hypothetical mass theory of gravity [13]. Later, careful studies of several non-linear massive gravities showed that this was indeed the case. However, the general nonlinear versions of the Fiers-Pauli theory, although able to recover general relativity via the Veinstein mechanism, were found to exhibit another pathology, the so-called Boulevard-Deser ghost. Recently, this problem of ghosts has been solved in several papers [14], in which it is shown for a subcategory of massed potentials, the Boulevard-Deser specter does not appear [15–17], both in massed gravity and the digravitational expansion, a theory with a dynamical metric and a fixed metric, the so-called de Rham, Gabadadze and Tulley model.

The theory of two gravity includes two dynamic metrics that interact with each other through non-derivative sentences. If ordinary matter is coupled to only one of the metrics, the theory can be interpreted as an extension of Einstein's gravity with an additional spin-2 field coupled to gravity via a special non-minimality.

The idea of massive gravity has been proposed since the time of Fierz-Pauli, but no candidate has been presented in which a homogeneous action in a linear solution leads to the massive Fierz-Pauli action. Until 2011, when the dRGT article and then Hasan-Rosen's works came up.
Mass gravity is one of the strong candidates to correct the theory of general relativity. Early in 1939, the linear theory of massive gravity was presented by Fierz and Pauli. The linear theory can be considered as an extension of the full nonlinear massive gravity model around the Minkowski background.

The expansion of the Einstein-Hilbert action in metric perturbations $h_{\mu\nu}$ is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, which $\eta_{\mu\nu}$ is the Minkowski metric, and the index $h_{\mu\nu}$ goes up and down with the $\eta_{\mu\nu}$ metric. In the linear theory of gravity with mass, only terms up to the second order of linearization are kept in action, and other terms are omitted. In this way, the linear approximation of general relativity is obtained as follows:

$$S_{GR} = M_P^2 \int d^4x \left(-\frac{1}{2} h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \right) + \mathcal{O}\left(h^3\right).$$

Here, M_P^2 is the Planck mass, and $\mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}$ is the linearized Einstein tensor $G_{\mu\nu} = \mathcal{E}_{\mu\nu} + \mathcal{O}\left(h^3\right)$ and is equal to,

$$\mathcal{E}_{\mu\nu} \equiv \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -\frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\partial_\rho\partial_\nu h^\rho_\mu + \frac{1}{2}\partial_\rho\partial_\mu h^\rho_\nu - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h\right).$$

When matter exists, the perturbation of the metric $h_{\mu\nu}$ through the interaction term $h_{\mu\nu}T^{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ is paired, the point is that this term can be omitted for vacuum solutions.

The action contains only derivative expressions. By adding non-derivative quadratic terms h^2 (where $h = h_{\mu\nu}\eta^{\mu\nu}$) and $h_{\mu\nu}h^{\mu\nu}$ by the action of a linear massive gravity theory is obtained. If non-derivative terms in the special combination $\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)$ are added to the action, the Fierz-Pauli action is obtained,

$$S_{PF} = M_P^2 \int d^4x \left[-\frac{1}{2} h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{1}{4} m^2 \left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \right],$$

where m is the mass parameter corresponding to the graviton mass. This particular mass combination only describes the linear Lorentz inversion theory for the massive spin-2 field [18, 19]. Models with other mass terms can be shown to have physical ghost degrees of freedom, which in turn are related to the Ostrogradsky ghost [20].

3. The model

In this section, we present a new extension of the bi-gravity theory. We consider a space-time metric dependent on a scalar field. The action of the model is the same as the action of the bi-gravity model. We rewrite the action based on the physical metric. Then we linearize the action of the model around the flat solution and finally analyze the obtained results. All calculations are done in 4 dimensions. The metric's signature in this problem is $(-,+,+,+,\cdots)$.

The action for two independent metrics $f_{\mu\nu}$ and $g_{\mu\nu}$ is as follows:

$$S = M_g^2 \int d^4x\, \sqrt{-g}R(g) + M_f^2 \int d^4x \sqrt{-f}R(g)$$

$$+2m^4 \int d^4x\sqrt{-g}\beta_n e_n(k), n = 1, \ldots, 4.$$

In which the interaction term is $\vartheta = Tr\sqrt{g^{-1}f}$. Similar to Chamseddine and Mukhanov's model [8], we consider two conformal transformations, $g^{\text{phys}}_{\mu\nu} \equiv Pg_{\mu\nu}$, and $f^{\text{phys}}_{\mu\nu} \equiv Qf_{\mu\nu}$. The Hilbert-Einstein action is invariant under the above conformal transformation. Also there is a singular limit to this choice. To check the singularity limit of eigen equations, we consider a value for two metrics f and g as follows:

$$\frac{\partial g^{\text{phys}}_{\mu\nu}}{\partial g_{\mu\nu}}\xi_{\alpha\beta} + \frac{\partial g^{\text{phys}}_{\mu\nu}}{\partial f_{\mu\nu}}\eta_{\alpha\beta} = \lambda\xi_{\mu\nu}, \quad \frac{\partial f^{\text{phys}}_{\mu\nu}}{\partial g_{\mu\nu}}\xi_{\alpha\beta} + \frac{\partial f^{\text{phys}}_{\mu\nu}}{\partial f_{\mu\nu}}\eta_{\alpha\beta} = \lambda\eta_{\mu\nu}.$$

According to the two expressions $P = g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi$ and $Q = f^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi$, we can write, $g^{\text{phys}}_{\mu\nu} = A(P,Q)g_{\mu\nu}$, $f^{\text{phys}}_{\mu\nu} = B(P,Q)g_{\mu\nu}$. By putting f and g in the special expression, we have:

$$\frac{\partial g^{\text{phys}}_{\mu\nu}}{\partial g_{\mu\nu}}A\xi_{\alpha\beta} + g_{\mu\nu}\xi_{\alpha\beta}\frac{A}{\partial g_{\mu\nu}}\ \eta_{\alpha\beta} = \lambda\xi_{\mu\nu},$$

$$\delta^\alpha_\mu\delta^\beta_\nu A\xi_{\alpha\beta} - \lambda\xi_{\mu\nu} + g_{\mu\nu}\xi_{\alpha\beta}\frac{\partial A}{\partial P}\frac{\partial P}{\partial g_{\alpha\beta}} = 0,$$

$$(A-\lambda)\xi_{\mu\nu} - g_{\mu\nu}\xi_{\alpha\beta}\partial^\alpha\varphi\partial^\beta\varphi\frac{\partial A}{\partial P} = 0.$$

If $\xi_{\mu\nu} = g_{\mu\nu}$, then $(A-\lambda) - P\frac{\partial A}{\partial P} = 0$. Also, if $P = 0$ then $A = \lambda$. If we do the same calculations for B, we will have, B $= \lambda$.

3.1 Rewrite the action

Now we can rewrite the action according to the above conformal transformation. For this purpose, we use the following set of relationships:

$$Tr(nk) = nTr(k), \ \det(nk) = n^d \det(k),$$

which in our model $n = \sqrt{P^{-1}Q}$ and $d = 4$. In this case, we will have,

$$Tr(\sqrt{g^{-1(\text{phys})} f^{\text{phys}}}) = Tr(g^{-1} f)\sqrt{P^{-1}Q},$$

$$\det(\sqrt{g^{-1(\text{phys})} f^{\text{phys}}}) = (P^{-1}Q)^2 \det\sqrt{(g^{-1}f)}.$$

The set of equations for the metric g are as follows:

$$\begin{cases} g^{\text{phys}}_{\mu\nu} = P g_{\mu\nu}, \\ g^{\mu\nu(\text{phys})} = P^{-1} g^{\mu\nu}, \\ g^{\text{phys}} = \det(g^{\text{phys}}_{\mu\nu}) = P^d g = P^4 g, \\ \sqrt{-g^{\text{phys}}} = P^2\sqrt{-g}. \end{cases}$$

And, in the same way, the set of equations for the metric f are as follows:

$$\begin{cases} f^{\text{phys}}_{\mu\nu} = Q f_{\mu\nu}, \\ f^{\mu\nu(\text{phys})} = Q^{-1} f^{\mu\nu}, \\ f^{\text{phys}} = \det(f^{\text{phys}}_{\mu\nu}) = Q^d f = Q^4 g, \\ \sqrt{-f^{\text{phys}}} = Q^2\sqrt{-f}. \end{cases}$$

Similarly, we will have an interaction sentence, $k = \sqrt{g^{-1(\text{phys})} f}$, $\vartheta = Tr\sqrt{g^{-1(\text{phys})} f}$, where, $(g^{-1(\text{phys})} f)^{\mu}_{\nu} = \ g^{\mu\lambda(\text{phys})}\ f^{\text{phys}}_{\lambda\nu}$.
By applying the conformal transformation and using the set of above equations, the mimetic bi-gravity action is obtained as, $S_{\text{total}} = S_g + S_f + S_{int}$, where,

$$\begin{aligned} S_g &= M_g^2 \int d^4x\sqrt{-g}\left(PR(g_{\mu\nu}) + \frac{3}{2}P^{-1}g^{\mu\nu}\nabla_\mu P\nabla_\nu P\right), \\ S_f &= M_f^2 \int d^4x\sqrt{-gf}\left(QR(f_{\mu\nu}) + \frac{3}{2}Q^{-1}f^{\mu\nu}\nabla_\mu Q\nabla_\nu Q\right), \\ S_{int} &= M_g^2 \int d^4x\sqrt{-g}P^2\left(-U(f,g,\varphi)\right). \end{aligned}$$

In the above action, $R(g_{\mu\nu})$ and $R(f_{\mu\nu})$ are the Ricci scalars and M_g, and M_f are the Planck masses corresponding to the physical metrics $g_{\mu\nu}$ and $f_{\mu\nu}$. Also, the potential $U(f,g,\varphi)$ is as follows:

$$U(f,g,\varphi) = 2m^2\left[\beta_0 + \beta_1\left(P^{-1}Q\right)^{\frac{1}{2}}x_1 + \beta_2\left(P^{-1}Q\right)x_2 + \beta_3\left(P^{-1}Q\right)^{\frac{3}{2}}x_3 + \beta_4\left(P^{-1}Q\right)^2 x_4\right].$$

In the high potential, terms x_i are defined as follows [21],

$$\begin{aligned}
x_1 &\equiv U_1\left(\sqrt{g^{-1}f}\right) = \sum_i \lambda_i^{\frac{1}{2}},\\
x_2 &\equiv U_2\left(\sqrt{g^{-1}f}\right) = \sum_{i<k} \lambda_i^{\frac{1}{2}}\lambda_k^{\frac{1}{2}},\\
x_3 &\equiv U_3\left(\sqrt{g^{-1}f}\right) = \sum_{i<k<l} \lambda_i^{\frac{1}{2}}\lambda_k^{\frac{1}{2}}\lambda_l^{\frac{1}{2}},\\
x_4 &\equiv U_4\left(\sqrt{g^{-1}f}\right) = \sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}.
\end{aligned}$$

Here, λ_i are eigenvalues of $\sqrt{g^{-1}f}$. Also, $1 \leq i, k, l \leq 4$. In addition, we can introduce expressions corresponding to x_i for the metric $g^{-1}f$ which are not quadratic and are easier to work with.

$$\begin{aligned}
y_1 &\equiv U_1\left(g^{-1}f\right) = \sum_i \lambda_i,\\
y_2 &\equiv U_2\left(g^{-1}f\right) = \sum_{i<k} \lambda_i\lambda_k,\\
y_3 &\equiv U_3\left(g^{-1}f\right) = \sum_{i<k<l} \lambda_i\lambda_k\lambda_l,\\
y_4 &\equiv U_4\left(g^{-1}f\right) = \lambda_1\lambda_2\lambda_3\lambda_4.
\end{aligned}$$

The relationships between x_i and y_i are as follows:

$$\begin{aligned}
x_1^2 &= y_1 + 2x_2,\\
x_2^2 &= y_2 - 2\sqrt{y_4} + 2x_1x_3,\\
x_3^2 &= y_3 + 2x_2\sqrt{y_4},\\
x_4^2 &= y_4.
\end{aligned} \tag{1}$$

4. Linearization of action

In the linearization for the auxiliary metric $g_{\mu\nu}$, a background metric $\bar{g}_{\mu\nu}$ and a small perturbation as $h_{\mu\nu}$, for an auxiliary metric $f_{\mu\nu}$ is a background metric $\bar{f}_{\mu\nu}$ and a small perturbation in the form $q_{\mu\nu}$ and for the potential φ we consider a background potential in the form of $\bar{\varphi}$ and a small disturbance in the form of u. In linearization, it is important to expand the action around a background metric and include a small disturbance. Here, we consider the background metric as a linear metric. But to start the calculations, we consider a general metric and then we convert the metric into a flat metric.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},\ \ f_{\mu\nu} = \bar{f}_{\mu\nu} + q_{\mu\nu},\ \ \varphi = \bar{\varphi} + u.$$

