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# Secure Domination Number in Kneser Graph

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**Abstract**

A vertex subset $S \subseteq V(G)$ is defined as a secure dominating set if, for any vertex $u$ outside of $S$, there is a neighbor $v \in S$ such that the set $(S \setminus \{v\}) \cup \{u\}$ remains a dominating set for the graph $G$. The minimum cardinality of such a set is termed the secure domination number and is denoted by $\gamma_s(G)$. In the present study, we establish specific bounds for the secure domination number of Kneser graphs $K(n,k)$. We further prove that for the condition $n \geq k^2 + 2k$, the value of $\gamma_s(K(n,k))$ is precisely $k+2$.

Keywords: Secure domination number, Kneser graph, Dominating set.

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## 1. Introduction

We define a graph $G = (V, E)$ as a structural pair consisting of a vertex set $V$, which may be infinite, and an edge set $E$. If an edge $e$ joins two vertices $v$ and $w$, we represent it as $e = \{v, w\}$. The degree of any vertex $x \in V$, written as $\deg(x)$, indicates the total number of vertices adjacent to it.

A subset of vertices forms a *clique* if every distinct pair of vertices within that subset is connected by an edge. A *graph homomorphism* $f$ from $G$ to $G'$ is a mapping $f : V(G) \to V(G')$ that maintains adjacency; specifically, if $\{x, y\} \in E(G)$, then $\{f(x), f(y)\} \in E(G')$. A set $S \subseteq V(G)$ is identified as a *dominating set* of $G$ if every vertex in $V \setminus S$ has at least one neighbor located in $S$. The

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minimum size of such a set is the *domination number*, denoted $\gamma(G)$. For additional foundational terminology, we refer the reader to [1, 2].

A *secure dominating set* is a subset $S \subseteq V(G)$ such that for every $u \notin S$, there is some $v \in S$ where $(S \setminus \{v\}) \cup \{u\}$ remains a dominating set. The *secure domination number*, $\gamma_s(G)$, represents the smallest cardinality of such a set. This concept was originally introduced by Cockayne et al. [3]. Numerous properties and bounds for $\gamma_s(G)$ have been explored across various graph classes [4–7].

Given positive integers $t, k, n$, and $\lambda$ where $0 \leq t \leq k \leq n$, a $\lambda$-$(n, k, t)$ *design* is a set of $k$-element blocks from the $n$-set $[n] = \{a_1, a_2, \ldots, a_n\}$ such that every $t$-element subset of $[n]$ appears in exactly $\lambda$ blocks. Specifically, a 2-$(k^2+