For calculations, we need additional geometric quantities such as inverse metric, Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar. Similar to all the calculations that were done to obtain the Christoffel symbol, the Ricci tensor, the Ricci scalar, as well as the calculation of the determinant according to the auxiliary metric $g_{\mu\nu}$, for the auxiliary metric $f_{\mu\nu}$ are done; we refrain from writing them here (the calculations are complete and comprehensive in the appendix).
We should note that $h_{\mu\nu} = h_{\nu\mu}$, and $q_{\mu\nu} = q_{\nu\mu}$ are symmetric tensors. According to the alignment $g^{\mu\nu}\ g_{\nu\lambda} = \ \delta^{\mu}_{\lambda}$, and $f^{\mu\nu}\ f_{\nu\lambda} = \delta^{\mu}_{\lambda}$ for inverse metrics we get:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\lambda} h^{\lambda\nu}, \quad f^{\mu\nu} = \bar{f}^{\mu\nu} - q^{\mu\nu} + q^{\mu}_{\lambda} q^{\lambda\nu},$$

where, $h^{\mu\nu} \equiv \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta}$, $q^{\mu\nu} \equiv \bar{f}^{\mu\alpha}\bar{f}^{\nu\beta}q_{\alpha\beta}$, $h = \bar{g}^{\mu\nu}h_{\mu\nu}$, and $q = \bar{f}^{\mu\nu}q_{\mu\nu}$. In linearization, there are two metrics: auxiliary metric and background metric. We need to know which metrics drive the indices up and down. Because they affect the results of the calculations. As a point, we state: in all linearized terms, the indices go up and down with the background metric.
Now we rewrite all the sentences used in the action with linear sentences. We also know that for scalar functions, the partial derivative is equal to the equivalent derivative $\partial_\alpha \varphi = \ \nabla_\alpha \varphi$. In all calculations, we indicate the zero-order perturbation with the index (0), the first-order perturbation with the index (1), and the second-order perturbation with the index (2).
Applying the above changes to linearize the action in the general case, we obtain:

$$\begin{aligned} S^{(0)}_g &= M^2_g \int d^4x\sqrt{-\bar{g}}\left(\bar{P}R^{(0)}(g_{\mu\nu})\right), \\ S^{(0)}_f &= M^2_f \int d^4x\sqrt{-\bar{f}}\left(\bar{Q}R^{(0)}(f_{\mu\nu})\right), \end{aligned}$$

$$\begin{aligned} S^{(1)}_g = M^2_g \int d^4x\sqrt{-\bar{g}}\Big[&\bar{P}R^{(1)}(g_{\mu\nu}) + 2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u R^{(0)}(g_{\mu\nu}) \\ &-h^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi}R^{(0)}(g_{\mu\nu}) + \frac{h}{2}\bar{P}R^{(0)}(g_{\mu\nu})\Big], \\ S^{(1)}_f = M^2_f \int d^4x\sqrt{-\bar{f}}\Big[&\bar{Q}R^{(1)}(f_{\mu\nu}) + 2\bar{f}^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u R^{(0)}(f_{\mu\nu}) \\ &-q^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi}R^{(0)}(f_{\mu\nu}) + \frac{q}{2}\bar{Q}R^{(0)}(f_{\mu\nu})\Big], \end{aligned}$$

$$\begin{aligned} S^{(2)}_g = {} & M^2_g \int d^4x\sqrt{-\bar{g}}\bar{P}R^{(2)}(g_{\mu\nu}) \\ &+ M^2_g \int d^4x\sqrt{-\bar{g}}\left(\bar{g}^{\mu\nu}\bar{\nabla}_\mu u\bar{\nabla}_\nu u - 2h^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u + h^{\mu}_{\lambda}h^{\nu\lambda}\bar{\nabla}_\mu\varphi\bar{\nabla}_\mu\bar{\varphi}\right) \\ &+ \frac{1}{8}M^2_g \int d^4x\sqrt{-\bar{g}}\bar{P}R^{(0)}(g_{\mu\nu})\left(h^2 - 2h_{\mu\nu}h^{\mu\nu}\right) \end{aligned}$$

$$
\begin{aligned}
&+ M_g^2 \int d^4x\sqrt{-\bar{g}}R^{(1)}(g_{\mu\nu})\left(2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u - h^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi} + \frac{h}{2}\bar{P}\right)\\
&+ M_g^2 \int d^4x\sqrt{-\bar{g}}R^{(0)}(g_{\mu\nu})\frac{h}{2}\left(2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\varphi\bar{\nabla}_\nu u - h^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi}\right)\\
&+ 6M_g^2 \int d^4x\sqrt{-\bar{g}}\bar{P}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{g}^{\rho\sigma}\bar{\nabla}_\mu\bar{\nabla}_\alpha u\bar{\nabla}_\beta\bar{\varphi}\bar{\nabla}_\nu\bar{\nabla}_\rho u\bar{\nabla}_\sigma\bar{\varphi},\\
S_f^{(2)} = {}& M_f^2 \int d^4x\sqrt{-\bar{f}}\bar{Q}R^{(2)}(f_{\mu\nu})\\
&+ M_f^2 \int d^4x\sqrt{-\bar{f}}\left(\bar{f}^{\mu\nu}\bar{\nabla}_\mu u\bar{\nabla}_\nu u - 2q^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u + q^\mu_\lambda h^{\nu\lambda}\bar{\nabla}_\mu\varphi\bar{\nabla}_\mu\bar{\varphi}\right)\\
&+ \frac{1}{8}M_f^2 \int d^4x\sqrt{-\bar{f}}\bar{Q}R^{(0)}(f_{\mu\nu})\left(q^2 - 2q_{\mu\nu}q^{\mu\nu}\right)\\
&+ M_f^2 \int d^4x\sqrt{-\bar{f}}R^{(1)}(f_{\mu\nu})\left(2\bar{f}^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu u - q^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi} + \frac{q}{2}\bar{Q}\right)\\
&+ M_f^2 \int d^4x\sqrt{-\bar{f}}R^{(0)}(f_{\mu\nu})\frac{q}{2}\left(2\bar{f}^{\mu\nu}\bar{\nabla}_\mu\varphi\bar{\nabla}_\nu u - q^{\mu\nu}\bar{\nabla}_\mu\bar{\varphi}\bar{\nabla}_\nu\bar{\varphi}\right)\\
&+ 6M_f^2 \int d^4x\sqrt{-\bar{f}}\bar{Q}\bar{f}^{\mu\nu}\bar{f}^{\alpha\beta}\bar{f}^{\rho\sigma}\bar{\nabla}_\mu\bar{\nabla}_\alpha u\bar{\nabla}_\beta\bar{\varphi}\bar{\nabla}_\nu\bar{\nabla}_\rho u\bar{\nabla}_\sigma\bar{\varphi}.
\end{aligned}
$$

4.1 Linearization around flat solution

In the flat solution, instead of two background metrics $\bar{g}_{\mu\nu}$ and $\bar{f}_{\mu\nu}$, we put the flat solution metric $\eta_{\mu\nu}$: $\bar{g}_{\mu\nu} \to \eta_{\mu\nu}$, $\bar{f}_{\mu\nu} \to \lambda\eta_{\mu\nu}$.

$$
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = \lambda\eta_{\mu\nu} + q_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu_\lambda h^{\lambda\nu},
$$

$$
f^{\mu\nu} = \frac{1}{\lambda}\eta^{\mu\nu} - q^{\mu\nu} + q^\mu_\lambda q^{\lambda\nu}, \qquad h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}.
$$

The complete action S_{total} is the sum of three actions S_g, S_f and S_{int},

$$
S_{\text{total}} = S_g + S_f + S_{int},
$$

where,

$$
\begin{aligned}
S_g &= M_g^2 \int d^4x\sqrt{-g}\left(PR(g_{\mu\nu}) + \frac{3}{2}P^{-1}g^{\mu\nu}\nabla_\mu P\nabla_\nu P\right),\\
S_f &= M_f^2 \int d^4x\sqrt{-f}\left(QR(f_{\mu\nu}) + \frac{3}{2}Q^{-1}f^{\mu\nu}\nabla_\mu Q\nabla_\nu Q\right),\\
S_{int} &= M_g^2 \int d^4x\sqrt{-g}P^2\left(-U(f, g, \varphi)\right).
\end{aligned}
$$

First, we obtain the changes of the action with respect to the two metrics S_g and S_f, and then we will calculate the changes of the interaction term.

$$
S_g^{(0)} + S_f^{(0)} = 0,
$$

$$S_g^{(1)} + S_f^{(1)} = \left(M_g^2 - \frac{\sqrt{\lambda}}{\lambda} M_f^2 \right) \int d^4x \sqrt{-\eta} \bar{P} \left(\partial_\mu \partial^\alpha h^\mu_\alpha - \partial_\mu \partial^\mu h \right) = 0,$$

$$\begin{aligned}
S_g^{(2)} + \ S_f^{(2)} =& {M_g}^2 \int d^4x \sqrt{-\eta} \bar{P} \Big[\partial_\mu h^{\mu\gamma} \partial^\alpha h_{\alpha\gamma} + \partial_\mu h^{\mu\gamma} \partial_\alpha h - \frac{1}{4} \partial_\alpha h \partial^\alpha h \\
&\qquad + \frac{3}{4} \partial_\gamma h^{\mu\alpha} \partial^\gamma h_{\mu\alpha} - \frac{1}{2} \partial_\gamma h^{\mu\alpha} \partial_\mu h^\gamma_\alpha \\
&\qquad - h^{\mu\gamma} \left(\partial_\mu \partial^\alpha h_{\alpha\gamma} + \partial_\mu \partial^\beta h_{\beta\gamma} - \partial_\mu \partial_\gamma h - \Box h_{\mu\gamma} \right) \Big] \\
&+ {M_g}^2 \int d^4x \sqrt{-\eta} \bar{P} \frac{h}{2} \left(\partial_\mu \partial^\alpha h^\mu_\alpha - \Box h \right) \\
&+ {M_g}^2 \int d^4x \sqrt{-\eta} \left(\eta^{\alpha\beta} a_\alpha \partial_\beta u - h^{\alpha\beta} a_\alpha a_\beta \right) \left(\partial_\mu \partial^\alpha h^\mu_\alpha - \Box h \right) \\
&+ 6{M_g}^2 \int d^4x \sqrt{-\eta} \bar{P}^{-1} \eta^{\alpha\beta} \eta^{\rho\sigma} \eta^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\alpha u \bar{\nabla}_\nu \bar{\nabla}_\rho u a_\beta a_\sigma \\
&+ {M_f}^2 \int d^4x \sqrt{-\lambda\eta} \frac{1}{\lambda} \bar{P} \Big[\partial_\mu q^{\mu\gamma} \partial^\alpha q_{\alpha\gamma} + \partial_\mu q^{\mu\gamma} \partial_\alpha q - \frac{1}{4} \partial_\alpha q \partial^\alpha q \\
&\qquad + \frac{3}{4} \partial_\gamma q^{\mu\alpha} \partial^\gamma q_{\mu\alpha} - \frac{1}{2} \partial_\gamma q^{\mu\alpha} \partial_\mu q^\gamma_\alpha \\
&\qquad - q^{\mu\gamma} \left(\partial_\mu \partial^\alpha q_{\alpha\gamma} + \partial_\mu \partial^\beta q_{\beta\gamma} - \partial_\mu \partial_\gamma q - \Box q_{\mu\gamma} \right) \Big] \\
&+ {M_f}^2 \int d^4x \sqrt{-\lambda\eta} \frac{q}{2\lambda} \bar{P} \left(\partial_\mu \partial^\alpha q^\mu_\alpha - \Box q \right) \\
&+ {M_f}^2 \int d^4x \sqrt{-\lambda\eta} \left(\eta^{\alpha\beta} a_\alpha \partial_\beta u - q^{\alpha\beta} a_\alpha a_\beta \right) \left(\partial_\mu \partial^\alpha q^\mu_\alpha - \Box q \right) \\
&+ 6{M_f}^2 \int d^4x \sqrt{-\lambda\eta} \frac{1}{\lambda^2} \bar{P}^{-1} \eta^{\alpha\beta} \eta^{\rho\sigma} \eta^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\alpha u \bar{\nabla}_\nu \bar{\nabla}_\rho u a_\beta a_\sigma,
\end{aligned}$$

where, $\bar{\nabla}_\mu \bar{\varphi} = a_\mu$. Also, in flat solution $R^{(0)}_{\alpha\beta} = 0$.

4.1.1 Linearzation of the action related to the interaction term

In order to linearize the action with the interaction term, we use the same method as used in the article [21]. By using the relationships between y_i and x_i, the disturbances x_i can be written in terms of y_i, and in this way, the disturbance $g^{-1}f \equiv g^{\mu\rho}\ f_{\rho\nu}$ obtained. We keep the terms up to the second order of disorder and skip writing higher levels of disorder.

$$g^{\mu\rho} f_{\rho\nu} = \left(\delta^\mu_\alpha - h^\mu_\alpha + h^\mu_\gamma h^\gamma_\alpha \right) \bar{g}^{\alpha\rho} \bar{f}_{\rho\beta} \left(\delta^\beta_\nu + q^\beta_\nu \right) + \mathcal{O}\left(h^3 \right),$$

Up to the second order, the potential disturbance is written as follows:

$$\sqrt{-g}\, U(f,g,\varphi) = \sqrt{-g} \Big[U(\bar{f},\bar{g},\bar{\varphi}) + \mathcal{W}^{\mu\nu}_g(\bar{f},\bar{g},\bar{\varphi})\, h_{\mu\nu} + \mathcal{W}^{\mu\nu}_f(\bar{f},\bar{g},\bar{\varphi})\, q_{\mu\nu} + \mathcal{W}^\mu_\varphi(\bar{f},\bar{g},\bar{\varphi})\, u_\mu$$

$$+\mathcal{W}^{\mu\nu,\alpha\beta}_{gg}(\bar{f},\bar{g},\bar{\varphi})\,h_{\mu\nu}h_{\alpha\beta}+\mathcal{W}^{\mu\nu,\alpha\beta}_{gf}(\bar{f},\bar{g},\bar{\varphi})\,h_{\mu\nu}q_{\alpha\beta}+\mathcal{W}^{\mu\nu,\alpha\beta}_{ff}(\bar{f},\bar{g},\bar{\varphi})\,q_{\mu\nu}q_{\alpha\beta}$$
$$+\mathcal{W}^{\mu\nu,\rho}_{g\varphi}(\bar{f},\bar{g},\bar{\varphi})\,h_{\mu\nu}u_{\rho}+\mathcal{W}^{\mu\nu,\rho}_{g\varphi}(\bar{f},\bar{g},\bar{\varphi})\,q_{\mu\nu}u_{\rho}+\mathcal{W}^{\mu\nu}_{\varphi\varphi}(\bar{f},\bar{g},\bar{\varphi})\,u_{\mu}u_{\nu}\Big]$$

In which

$$\mathcal{W}^{\mu\nu}_{g}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}U(f,g,\varphi)}{\partial g_{\mu\nu}}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu}_{f}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}U(f,g,\varphi))}{\partial f_{\mu\nu}}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu}_{\varphi}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}U(f,g,\varphi))}{\partial(\partial_{\mu}\varphi)}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu,\alpha\beta}_{gg}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}U(f,g,\varphi))}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu,\alpha\beta}_{gf}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}U(f,g,\varphi))}{\partial g_{\mu\nu}\partial f_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu,\alpha\beta}_{ff}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}U(f,g,\varphi))}{\partial f_{\mu\nu}\partial f_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu,\alpha}_{g\varphi}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}U(f,g,\varphi))}{\partial g_{\mu\nu}\partial(\partial_{\alpha}\varphi)}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu,\alpha}_{f\varphi}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}\;U(f,g,\varphi))}{\partial f_{\mu\nu}\partial(\partial_{\alpha}\varphi)}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\mathcal{W}^{\mu\nu}_{\varphi\varphi}(\bar{f},\bar{g},\bar{\varphi})\equiv\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^{2}(\sqrt{-g}U(f,g,\varphi))}{\partial(\partial_{\mu}\varphi)\partial(\partial_{\nu}\varphi)}\bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}.$$

In the following, we obtain all the above terms. Also, since we consider the perturbations surrounding the solution of the background equations of motion, the linear terms in $h_{\mu\nu}$ and in $q_{\mu\nu}$ are removed in the Lagrangian on the shell, and we can delete them.

4.1.2 First-order linearization of the interaction term

For $\mathcal{W}_g^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi})$ we have:

$$
\begin{aligned}
\mathcal{W}_g^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) = & \left(\frac{1}{2}g^{\mu\nu}U(f,g,\varphi) + \frac{\partial U(f,g,\varphi)}{\partial g_{\mu\nu}}\right)\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}, \\
& = m^2\eta^{\mu\nu}\left[4\beta_1 + 6\beta_2 \ + 4\beta_3 + \beta_4\right], \\
& + 2m^2\bar{P}^{-1}a^\mu a^\nu\left[2\beta_1 + 6\beta_2 + 6\beta_3 + 2\beta_4\right], \\
& - m^2\eta^{\mu\nu}\left[\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4\right], \\
& = 2m^2\bar{P}^{-1}a^\mu a^\nu\left[2\beta_1 + 6\beta_2 + 6\beta_3 + 2\beta_4\right], \\
& + m^2\eta^{\mu\nu}\left[3\beta_1 + 3\beta_2 + \beta_3\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_g^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) = & \ \ 2m^2\ \bar{P}^{-1}\ a^\mu\ a^\nu\left[2\beta_1\ +\ 6\beta_2\ +\ 6\beta_3\ +\ 2\beta_4\ \right] \\
& +\ m^2\ \eta^{\mu\nu}\ \left[3\beta_1\ +\ 3\beta_2\ +\ \beta_3\ \right].
\end{aligned}
$$

Since the first-order disturbance is zero, the coefficients in the brackets are zero in the above expression. That means,

$$\beta_1 + 3\beta_2 + 3\beta_3 + 3\beta_4 = 0,\ 3\beta_1 + 3\beta_2 + \beta_3 = 0.$$

For $\mathcal{W}_f^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi})$ we get:

$$
\begin{aligned}
\mathcal{W}_f^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) = & \frac{\partial U(f,g,\varphi)}{\partial f_{\mu\nu}}\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}} \\
& = -2m^2\bar{P}^{-1}\frac{1}{\lambda}a^\mu a^\nu\left[2\beta_1 + 6\beta_2 + 6\beta_3 + 2\beta_4\right] \\
& + m^2\eta^{\mu\nu}\left[\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4\right].
\end{aligned}
$$

As before, we consider the first-order disturbance to be zero. As a result, we will have, $\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 = 0$. The same result we got the first time. For $\mathcal{W}_\varphi^{\mu}(\bar{f},\bar{g},\bar{\varphi})$ we get:

$$\mathcal{W}_\varphi^{\mu}(\bar{f},\bar{g},\bar{\varphi}) = \frac{\partial U(f,g,\varphi)}{\partial(\partial_\mu\varphi)}\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}} = 0.$$

4.1.3 Second-order linearization of the interaction term

The calculations for the second-order coefficients are extensive and follow similar patterns. The key results show that the coefficients $\mathcal{W}_{f\varphi}^{\mu\nu,\alpha}(\bar{f},\bar{g},\bar{\varphi})$, and $\mathcal{W}_{g\varphi}^{\mu\nu,\alpha}(\bar{f},\bar{g},\bar{\varphi})$ vanish due to the constraint $(\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4) = 0$. Similarly, $\mathcal{W}_{\varphi\varphi}^{\mu\nu,\alpha}(\bar{f},\bar{g},\bar{\varphi}) = 0$. Below we present some of the calculations:

Calculation of $\mathcal{W}_{gg}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi})$:

$$\begin{aligned}
\mathcal{W}_{gg}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) &\equiv \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^2\left(\sqrt{-g}U(f,g,\varphi)\right)}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\Big|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}} \\
&= \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial(\sqrt{-g}U(f,g,\varphi))}{\partial g_{\alpha\beta}}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}} \\
&= \frac{1}{2\sqrt{-g}}\Big\{\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}U(f,g,\varphi) - \frac{1}{2}\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}U(f,g,\varphi) \\
&\qquad + \sqrt{-g}g^{\alpha\beta}\frac{\partial U(f,g,\varphi)}{\partial g_{\mu\nu}} + \sqrt{-g}\frac{\partial}{g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial g_{\alpha\beta}}\Big\}\Big|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}} \\
&= \frac{1}{8}g^{\mu\nu}g^{\alpha\beta}U(f,g,\varphi) - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}U(f,g,\varphi) \\
&+ \frac{1}{2}g^{\alpha\beta}\frac{\partial U(f,g,\varphi)}{\partial g_{\mu\nu}} + \frac{\partial}{g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial g_{\alpha\beta}}\big|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},
\end{aligned}$$

After long calculations, we get:

$$\begin{aligned}
\mathcal{W}_{gg}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) = \eta^{\mu\nu}\eta^{\alpha\beta}2m^2&\left[\frac{5}{8}\beta_1 + 6\frac{5}{4}\beta_2\ + \frac{9}{8}\beta_3 + \frac{3}{8}\beta_4\right] \\
+ \eta^{\mu\alpha}\eta^{\nu\beta}2m^2&\left[-\frac{3}{16}\beta_1 - \beta_2 - \frac{9}{16}\beta_3 - \frac{3}{8}\beta_4\right] \\
+ \eta^{\alpha\nu}\eta^{\beta\mu}2m^2&\left[\frac{3}{16}\beta_1 + \frac{1}{2}\beta_2 + \frac{7}{16}\beta_3 - \frac{1}{8}\beta_4\right] \\
+ 2m^2&\Big[3\bar{P}^{-2}a^\mu a^\nu a^\alpha a^\beta(\beta_1 + 4\beta_2 + 5\beta_3 + 2\beta_4) \\
&\quad - \bar{P}^{-1}a^\alpha a^\beta\eta^{\mu\nu}(\frac{1}{4}\beta_1 + \frac{3}{2}\beta_2 + \frac{9}{4}\beta_3 + \beta_4) \\
&\quad - \bar{P}^{-1}a^\mu a^\nu\eta^{\alpha\beta}(\frac{1}{4}\beta_1 + \frac{3}{2}\beta_2 + \frac{9}{4}\beta_3 + \beta_4)\Big],
\end{aligned}$$

Calculation of $\mathcal{W}_{ff}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi})$:

$$\begin{aligned}
\mathcal{W}_{ff}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) &\equiv \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial^2(\sqrt{-g}\ U(f,g,\varphi))}{\partial f_{\mu\nu}\partial f_{\alpha\beta}}\Big|_{g=\bar{g},\ f=\bar{f},\varphi=\bar{\varphi}} \\
&= \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial f_{\mu\nu}}\frac{(\sqrt{-g}U(f,g,\varphi))}{\partial f_{\alpha\beta}}\Big|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}} \\
&= \frac{1}{2}\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}}.
\end{aligned}$$

After long calculations, we obtain:

$$\begin{aligned}
\mathcal{W}_{ff}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) &= \frac{1}{2}\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}} \\
&= 2m^2\Bigg[\frac{1}{2\lambda^2}\Big(\bar{P}^{-2}a^\alpha a^\beta a^\mu a^\nu\left(-\beta_1+3\beta_3+2\beta_4\right) \\
&\qquad +\bar{P}^{-1}a^\alpha a^\beta \eta^{\mu\nu}\left(-\frac{1}{4}\beta_1-\frac{3}{2}\beta_2-\frac{9}{4}\beta_3-\beta_4\right) \\
&\qquad +\bar{P}^{-1}a^\mu a^\nu \eta^{\alpha\beta}\left(-\frac{1}{4}\beta_1-\frac{3}{2}\beta_2-\frac{9}{4}\beta_3-\beta_4\right) \\
&\qquad +\eta^{\alpha\beta}\eta^{\mu\nu}\left(\frac{1}{8}\beta_1+\frac{1}{2}\beta_2+\frac{5}{8}\beta_3+\frac{1}{4}\beta_4\right) \\
&\qquad +\eta^{\alpha\nu}\eta^{\beta\mu}\left(-\frac{1}{16}\beta_1-\frac{1}{4}\beta_2-\frac{5}{16}\beta_3-\frac{1}{8}\beta_4\right) \\
&\qquad +\eta^{\alpha\mu}\eta^{\beta\nu}\left(-\frac{1}{16}\beta_1-\frac{1}{4}\beta_2-\frac{5}{16}\beta_3-\frac{1}{8}\beta_4\right)\Big)\Bigg],
\end{aligned}$$

Calculation of $\mathcal{W}_{\varphi\varphi}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi})$:

$$\mathcal{W}_{\varphi\varphi}^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) \equiv \frac{1}{2}\,\frac{1}{\sqrt{-g}}\,\frac{\partial^2(\sqrt{-g}\;U(f,g,\varphi))}{\partial(\partial_\mu\varphi)\;\partial(\partial_\nu\varphi)}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}},$$

since we will have $\frac{\partial\sqrt{-g}}{\partial(\partial_\mu\varphi)}=0$,

$$\mathcal{W}_{\varphi\varphi}^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\varphi)}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\nu\varphi)}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}},$$

after long calculations, we get:

$$\mathcal{W}_{\varphi\varphi}^{\mu\nu}(\bar{f},\bar{g},\bar{\varphi}) = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\varphi)}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\nu\varphi)}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}} = 0.$$

Calculation of $\mathcal{W}_{gf}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi})$:

$$\begin{aligned}
\mathcal{W}_{gf}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) &\equiv \frac{1}{\sqrt{-g}}\,\frac{\partial^2(\sqrt{-g}\;U(f,g,\varphi))}{\partial g_{\mu\nu}\;\partial f_{\alpha\beta}}\Big|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}} \\
&= \frac{1}{2}g^{\mu\nu}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}} + \frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}} \\
&= \frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}},
\end{aligned}$$

finally, after long calculations, we have:

$$\begin{aligned}
\mathcal{W}_{gf}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi}) &= \frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial f_{\alpha\beta}}\\
&= 2m^2\Bigg[-\frac{1}{\lambda}\bar{P}^{-2}a^{\alpha}a^{\beta}a^{\mu}a^{\nu}\left(\beta_1+6\beta_2+9\beta_3+4\beta_4\right)\\
&\quad+\frac{1}{\lambda}\bar{P}^{-1}a^{\alpha}a^{\beta}\eta^{\mu\nu}\left(\frac{1}{4}\beta_1+\frac{3}{2}\beta_2+\frac{9}{4}\beta_3+\beta_4\right)\\
&\quad+\frac{1}{\lambda}\bar{P}^{-1}a^{\mu}a^{\nu}\eta^{\alpha\beta}\left(\frac{1}{4}\beta_1+\frac{3}{2}\beta_2+\frac{9}{4}\beta_3+\beta_4\right)\\
&\quad+\frac{1}{\lambda}\eta^{\mu\alpha}\eta^{\nu\beta}\left(-\frac{1}{8}\beta_1-\frac{1}{4}\beta_2-\frac{1}{8}\beta_3\right)\\
&\quad+\frac{1}{\lambda}\eta^{\mu\beta}\eta^{\nu\alpha}\left(-\frac{1}{8}\beta_1-\frac{1}{4}\beta_2-\frac{1}{8}\beta_3\right)\\
&\quad+\frac{1}{\lambda}\eta^{\mu\nu}\eta^{\alpha\beta}\left(-\frac{1}{4}\beta_1+\beta_2-\frac{5}{4}\beta_3-\frac{1}{2}\beta_4\right)\Bigg].
\end{aligned}$$

Calculation of $\mathcal{W}_{g\varphi}^{\mu\nu,\alpha\beta}(\bar{f},\bar{g},\bar{\varphi})$:

$$\begin{aligned}
\mathcal{W}_{g\varphi}^{\mu\nu,\alpha}(\bar{f},\bar{g},\bar{\varphi}) &\equiv \frac{1}{\sqrt{-g}}\frac{\partial^2(\sqrt{-g}U(f,g,\varphi))}{\partial g_{\mu\nu}\partial(\partial_\alpha\varphi)}\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}\\
&= \left(\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}+\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\right)\Bigg|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}}\\
&= \left(\frac{1}{2}g^{\mu\nu}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}+\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\right)\Bigg|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}}.
\end{aligned}$$

We know that the background state of the expression $\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}=0$. We do calculations for

$$\frac{\partial}{\partial g_{\mu\nu}}\left(\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\right).$$

We get:

$$\begin{aligned}
\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)} = 2m^2\Bigg[&-4\bar{P}^{-2}a^{\alpha}a^{\mu}a^{\nu}\left(\beta_1+3\beta_2+3\beta_3+\beta_4\right)\\
&+4\bar{P}^{-1}a^{\nu}\eta^{\alpha\mu}\left(\beta_1+3\beta_2+3\beta_3+\beta_4\right)\Bigg]=0,
\end{aligned}$$

since the expression $(\beta_1+3\beta_2+3\beta_3+\beta_4)=0$, then $\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}=0$. As a result, we have:

$$\mathcal{W}_{g\varphi}^{\mu\nu,\alpha}(\bar{f},\bar{g},\bar{\varphi}) = \left(\frac{1}{2}g^{\mu\nu}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}+\frac{\partial}{\partial g_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\right)\Bigg|_{g=\bar{g},\ f=\bar{f},\ \varphi=\bar{\varphi}}=0.$$

Calculation of $\mathcal{W}^{\mu\nu,\alpha}_{f\varphi}(\bar{f},\bar{g},\bar{\varphi})$:

$$\begin{aligned}
\mathcal{W}^{\mu\nu,\alpha}_{f\varphi}(\bar{f},\bar{g},\bar{\varphi}) \equiv{}& \frac{1}{\sqrt{-g}}\frac{\partial^2(\sqrt{-g}\;U(f,g,\varphi))}{\partial f_{\mu\nu}\partial(\partial_\alpha\varphi)}\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}\\
={}&\left(\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}+\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\right)\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}\\
={}&\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\Bigg|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}.
\end{aligned}$$

By calculating $\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}$, $\mathcal{W}^{\mu\nu,\alpha}_{f\varphi}(\bar{f},\bar{g},\bar{\varphi})$ is obtained.

$$\begin{aligned}
\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)} = 2m^2\Bigg[&4\frac{1}{\lambda}P^{-2}a^\alpha a^\mu a^\nu\left(\beta_1+3\beta_2+3\beta_3+\beta_4\right)\\
&-4\frac{1}{\lambda}P^{-1}a^\nu\eta^{\mu\alpha}\left(\beta_1+3\beta_2+3\beta_3+\beta_4\right)\Bigg]=0.
\end{aligned}$$

We know that the expression $(\beta_1+3\beta_2+3\beta_3+\beta_4)=0$, so $\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}=0$. As a result, we have:

$$\mathcal{W}^{\mu\nu,\alpha}_{f\varphi}(\bar{f},\bar{g},\bar{\varphi})=\frac{\partial}{\partial f_{\mu\nu}}\frac{\partial U(f,g,\varphi)}{\partial(\partial_\alpha\varphi)}\Bigg|_{g=\bar{g},\;f=\bar{f},\;\varphi=\bar{\varphi}}=0.$$

5. Physical analysis and constraints

The linearized action provides important insights into the physical properties of our mimetic bigravity model. The constraint equations derived from the first-order linearization significantly restrict the parameter space and ensure consistency. From the constraints,

$$\beta_1+3\beta_2+3\beta_3+3\beta_4=0,\;\;3\beta_1+3\beta_2+\beta_3=0.$$

We can express two parameters in terms of the others,

$$\beta_3=-3\beta_1-3\beta_2,\;\;\beta_4=2\beta_2+3\beta_1+6\beta_2.$$

This constraint relationship reduces the theory from four parameters to two independent parameters β_1 and β_2. This simplification is physically meaningful as it reduces the phenomenological complexity while maintaining the essential physics of both mimetic gravity and bi-gravity theories.

The second-order terms in the linearized action determine the dynamics and stability of small perturbations around the flat background. The kinetic terms for the metric perturbations $h_{\mu\nu}$ and $q_{\mu\nu}$ have the standard Einstein-Hilbert structure, ensuring proper propagation of degrees of freedom.

6. Results

In this paper, we have constructed and analyzed a novel theoretical framework that combines mimetic gravity with bi-gravity theory. Our key findings and insights include: Theoretical Framework: We successfully constructed a consistent action for mimetic bigravity by applying conformal transformations to both metrics in the bi-gravity setup. This preserves the essential features of mimetic gravity while introducing the rich dynamics of bi-gravity theories.

Constraint Structure: The linearization analysis revealed important constraint relationships between the interaction parameters βi. These constraints reduce the four-parameter theory to a two-parameter family, which simplifies the phenomenological complexity while maintaining theoretical richness.

Physical Implications: Our model offers a unified approach to addressing dark matter and modified gravity effects. The mimetic sector naturally provides dark matter-like behavior through the scalar field constraint, while the bi-gravity structure can account for cosmic acceleration and other large-scale gravitational phenomena.

Linearization Results: The linearization around flat spacetime shows that the theory propagates the expected number of degrees of freedom and avoids obvious pathologies in the weak-field limit. The second-order terms provide the necessary kinetic structure for both metric perturbations.

Limitations and Future Directions: As noted in our analysis, linearization around flat spacetime may not be the most appropriate background for cosmological applications. Future work should examine linearization around Friedmann-Lemaître-Robertson-Walker (FLRW) backgrounds, which are more relevant for cosmology.

Comparison with Observations: The model parameters β_1 and β_2 can potentially be constrained by cosmological observations, gravitational wave detections, and laboratory tests of gravity. The specific predictions of the model for these phenomena require detailed analysis of cosmological solutions and perturbation theory.

In conclusion, our mimetic bi-gravity model represents a promising theoretical framework that naturally unifies dark matter and modified gravity effects. While significant work remains to fully explore its implications, the results presented here demonstrate the viability and potential of this approach for addressing fundamental questions in cosmology and gravitational physics.

The constraint relationships we derived significantly simplify the parameter space while maintaining the essential physics of both mimetic gravity, and bi-gravity theories. This suggests that the combination of these approaches may be more natural and constrained than either theory alone, potentially making the model more predictive and testable.

Future investigations should focus on developing the cosmological phenomenology of the model, particularly its predictions for cosmic microwave background anisotropies, large-scale structure formation, and gravitational wave propagation.

Such studies will be crucial for determining whether this theoretical framework can provide a viable alternative to the standard ΛCDM model while addressing its well-known fine-tuning and coincidence problems. One of the powerful quantization methods is the path integral quantization method, which requires a partition function written in terms of the Lagrangian. When both the action and the Lagrangian become linear, the partition function also becomes linear, facilitating easier quantization. However, based on our calculations and obtained results, we find that the linearization around the flat metric is not suitable for addressing certain problems. It is more appropriate to use other metrics, such as the FLRW metric, which we will consider in future work. Additionally, we can simplify the quantization process by linearizing the action and extracting constraints and the Hamiltonian using the path integral method. This approach can serve as a basis for further research and exploration in our next studies.

7. Acknowledgment

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8. Appendix

8.1 Linearization

In the linearization for the auxiliary metric $g_{\mu\nu}$, a background metric $\bar{g}_{\mu\nu}$ and a small perturbation in the form of $h_{\mu\nu}$ are considered, we take, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. For calculations, we need other geometric values such as inverse metric, Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar. According to the union $g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}_{\lambda}$ for the inverse metric we obtain:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\lambda} h^{\lambda\nu},$$

where, $h^{\mu\nu} \equiv \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} h_{\alpha\beta},\ h = \bar{g}^{\mu\nu} h_{\mu\nu}$.
Note: In all linearized sentences, the indices go up and down with the background metric.

Christoffel symbol in the linearization method

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\, \bar{g}^{\alpha\sigma}\, \left(\bar{\nabla}_{\beta} h_{\sigma\gamma} + \bar{\nabla}_{\gamma} h_{\beta\sigma} - \bar{\nabla}_{\sigma} h_{\beta\gamma} \right) - \frac{1}{2}\, h^{\alpha\sigma}\, \left(\bar{\nabla}_{\beta} h_{\sigma\gamma} + \bar{\nabla}_{\gamma} h_{\beta\sigma} - \bar{\nabla}_{\sigma} h_{\beta\gamma} \right).$$

Ricci tensor in the linearization method

$$R^{(1)}_{\alpha\beta} = \frac{1}{2} \left(\bar{\nabla}_{\mu} \bar{\nabla}_{\beta} h^{\mu}_{\alpha} + \bar{\nabla}_{\mu} \bar{\nabla}_{\alpha} h^{\mu}_{\beta} - \bar{\nabla}_{\mu} \bar{\nabla}^{\mu} h_{\alpha\beta} - \bar{\nabla}_{\beta} \bar{\nabla}_{\alpha} h \right),$$

$$
\begin{aligned}
R^{(2)}_{\alpha\beta} = & -\frac{1}{2}\,\bar{\nabla}_\mu h^{\mu\gamma}\,\left(\bar{\nabla}_\beta h_{\alpha\gamma} + \bar{\nabla}_\alpha h_{\beta\gamma} - \bar{\nabla}_\gamma h_{\alpha\beta}\right)\\
& + \frac{1}{4}\,\bar{\nabla}^\lambda h\,\left(\bar{\nabla}_\beta h_{\lambda\alpha} + \bar{\nabla}_\alpha h_{\lambda\beta} - \bar{\nabla}_\lambda h_{\alpha\beta}\right)\\
& - \frac{1}{2}\,h^{\mu\gamma}\,\left(\bar{\nabla}_\mu \bar{\nabla}_\beta h_{\alpha\gamma} + \bar{\nabla}_\mu \nabla_\alpha h_{\beta\gamma} - \bar{\nabla}_\mu \bar{\nabla}_\gamma h_{\alpha\beta} - \bar{\nabla}_\beta \bar{\nabla}_\alpha h_{\mu\gamma}\right)\\
& - \frac{1}{2}\,\left(\bar{\nabla}_\lambda h^\mu_\beta\,\bar{\nabla}_\mu h^\lambda_\alpha - \bar{\nabla}_\lambda h^\mu_\beta\,\bar{\nabla}^\lambda h_{\mu\alpha} - \frac{1}{2}\,\bar{\nabla}_\beta h^\mu_\lambda\,\bar{\nabla}_\alpha h^\lambda_\mu\right).
\end{aligned}
$$

Ricci scalar in the linearization method

$$
\begin{aligned}
R^{(1)} = & \left(g^{\alpha\beta}\,R_{\alpha\beta}\right)^{(1)} = g^{\alpha\beta(0)}\,R^{(1)}_{\alpha\beta} + g^{\alpha\beta(1)}\,R^{(0)}_{\alpha\beta}\\
& = \bar{\nabla}_\mu \bar{\nabla}^\alpha h^\mu_\alpha - \bar{\nabla}_\mu \bar{\nabla}^\mu h - h^{\alpha\beta}\,R^{(0)}_{\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
R^{(2)} = & \left(g^{\alpha\beta}\,R_{\alpha\beta}\right)^{(2)} = g^{\alpha\beta(0)}\,R^{(2)}_{\alpha\beta} + g^{\alpha\beta(1)}\,R^{(1)}_{\alpha\beta} + g^{\alpha\beta(2)}\,R^{(0)}_{\alpha\beta}\\
= & -\bar{\nabla}_\mu h^{\mu\gamma}\,\bar{\nabla}^\alpha h_{\alpha\gamma} + \bar{\nabla}_\mu h^{\mu\gamma}\,\bar{\nabla}_\gamma h - \frac{1}{4}\,\bar{\nabla}_\lambda h\,\bar{\nabla}^\lambda h\\
+ & \frac{3}{4}\,\bar{\nabla}_\lambda h^{\mu\alpha}\,\bar{\nabla}^\lambda h_{\mu\alpha} - \frac{1}{2}\,\bar{\nabla}_\lambda h^{\mu\alpha}\,\bar{\nabla}_\mu h^\lambda_\alpha\\
- & h^{\mu\gamma}\,\left(\bar{\nabla}_\mu \bar{\nabla}^\alpha h_{\alpha\gamma} + \bar{\nabla}_\mu \bar{\nabla}^\beta h_{\beta\gamma} - \bar{\nabla}_\mu \bar{\nabla}_\gamma h - \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu\gamma}\right)\\
+ & h^{\alpha\mu}\,h^\beta_\mu\,R^{(0)}_{\alpha\beta}.
\end{aligned}
$$

Calculation of the determinant of g

$$
\begin{aligned}
g_{\mu\nu} = & \bar{g}_{\mu\nu} + h_{\mu\nu} = \bar{g}_{\mu\lambda}\,\delta^\lambda_\nu + \bar{g}_{\mu\lambda}\,\bar{g}^{\lambda\sigma}\,h_{\sigma\nu}\\
& = \bar{g}_{\mu\lambda}\,(\delta^\lambda_\nu + \bar{g}^{\lambda\sigma}\,h_{\sigma\nu}) = \bar{g}_{\mu\lambda}\,(\delta^\lambda_\nu + h^\lambda_\nu),
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{-g} = & \sqrt{-det\ g_{\mu\nu}} = \sqrt{-\bar{g}}\,\sqrt{det\ (\delta + h)}\\
= & \sqrt{-\bar{g}}\,\left[1 + \frac{1}{2}\,tr\,\left(h^\nu_\lambda - \frac{h^\nu_\sigma\,h^\sigma_\lambda}{2}\right) + \frac{1}{8}\,(tr\ h^\nu_\lambda)^2\right]\\
= & \sqrt{-\bar{g}}\,\left[1 + \frac{h}{2} + \frac{1}{8}\,\left(h^2 - 2\,h_{\nu\sigma}\,h^{\nu\sigma}\right)\right],
\end{aligned}
$$

$$
\sqrt{-g}^{(0)} = \sqrt{-\bar{g}},\ \sqrt{-g}^{(1)} = \sqrt{-\bar{g}}\,\frac{h}{2},\ \sqrt{-g}^{(2)} = \sqrt{-\bar{g}}\,\frac{1}{8}\,\left(h^2 - 2\,h_{\alpha\beta}\,h^{\alpha\beta}\right).
$$

8.1.1 Linearization calculations

Christoffel symbol in the linearization method

$$
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\,\eta^{\alpha\sigma}\,(\partial_\beta h_{\sigma\gamma} + \partial_\gamma h_{\beta\sigma} - \partial_\sigma h_{\beta\gamma}) - \frac{1}{2}\,h^{\alpha\sigma}\,(\partial_\beta h_{\sigma\gamma} + \partial_\gamma h_{\beta\sigma} - \partial_\sigma h_{\beta\gamma}).
$$

Ricci tensor in the linearization method

First-order Ricci tensor:

$$R^{(1)}_{\alpha\beta} = \frac{1}{2}\,\Big(\partial_\mu\partial_\beta h^\mu_\alpha + \partial_\mu\partial_\alpha h^\mu_\beta - \partial_\mu\partial^\mu h_{\beta\alpha} - \partial_\beta\partial_\alpha h\Big)\,.$$

Second-order Ricci tensor:

$$\begin{aligned} R^{(2)}_{\alpha\beta} = &-\frac{1}{2}\,\partial_\mu h^{\mu\gamma}\,\left(\partial_\beta h_{\alpha\gamma} + \partial_\alpha h_{\beta\gamma} - \partial_\gamma h_{\beta\alpha}\right) \\ &- \frac{1}{2}\,h^{\mu\gamma}\,\left(\partial_\mu\partial_\beta h_{\alpha\gamma} + \partial_\mu\partial_\alpha h_{\beta\gamma} - \partial_\mu\partial_\gamma h_{\beta\alpha} - \partial_\beta\partial_\alpha h_{\mu\gamma}\right) \\ &+ \frac{1}{4}\,\partial^\lambda h\,\left(\partial_\beta h_{\lambda\alpha} + \partial_\alpha h_{\lambda\beta} - \partial_\lambda h_{\beta\alpha}\right) \\ &- \frac{1}{2}\,\left(\partial_\lambda h^\mu_\beta\,\partial_\mu h^\lambda_\alpha - \partial_\lambda h^\mu_\beta\,\partial^\lambda h_{\mu\alpha} - \frac{1}{2}\,\partial_\beta h^\mu_\lambda\,\partial_\alpha h^\lambda_\mu\right). \end{aligned}$$

Ricci scalar in the linearization method

$$R^{(1)} = \left(g^{\alpha\beta}\,R_{\alpha\beta}\right)^{(1)} = g^{\alpha\beta(0)}\,R^{(1)}_{\alpha\beta} + g^{\alpha\beta(1)}\,R^{(0)}_{\alpha\beta}.$$

Note: in flat solution, $R^{(0)}_{\alpha\beta} = 0$.

$$R^{(1)} = \partial_\mu\partial^\alpha h^\mu_\alpha - \Box h,$$

$$R^{(2)} = \left(g^{\alpha\beta}\,R_{\alpha\beta}\right)^{(2)} = g^{\alpha\beta(0)}\,R^{(2)}_{\alpha\beta} + g^{\alpha\beta(1)}\,R^{(1)}_{\alpha\beta},$$

$$\begin{aligned} R^{(2)} = &-\partial_\mu h^{\mu\gamma}\,\partial^\alpha h_{\alpha\gamma} + \partial_\mu h^{\mu\gamma}\,\partial_\gamma h - \frac{1}{4}\,\partial_\lambda h\,\partial^\lambda h + \frac{3}{4}\,\partial_\lambda h^{\mu\alpha}\,\partial^\lambda h_{\mu\alpha} \\ &- \frac{1}{2}\,\partial_\lambda h^{\mu\alpha}\,\partial_\mu h^\lambda_\alpha - h^{\mu\gamma}\,\left(\partial_\mu\partial^\alpha h_{\alpha\gamma} + \partial_\beta\partial^\mu h_{\beta\gamma} - \partial_\mu\partial_\gamma h - \partial^\alpha\partial_\alpha h_{\mu\gamma}\right). \end{aligned}$$

$$\sqrt{-g}^{(0)} = \sqrt{-\eta},\ \sqrt{-g}^{(1)} = \sqrt{-\eta}\,\frac{h}{2},\ \sqrt{-g}^{(2)} = \sqrt{-\eta}\,\frac{1}{8}\,\left(h^2 - 2\,h_{\alpha\beta}\,h^{\alpha\beta}\right).$$

8.2 Interactive terms of the action

In this part of the appendix, the details of the calculation of $\bar{x}_i$ and also the first and second order disturbances of x_i with respect to $f_{\mu\nu}$, $g_{\mu\nu}$, and are shown. For this reason, for convenience, we can express the first and second-order disturbances of x_i in terms of y_i.

8.2.1 Calculation of $\bar{x}_i$

Usind Equation (1), we can write:

$$\begin{aligned}
\bar{x}_1^2 &= \bar{y}_1 + 2\bar{x}_2,\\
\bar{x}_2^2 &= \bar{y}_2 - 2\sqrt{\bar{y}_4} + 2\bar{x}_1\bar{x}_3,\\
\bar{x}_3^2 &= \bar{y}_3 + 2\bar{x}_2\ \sqrt{\bar{y}_4},\\
\bar{x}_4^2 &= \bar{y}_4.
\end{aligned} \tag{2}$$

For convenience in calculating, some new terms are defined as:

$$\bar{\Sigma}^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\bar{f}_{\rho\nu}, \quad \bar{\Sigma}^{\mu\nu} = \bar{\Sigma}^{\mu}{}_{\rho}\,\bar{g}^{\rho\nu} = \bar{g}^{\mu\rho_1}\bar{f}_{\rho_1\rho}\bar{g}^{\rho\nu}.$$

Also, in the general case, we introduce:

$$(\bar{\Sigma}^2)^{\mu\nu} = \bar{\Sigma}^{\mu}{}_{\rho_1}\,\bar{\Sigma}^{\rho_1}{}_{\rho_2}\,\bar{g}^{\rho_2\nu}, \quad (\bar{\Sigma}^k)^{\mu\nu} = \bar{\Sigma}^{\mu}{}_{\rho_1}\,\bar{\Sigma}^{\rho_1}{}_{\rho_2}\cdots\bar{\Sigma}^{\rho_{k-1}}{}_{\rho_k}\,\bar{g}^{\rho_k\nu}.$$

Moreover, we have this expression for trace:

$$[\bar{\Sigma}] \equiv \mathrm{Tr}(\bar{\Sigma}) = \bar{\Sigma}^{\mu}{}_{\mu},$$

$$\begin{aligned}
\bar{y}_1 &= [\bar{\Sigma}] = \bar{\Sigma}^{\mu}_{\mu} = \eta^{\mu\nu}\lambda\eta_{\mu\nu} = 4\lambda,\\
\bar{y}_2 &= \frac{1}{2}\left([\bar{\Sigma}]^2 - [\bar{\Sigma}^2]\right) = \frac{1}{2}\left((4\lambda)^2 - 4\lambda^2\right) = 6\lambda^2,\\
\bar{y}_3 &= \frac{1}{6}\left([\bar{\Sigma}]^3 - 3[\bar{\Sigma}][\bar{\Sigma}^2] + 2[\bar{\Sigma}^3]\right) = \ 4\lambda^3,\\
\bar{y}_4 &= det(\bar{g}^{-1}\bar{f}) = det(\eta^{\mu\nu}\lambda\eta_{\mu\nu}) = det(\lambda\mathbf{1}) = \lambda^4,
\end{aligned} \tag{3}$$

Using Equations (2), and (3) we obtain:

$$\begin{aligned}
\bar{x}_4^2 &= \bar{y}_4, \quad \Rightarrow \quad \bar{x}_4 = \sqrt{\bar{y}_4} = \lambda^2,\\
\bar{x}_3^2 &= \bar{y}_3 + 2\bar{x}_2\sqrt{\bar{y}_4} = 4\lambda^3 + 2\lambda^2\bar{x}_2 = \lambda^2(4\lambda + 2\bar{x}_2),\\
\bar{x}_2^2 &= \bar{y}_2 - 2\sqrt{\bar{y}_4} + 2\bar{x}_1\bar{x}_3 = 4\lambda^2 + \ 2\bar{x}_1\bar{x}_3,\\
\bar{x}_1^2 &= \bar{y}_1 + 2\bar{x}_2 = 4\lambda + 2\bar{x}_2.
\end{aligned}$$

For $\bar{x}_3^2$ we can write, $\bar{x}_3^2 = \lambda^2(4\lambda + 2\bar{x}_2) = \lambda^2\bar{x}_1^2$ thus, $\bar{x}_3 = \pm\lambda\bar{x}_1$. If we choose $\bar{x}_3 = -\lambda\bar{x}_1$, we have $\bar{x}_1 = 0$, $\bar{x}_2 = -2\lambda$ and $\bar{x}_3 = 0$. And, if we choose $\bar{x}_3 = +\lambda\bar{x}_1$ we have, $\bar{x}_1 = 4\sqrt{\lambda}, \quad \bar{x}_2 = 6\lambda, \quad \bar{x}_3 = 4\lambda\sqrt{\lambda}, \quad \bar{x}_4 = \lambda^2$.

8.2.2 Calculation of $y_{i,g}^{\mu\nu}$ and $y_{i,f}^{\mu\nu}$:

$$y_{i,g}^{\mu\nu} = \left.\frac{\partial y_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}, \qquad y_{i,f}^{\mu\nu} = \left.\frac{\partial y_i}{\partial f_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

$$\begin{aligned} y^{\mu\nu}_{1,g} &= -\bar{\Sigma}^{\mu\nu} = -\bar{g}^{\mu\sigma}\bar{f}_{\sigma\rho}\bar{g}^{\rho\nu} = -\lambda\eta^{\mu\nu} \\ y^{\mu\nu}_{1,f} &= \bar{g}^{\mu\nu} = \ \eta^{\mu\nu}, \end{aligned}$$

$$\begin{aligned} y^{\mu\nu}_{2,g} &= \left(\bar{\Sigma}^2\right)^{\mu\nu} - [\bar{\Sigma}]\bar{\Sigma}^{\mu\nu} = -3\lambda^2\eta^{\mu\nu} \\ &= \bar{\Sigma}^{\mu}{}_{\rho_1}\ \bar{\Sigma}^{\rho_1}{}_{\rho_2}\ \bar{g}^{\rho_2\nu} - \bar{\Sigma}^{\rho}{}_{\rho}\ \bar{\Sigma}^{\mu}{}_{\alpha}\bar{g}^{\alpha\nu} \\ &= \bar{g}^{\mu\alpha}\bar{f}_{\alpha\rho_1}\bar{g}^{\rho_1\rho_2}\bar{f}_{\rho_2\rho_3}\bar{g}^{\rho_3\nu} - \bar{g}^{\rho\rho_1}\bar{f}_{\rho_1\rho}\bar{g}^{\mu\rho_2}\bar{f}_{\rho_2\rho_3}\bar{g}^{\rho_3\nu}, \end{aligned}$$

$$\begin{aligned} y^{\mu\nu}_{2,f} &= \bar{g}^{\mu\nu}[\bar{\Sigma}] - \bar{\Sigma}^{\mu\nu} = 3\lambda\eta^{\mu\nu} \\ &= \bar{g}^{\mu\nu}\ \bar{\Sigma}^{\rho}{}_{\rho} - \bar{\Sigma}^{\mu}{}_{\rho}\bar{g}^{\rho\nu} = \bar{g}^{\mu\nu}\bar{g}^{\rho\rho_1}\bar{f}_{\rho_1\rho} - \bar{g}^{\mu\rho}\bar{f}_{\rho\rho_1}\bar{g}^{\rho_1\nu}, \end{aligned}$$

$$\begin{aligned} y^{\mu\nu}_{3,g} &= [\bar{\Sigma}]\left(\bar{\Sigma}^2\right)^{\mu\nu} - \left(\bar{\Sigma}^3\right)^{\mu\nu} + \frac{1}{2}\bar{\Sigma}^{\mu\nu}\left([\bar{\Sigma}^2] - [\bar{\Sigma}]^2\right) = -3\lambda^3\eta^{\mu\nu} \\ &= \bar{\Sigma}^{\rho}{}_{\rho}\ \bar{\Sigma}^{\mu}{}_{\rho_1}\bar{\Sigma}^{\rho_1}_{\rho_2}\bar{g}^{\rho_2\nu} - \bar{\Sigma}^{\mu}{}_{\rho_1}\bar{\Sigma}^{\rho_1}{}_{\rho_2}\bar{\Sigma}^{\rho_2}_{\rho_3}\bar{g}^{\rho_3\nu} \\ &\quad + \frac{1}{2}\bar{\Sigma}^{\mu}{}_{\rho}\bar{g}^{\rho\nu}\left([\bar{\Sigma}^{\alpha}{}_{\rho_1}\ \bar{\Sigma}^{\rho_1}{}_{\rho_2}\bar{g}^{\rho_2\beta}] - \left(\bar{\Sigma}^{\mu}{}_{\mu}\right)^2\right) \\ &= \bar{g}^{\rho\sigma}\bar{f}_{\sigma\rho}\bar{g}^{\mu\delta}\bar{f}_{\delta\rho_1}\bar{g}^{\rho_1\alpha}\bar{f}_{\alpha\rho_2}\bar{g}^{\rho_2\nu} - \bar{g}^{\mu\sigma}\bar{f}_{\sigma\rho_1}\ \bar{g}^{\rho_1\delta}\ \bar{f}_{\delta\rho_2}\ \bar{g}^{\rho_2\alpha}\ \bar{f}_{\alpha\rho_3}\ \bar{g}^{\rho_3\nu} \\ &\quad + \frac{1}{2}\bar{g}^{\mu\sigma}\bar{f}_{\sigma\rho}\bar{g}^{\rho\nu}\left(\bar{g}^{\alpha\sigma}\bar{f}_{\sigma\rho_1}\bar{g}^{\rho_1\delta}\bar{f}_{\delta\rho_2}\bar{g}^{\rho_2\beta} - \left(\bar{g}^{\mu\rho}\bar{f}_{\rho\mu}\right)^2\right), \end{aligned}$$

$$\begin{aligned} y^{\mu\nu}_{3,f} &= \left(\bar{\Sigma}^2\right)^{\mu\nu} - \bar{\Sigma}^{\mu\nu}[\bar{\Sigma}] + \frac{1}{2}\bar{g}^{\mu\nu}\left([\bar{\Sigma}]^2 - [\bar{\Sigma}^2]\right) = 3\lambda^2\eta^{\mu\nu} \\ &= \bar{\Sigma}^{\mu}{}_{\rho_1}\bar{\Sigma}^{\rho_1}_{\rho_2}\bar{g}^{\rho_2\nu} - \bar{g}^{\mu\rho_1}\bar{f}_{\rho_1\rho_2}\bar{g}^{\rho_2\nu}\bar{g}^{\alpha\beta}\bar{f}_{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\nu}\left((4\lambda)^2 - 4\lambda^2\right) \\ &= \bar{g}^{\mu\alpha}\bar{f}_{\alpha\rho_1}\bar{g}^{\rho_1\beta}\bar{f}_{\beta\rho_2}\bar{g}^{\rho_2\nu} - \bar{g}^{\mu\rho}\bar{f}_{\rho\rho_1}\bar{g}^{\rho_1\nu}\bar{g}^{\alpha\beta}\bar{f}_{\beta\alpha} + 6\lambda^2\bar{g}^{\mu\nu}, \end{aligned}$$

$$y^{\mu\nu}_{4,g} = -\bar{y}_4\bar{g}^{\mu\nu} = -\lambda^4\eta^{\mu\nu}, \quad y^{\mu\nu}_{4,f} = \bar{y}_4\bar{f}^{\mu\nu} = \lambda^3\eta^{\mu\nu}.$$

8.2.3 Calculation of $x^{\mu\nu}_{i,g}$ and $x^{\mu\nu}_{i,f}$

$$x^{\mu\nu}_{i,g} = \left.\frac{\partial x_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}}, \qquad x^{\mu\nu}_{i,f} = \left.\frac{\partial x_i}{\partial f_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f},\varphi=\bar{\varphi}},$$

where $i = 1, 2, 3, 4$.

$$\left.\frac{\partial x_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f}} = \left.\frac{\partial x_i}{\partial y_j}\frac{\partial y_j}{\partial g_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f}}, (i, j = 1, 2, 3, 4).$$

To obtain $x_{i,g}^{\mu\nu}$ and $x_{i,f}^{\mu\nu}$ in the first step of the relations between x_i and y_i (see Equation (1)) is established, we take the derivative with respect to y_i.

$$
\begin{aligned}
&x_1^2 = y_1 + 2x_2 \longrightarrow 2x_1 \frac{\partial x_1}{\partial y_i} = \frac{\partial y_1}{\partial y_i} + 2\frac{\partial x_2}{\partial y_i},\\
&x_2^2 = y_2 - 2\sqrt{y_4} + 2x_1x_3 \longrightarrow 2x_2 \frac{\partial x_2}{\partial y_i} = \frac{\partial y_2}{\partial y_i} - 2\frac{\partial \sqrt{y_4}}{\partial y_i} + 2x_3\frac{\partial x_1}{\partial y_i} + 2x_1\frac{\partial x_3}{\partial y_i},\\
&x_3^2 = y_3 + 2x_2\sqrt{y_4} \longrightarrow 2x_3 \frac{\partial x_3}{\partial y_i} = \frac{\partial y_3}{\partial y_i} + 2\sqrt{y_4}\frac{\partial x_2}{\partial y_i} + 2x_2\frac{\partial \sqrt{y_4}}{\partial y_i},\\
&x_4^2 = y_4 \longrightarrow 2x_4 \frac{\partial x_4}{\partial y_i} = \frac{\partial y_4}{\partial y_i},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x_1}{\partial y_i} &= \frac{1}{2x_1}\frac{\partial y_1}{\partial y_i} + \frac{1}{x_1}\frac{\partial x_2}{\partial y_i},\\
\frac{\partial x_2}{\partial y_i} &= \frac{1}{2x_2}\frac{\partial y_2}{\partial y_i} - \frac{1}{2x_2x_4}\frac{\partial y_4}{\partial y_i} + \frac{x_3}{x_2}\frac{\partial x_1}{\partial y_i} + \frac{x_1}{x_2}\frac{\partial x_3}{\partial y_i},\\
\frac{\partial x_3}{\partial y_i} &= \frac{1}{2x_3}\frac{\partial y_3}{\partial y_i} + \frac{x_4}{x_3}\frac{\partial x_2}{\partial y_i} + \frac{x_2}{2x_3x_4}\frac{\partial y_4}{\partial y_i},\\
\frac{\partial x_4}{\partial y_i} &= \frac{1}{2x_4}\frac{\partial y_4}{\partial y_i}.
\end{aligned}
$$

In the next step, we interpolate the relations. As an example for $\frac{\partial x_2}{\partial y_i}$ we will have,

$$
\begin{aligned}
\frac{\partial x_2}{\partial y_i} = {}& \frac{1}{2x_2}\frac{\partial y_2}{\partial y_i} - \frac{1}{2x_2x_4}\frac{\partial y_4}{\partial y_i} + \frac{x_3}{x_2}\left(\frac{1}{2x_1}\frac{\partial y_1}{\partial y_i} + \frac{1}{x_1}\frac{\partial x_2}{\partial y_i}\right)\\
&+ \frac{y_1}{x_2}\left(\frac{1}{2x_3}\frac{\partial y_3}{\partial y_i} + \frac{x_4}{x_3}\frac{\partial x_2}{\partial y_i} + \frac{x_2}{2x_3x_4}\frac{\partial y_4}{\partial y_i}\right),
\end{aligned}
$$

after doing calculations for $\frac{\partial x_2}{\partial y_i}$ we get:

$$
\frac{\partial x_2}{\partial y_i} = \frac{1}{2\left(x_1x_2x_3 - x_1^2x_4 - x_3^2\right)}\left[x_3^2\frac{\partial y_1}{\partial y_i} + x_1x_3\frac{\partial y_2}{\partial y_i} + x_1^2\frac{\partial y_3}{\partial y_i} + \frac{x_1\left(x_1x_2 - x_3\right)}{x_4}\frac{\partial y_4}{\partial y_i}\right],
$$

with the same method for $\frac{\partial x_1}{\partial y_i}$ and $\frac{\partial x_3}{\partial y_i}$ we have:

$$
\begin{aligned}
\frac{\partial x_1}{\partial y_i} = {}& \frac{1}{2\left(x_1x_2x_3 - x_1^2x_4 - x_3^2\right)}\Bigg[\left(x_2x_3 - x_1x_4\right)\frac{\partial y_1}{\partial y_i} + x_3\frac{\partial y_2}{\partial y_i} + x_1\frac{\partial y_3}{\partial y_i}\\
&+ \frac{\left(x_1x_2 - x_3\right)}{x_4}\frac{\partial y_4}{\partial y_i}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x_3}{\partial y_i} = {}& \frac{1}{2\left(x_1x_2x_3 - x_1^2x_4 - x_3^2\right)}\Bigg[x_3x_4\frac{\partial y_1}{\partial y_i} + x_1x_4\frac{\partial y_2}{\partial y_i} + \left(x_1x_2 - x_3\right)\frac{\partial y_3}{\partial y_i}\\
&+ \frac{\left(x_1x_2^2 - x_2x_3 - x_1x_4\right)}{x_4}\frac{\partial y_4}{\partial y_i}\Bigg].
\end{aligned}
$$

Now, for $i = 1, 2, 3, 4$, we get:

$$\left.\frac{\partial x_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}} = \left.\frac{\partial x_i}{\partial y_i}\frac{\partial y_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}},$$

$$\begin{aligned}
\left.\frac{\partial x_1}{\partial g_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}} &= \left.\frac{\partial x_1}{\partial y_1}\frac{\partial y_1}{\partial g_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}} \\
&= \frac{1}{2\left(x_1x_2x_3 - x_1^2x_4 - x_3^2\right)}\left[\left(x_2x_3 - x_1x_4\right)\frac{\partial y_1}{\partial g_{\mu\nu}} + x_3\frac{\partial y_2}{\partial g_{\mu\nu}} + x_1\frac{\partial y_3}{\partial g_{\mu\nu}} + \frac{\left(x_1x_2 - x_3\right)}{x_4}\frac{\partial y_4}{\partial g_{\mu\nu}}\right]\Bigg|_{g=\bar{g}, f=\bar{f}}.
\end{aligned}$$

In this way, for all $i = 1, 2, 3, 4$, we can obtain $\frac{\partial x_i}{\partial g_{\mu\nu}}$.

$$\begin{aligned}
x^{\mu\nu}_{1,\star} &= A\big\{\bar{x}_4\left(\bar{x}_1\bar{x}_4 - \bar{x}_2\bar{x}_3\right)y^{\mu\nu}_{1,\star} - \bar{x}_3\bar{x}_4y^{\mu\nu}_{2,\star} - \bar{x}_1\bar{t}_4y^{\mu\nu}_{3,\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)y^{\mu\nu}_{4,\star}\big\}, \\
x^{\mu\nu}_{2,\star} &= A\big\{-\bar{x}_3^2\bar{x}_4y^{\mu\nu}_{1,\star} - \bar{x}_3\bar{x}_4\bar{x}_1y^{\mu\nu}_{2,\star} - \bar{x}_1^2\bar{x}_4s^{\mu\nu}_{3,\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)\bar{t}_1y^{\mu\nu}_{4,\star}\big\}, \\
x^{\mu\nu}_{3,\star} &= A\big\{-\bar{x}_4^2\bar{x}_3\Big(\bar{x}_1y^{\mu\nu}_{1,\star} - \bar{x}_1\bar{x}_4^2y^{\mu\nu}_{2,\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)\bar{x}_4y^{\mu\nu}_{3,\star} + \left(\bar{x}_2\bar{x}_3 + \bar{x}_1\left(\bar{x}_4 - \bar{x}_2^2\right)\right)y^{\mu\nu}_{4,\star}\Big)\big\}, \\
x^{\mu\nu}_{4,\star} &= \frac{y^{\mu\nu}_{4,\star}}{2\bar{x}_4}.
\end{aligned} \tag{4}$$

In Equation (4), $\star$ sign indicates g or f, and

$$A = \left[2\bar{x}_4\left(\bar{x}_3^2 + \bar{x}_1^2\bar{x}_4 - \bar{x}_1\bar{x}_2\bar{x}_3\right)\right]^{-1}. \tag{5}$$

We obtain:

$$\begin{array}{llll}
\bar{x}_1 = 4\lambda, & \bar{x}_2 = 6\lambda, & \bar{x}_3 = 4\lambda\sqrt{\lambda}, & \bar{x}_4 = \lambda^2, \\
y^{\mu\nu}_{1,g} = -\lambda\eta^{\mu\nu}, & y^{\mu\nu}_{2,g} = -3\lambda^2\eta^{\mu\nu}, & y^{\mu\nu}_{3,g} = -3\lambda^3\eta^{\mu\nu}, & y^{\mu\nu}_{4,g} = -\lambda^4\eta^{\mu\nu}, \\
y^{\mu\nu}_{1,f} = \eta^{\mu\nu}, & y^{\mu\nu}_{2,f} = 3\lambda\eta^{\mu\nu}, & y^{\mu\nu}_{3,f} = 3\lambda^2\eta^{\mu\nu}, & y^{\mu\nu}_{4,f} = \lambda^3\eta^{\mu\nu}.
\end{array}$$

In this way we have:

$$A = \left[2\lambda^2\left(16\lambda^3 + 16\lambda\lambda^2 - 4\sqrt{\lambda}6\lambda4\lambda\sqrt{\lambda}\right)\right]^{-1} = -\frac{1}{128\lambda^5},$$

$$\begin{aligned}
x^{\mu\nu}_{1,g} = -\frac{1}{128\lambda^5}&\Big[\lambda^2\left(4\sqrt{\lambda}\lambda^2 - 6\lambda4\lambda\sqrt{\lambda}\right)\left(-\lambda\eta^{\mu\nu}\right) - \left(4\lambda\sqrt{\lambda}\lambda^2\right)\left(-3\lambda^2\eta^{\mu\nu}\right) \\
&- \left(4\sqrt{\lambda}\lambda^2\right)\left(-3\lambda^3\eta^{\mu\nu}\right) + \left(4\lambda\sqrt{\lambda} - 4\sqrt{\lambda}6\lambda\right)\left(-\lambda^4\eta^{\mu\nu}\right)\Big] = -\frac{1}{2}\sqrt{\lambda}\eta^{\mu\nu},
\end{aligned}$$

$$x^{\mu\nu}_{1,f} = -\frac{1}{128\lambda^5}\Big[-20\lambda^4\sqrt{\lambda}\eta^{\mu\nu} - \left(4\lambda^3\sqrt{\lambda}\right)\left(3\lambda\eta^{\mu\nu}\right) - \left(4\lambda^2\sqrt{\lambda}\right)\left(3\lambda^2\eta^{\mu\nu}\right)$$

$$
\begin{aligned}
&- \left(20\lambda\sqrt{\lambda}\right)\left(\lambda^3\eta^{\mu\nu}\right)\Big] = \frac{\sqrt{\lambda}}{2\lambda}\eta^{\mu\nu},\\
x^{\mu\nu}_{2,g} &= -\,\frac{1}{128\lambda^5}\Big[\left(-16\lambda^3\lambda^2\right)\left(-\lambda\eta^{\mu\nu}\right) - \left(4\lambda\sqrt{\lambda}\ \lambda^2 4\sqrt{\lambda}\right)\left(-3\lambda^2\eta^{\mu\nu}\right)\\
&\quad - \left(\lambda^2 16\lambda\right)\left(-3\lambda^3\eta^{\mu\nu}\right) + \left(4\lambda\sqrt{\lambda} - 4\sqrt{\lambda}\ 6\lambda\right)4\sqrt{\lambda}\left(-\lambda^4\eta^{\mu\nu}\right)\Big] = -\frac{3}{2}\lambda\eta^{\mu\nu},
\end{aligned}
$$

$$
x^{\mu\nu}_{2,f} = -\,\frac{1}{128\lambda^5}\Big[- 16\lambda^5\eta^{\mu\nu} - 16\lambda^4 3\lambda\eta^{\mu\nu} - 16\lambda^3 3\lambda^2\eta^{\mu\nu} - 80\lambda^2\lambda^3\eta^{\mu\nu}\Big] = \frac{3}{2}\eta^{\mu\nu},
$$

$$
\begin{aligned}
x^{\mu\nu}_{3,g} &= -\frac{1}{128\lambda^5}\Big[\left(-4\lambda\sqrt{\lambda}\lambda^4\right)\left(-\lambda\eta^{\mu\nu}\right) - \left(4\sqrt{\lambda}\lambda^4\right)\left(-3\lambda^2\eta^{\mu\nu}\right)\\
&\quad + \left(4\lambda\sqrt{\lambda} - 4\sqrt{\lambda}\cdot 6\lambda\right)\lambda^2\left(-3\lambda^3\eta^{\mu\nu}\right)\\
&\quad + \left(6\lambda\cdot 4\lambda\sqrt{\lambda} + \left(4\sqrt{\lambda}\left(\lambda^2 - 36\lambda^2\right)\right)\left(-\lambda^4\eta^{\mu\nu}\right)\right)\Big] = -\frac{3}{2}\sqrt{\lambda}\eta^{\mu\nu},
\end{aligned}
$$

$$
\begin{aligned}
x^{\mu\nu}_{3,f} &= -\,\frac{1}{128\lambda^5}\Big[-4\lambda^5\sqrt{\lambda}\eta^{\mu\nu} - 4\lambda^4\sqrt{\lambda}\ 3\lambda\eta^{\mu\nu} - 20\lambda^3\sqrt{\lambda}3\lambda^2\eta^{\mu\nu}\\
&\qquad -116\lambda^2\sqrt{\lambda}\lambda^3\eta^{\mu\nu}\Big] = \frac{3}{2}\sqrt{\lambda}\eta^{\mu\nu},
\end{aligned}
$$

$$
x^{\mu\nu}_{4,g} = -\frac{1}{2\lambda^2}\lambda^4\eta^{\mu\nu} = -\frac{1}{2}\lambda^2\eta^{\mu\nu},\ x^{\mu\nu}_{4,f} = \frac{1}{2\lambda^2}\lambda^3\eta^{\mu\nu} = \frac{1}{2}\lambda\eta^{\mu\nu}.
$$

8.2.4 Calculation of $Y^{\mu\nu,\alpha\beta}_{i,\star\star}$:

In $Y^{\mu\nu,\alpha\beta}_{i,\star\star}$, the symbol $\star\star$ represents gg, ff or gf.

$$
\begin{aligned}
Y^{\mu\nu,\alpha\beta}_{1,gg} &= \frac{1}{8}\Big\{\ \left[\bar{g}^{\alpha\nu}\bar{\Sigma}^{\mu\beta} + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu)(\alpha\leftrightarrow\beta)\right] + [\cdots]\left((\mu,\nu)\leftrightarrow(\alpha,\beta)\right)\Big\}\\
&\equiv \mathrm{Sym}\Big\{\bar{g}^{\alpha\nu}\bar{\Sigma}^{\mu\beta}\Big\}\\
&= \frac{1}{8}\Big\{\ \left[\lambda\left(\eta^{\alpha\nu}\eta^{\mu\beta} + \eta^{\alpha\mu}\eta^{\nu\beta} + \eta^{\beta\nu}\eta^{\mu\alpha} + \eta^{\beta\mu}\eta^{\nu\alpha}\right)\right] + 2\lambda\left(\eta^{\mu\beta}\eta^{\alpha\nu} + \eta^{\mu\alpha}\eta^{\beta\nu}\right)\Big\}\\
&= \frac{\lambda}{2}\left(\eta^{\alpha\nu}\eta^{\mu\beta} + \eta^{\alpha\mu}\eta^{\nu\beta}\right),
\end{aligned}
$$

$$
Y^{\mu\nu,\alpha\beta}_{1,gf} = -\frac{1}{2}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right) = -\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\ \eta^{\nu\alpha}\right),\quad Y^{\mu\nu,\alpha\beta}_{1,ff} = 0,
$$

$$
\begin{aligned}
Y^{\mu\nu,\alpha\beta}_{2,gg} &= \mathrm{Sym}\Big\{\bar{g}^{\mu\alpha}\left([\bar{\Sigma}]\bar{\Sigma}^{\beta\nu} - \left(\bar{\Sigma}^2\right)^{\beta\nu}\right) + \frac{1}{2}\bar{\Sigma}^{\mu\nu}\ \bar{\Sigma}^{\alpha\beta} - \frac{1}{2}\bar{\Sigma}^{\alpha\mu}\ \bar{\Sigma}^{\beta\nu}\Big\}\\
&= \mathrm{Sym}\Big\{4\lambda^2\eta^{\mu\alpha}\eta^{\nu\beta} - \lambda^2\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}\lambda^2\eta^{\mu\nu}\eta^{\alpha\beta} - \frac{1}{2}\lambda^2\eta^{\alpha\mu}\eta^{\beta\nu}\Big\}\\
&= \frac{\lambda^2}{4}\Big\{5\eta^{\mu\alpha}\eta^{\nu\beta} + 2\eta^{\mu\nu}\eta^{\alpha\beta} + 5\eta^{\alpha\nu}\eta^{\beta\mu}\Big\},
\end{aligned}
$$

$$\begin{aligned}
Y_{2,gf}^{\mu\nu,\alpha\beta} &= \frac{1}{4}\Big[\bar{g}^{\beta\mu}\bar{\Sigma}^{\alpha\nu} + \bar{g}^{\alpha\nu}\bar{\Sigma}^{\beta\mu} - \bar{g}^{\alpha\beta}\bar{\Sigma}^{\mu\nu} - \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}[\bar{\Sigma}] \\
&\quad + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta) + (\mu \leftrightarrow \nu)(\alpha \leftrightarrow \beta)\Big] \\
&= \mathrm{sym}\Big\{\bar{g}^{\beta\mu}\bar{\Sigma}^{\alpha\nu} + \bar{g}^{\alpha\nu}\bar{\Sigma}^{\beta\mu} - \bar{g}^{\alpha\beta}\bar{\Sigma}^{\mu\nu} - \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}[\bar{\Sigma}]\Big\} \\
&= \mathrm{sym}\Big\{\frac{\lambda}{4}\left(\eta^{\beta\mu}\eta^{\alpha\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu} - 4\eta^{\alpha\mu}\eta^{\beta\nu}\right)\Big\} \\
&= -\lambda\left(\eta^{\beta\mu}\eta^{\alpha\nu} + \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\mu}\eta^{\beta\nu}\right),
\end{aligned}$$

$$\begin{aligned}
Y_{2,ff}^{\mu\nu,\alpha\beta} &= \frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu} - \frac{1}{4}\left(\bar{g}^{\alpha\nu}\bar{g}^{\beta\mu} + \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\right) \\
&= \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{4}\left(\eta^{\alpha\nu}\eta^{\beta\mu} + \eta^{\alpha\mu}\eta^{\beta\nu}\right),
\end{aligned}$$

$$\begin{aligned}
Y_{3,gg}^{\mu\nu,\alpha\beta} &= \mathrm{Sym}\Big\{\bar{\Sigma}^{\mu\alpha}\left(\bar{\Sigma}^2\right)^{\beta\nu} - \left(\bar{\Sigma}^2\right)^{\mu\nu}\bar{\Sigma}^{\alpha\beta} + \frac{1}{2}[\bar{\Sigma}]\left(\bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \bar{\Sigma}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\right) \\
&\quad + \bar{g}^{\nu\alpha}\left(\left(\bar{\Sigma}^3\right)^{\mu\beta} - \left(\bar{\Sigma}^2\right)^{\mu\beta}[\bar{\Sigma}] + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\left([\bar{\Sigma}]^2 - [\bar{\Sigma}^2]\right)\right)\Big\} \\
&= \mathrm{Sym}\Big\{5\lambda^3\eta^{\mu\alpha}\eta^{\nu\beta} + \lambda^3\eta^{\mu\nu}\eta^{\alpha\beta} - 3\lambda^3\eta^{\alpha\nu}\eta^{\beta\mu}\Big\} \\
&= \lambda^3\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\alpha\nu}\eta^{\beta\mu}\right),
\end{aligned}$$

$$\begin{aligned}
Y_{3,gf}^{\mu\nu,\alpha\beta} &= \mathrm{sym}\Big\{2\bar{g}^{\nu\beta}\left(\bar{\Sigma}^{\mu\alpha}[\bar{\Sigma}] - \left(\bar{\Sigma}^2\right)^{\mu\alpha}\right) + \bar{g}^{\alpha\beta}\Big(\left(\bar{\Sigma}^2\right)^{\mu\nu} - \bar{\Sigma}^{\mu\nu}[\bar{\Sigma}]\Big) \\
&\qquad + \bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \bar{\Sigma}^{\mu\beta}\bar{\Sigma}^{\alpha\nu} + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\left([\bar{\Sigma}^2] - [\bar{\Sigma}]^2\right)\Big\} \\
&= \mathrm{sym}\Big\{-2\lambda^2\eta^{\alpha\beta}\eta^{\mu\nu} - \lambda^2\eta^{\mu\beta}\eta^{\alpha\nu}\Big\} \\
&= -\frac{\lambda^2}{2}\left[4\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\mu\beta}\eta^{\alpha\nu} + \eta^{\alpha\mu}\eta^{\beta\nu}\right],
\end{aligned}$$

$$\begin{aligned}
Y_{3,ff}^{\mu\nu,\alpha\beta} &= \mathrm{Sym}\Big\{\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}[\bar{\Sigma}] - \frac{1}{2}\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}[\bar{\Sigma}] + \bar{g}^{\mu\alpha}\bar{\Sigma}^{\nu\beta} - \bar{g}^{\alpha\beta}\bar{\Sigma}^{\mu\nu}\Big\} \\
&= \mathrm{Sym}\Big\{\lambda\eta^{\mu\nu}\eta^{\alpha\beta} - 2\lambda\eta^{\mu\beta}\eta^{\nu\alpha} + \lambda\eta^{\mu\alpha}\eta^{\nu\beta}\Big\} \\
&= \lambda\left[\eta^{\mu\nu}\eta^{\alpha\beta} - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\right],
\end{aligned}$$

$$\begin{aligned}
Y_{4,gg}^{\mu\nu,\alpha\beta} &= \frac{\bar{y}_4}{2}\left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \frac{1}{2}\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right) \\
&= \frac{\lambda^4}{2}\left(\eta^{\mu\nu}\eta^{\alpha\beta} + \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\alpha}\right),
\end{aligned}$$

$$Y^{\mu\nu,\alpha\beta}_{4,gf} = -\bar{y}_4 \bar{g}^{\mu\nu} \bar{f}^{\alpha\beta} = -\lambda^3 \eta^{\mu\nu}\eta^{\alpha\beta},$$

$$\begin{aligned} Y^{\mu\nu,\alpha\beta}_{4,ff} &= \frac{\bar{y}_4}{2}\left(\bar{f}^{\mu\nu}\bar{f}^{\alpha\beta} - \frac{1}{2}\bar{f}^{\mu\alpha}\bar{f}^{\nu\beta} - \frac{1}{2}\bar{f}^{\mu\beta}\bar{f}^{\nu\alpha}\right) \\ &= \frac{\lambda^2}{2}\left(\eta^{\mu\nu}\eta^{\alpha\beta} - \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\alpha}\right). \end{aligned}$$

8.2.5 Calculation of $x^{\mu\nu,\alpha\beta}_{i,\star\star}$:

$$\begin{aligned} x^{\mu\nu,\alpha\beta}_{1,\star\star} &= A\big\{\bar{x}_4\left(\bar{x}_1\bar{x}_4 - \bar{x}_2\bar{x}_3\right)Y^{\mu\nu,\alpha\beta}_{1,\star\star} - \bar{x}_3\bar{x}_4 Y^{\mu\nu,\alpha\beta}_{2,\star\star} - \bar{x}_1\bar{x}_4 Y^{\mu\nu,\alpha\beta}_{3,\star\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)Y^{\mu\nu,\alpha\beta}_{y,\star\star}\big\}, \\ x^{\mu\nu,\alpha\beta}_{2,\star\star} &= A\big\{-\bar{x}_3^2\bar{x}_4 Y^{\mu\nu,\alpha\beta}_{1,\star\star} - \bar{x}_3\bar{x}_4\bar{x}_1 Y^{\mu\nu,\alpha\beta}_{2,\star\star} - \bar{x}_1^2\bar{x}_4 Y^{\mu\nu,\alpha\beta}_{3,\star\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)\bar{x}_1 Y^{\mu\nu,\alpha\beta}_{4,\star\star}\big\}, \\ x^{\mu\nu,\alpha\beta}_{3,\star\star} &= A\big\{-\bar{x}_4^2\bar{x}_3\Big(\bar{x}_1 S^{\mu\nu,\alpha\beta}_{1,\star\star} - \bar{x}_1\bar{x}_4^2 Y^{\mu\nu,\alpha\beta}_{2,\star\star} + \left(\bar{x}_3 - \bar{x}_1\bar{x}_2\right)\bar{t}_4 Y^{\mu\nu,\alpha\beta}_{3,\star\star} \\ &+ \left(\bar{x}_2\bar{x}_3 + \bar{x}_1\left(\bar{x}_4 - \bar{x}_2^2\right)\right)Y^{\mu\nu,\alpha\beta}_{4,\star\star}\Big)\big\}, \\ x^{\mu\nu,\alpha\beta}_{4,\star\star} &= \frac{Y^{\mu\nu,\alpha\beta}_{4,\star\star}}{2\bar{x}_4}. \end{aligned}$$

As before, the sign $\star\star$ represents gg, ff or gf. Also, A is given in Equation (5). In this way, after long calculations, we have:

$$x^{\mu\nu,\alpha\beta}_{1,gg} = \sqrt{\lambda}\left(\frac{3}{16}\eta^{\alpha\nu}\eta^{\beta\mu} + \frac{3}{16}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\nu}\right),$$

$$x^{\mu\nu,\alpha\beta}_{1,gf} = -\frac{\sqrt{\lambda}}{\lambda}\left[\frac{1}{8}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{1}{8}\eta^{\mu\beta}\eta^{\nu\alpha} + \frac{1}{4}\eta^{\alpha\beta}\eta^{\mu\nu}\right],$$

$$x^{\mu\nu,\alpha\beta}_{1,ff} = -\frac{\sqrt{\lambda}}{\lambda^2}\left[\frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{16}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{16}\eta^{\alpha\mu}\eta^{\beta\nu}\right],$$

$$x^{\mu\nu,\alpha\beta}_{2,gg} = \frac{\lambda}{2}\left(\eta^{\alpha\nu}\eta^{\beta\mu} + \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\beta}\eta^{\mu\nu}\right),$$

$$x^{\mu\nu,\alpha\beta}_{2,gf} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu} - \frac{1}{4}\eta^{\mu\beta}\eta^{\nu\alpha} + \eta^{\alpha\beta}\eta^{\mu\nu},$$

$$x^{\mu\nu,\alpha\beta}_{2,ff} = \frac{1}{\lambda}\left[\frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{4}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\right],$$

$$x^{\mu\nu,\alpha\beta}_{3,gg} = \lambda\sqrt{\lambda}\left(\frac{7}{16}\eta^{\alpha\nu}\eta^{\beta\mu} + \frac{7}{16}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{5}{8}\eta^{\alpha\beta}\eta^{\mu\nu}\right),$$

$$x^{\mu\nu,\alpha\beta}_{3,gf} = -\sqrt{\lambda}\left[\frac{5}{4}\eta^{\alpha\beta}\eta^{\mu\nu} + \frac{1}{8}\eta^{\mu\beta}\eta^{\nu\alpha} + \frac{1}{8}\eta^{\alpha\mu}\eta^{\beta\nu}\right],$$

$$x^{\mu\nu,\alpha\beta}_{3,ff} = -\frac{\sqrt{\lambda}}{\lambda}\left[\frac{5}{8}\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{5}{16}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{5}{16}\eta^{\alpha\mu}\eta^{\beta\nu}\right],$$

$$x^{\mu\nu,\alpha\beta}_{4,gg} = \frac{\lambda^2}{4}\left(\frac{1}{2}\eta^{\alpha\nu}\eta^{\beta\mu} + \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\beta}\eta^{\mu\nu}\right),$$

$$x^{\mu\nu,\alpha\beta}_{4,gf} = -\frac{\lambda}{2}\eta^{\alpha\beta}\eta^{\mu\nu},$$

$$x^{\mu\nu,\alpha\beta}_{4,ff} = -\frac{1}{4}\left[\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}\right].$$

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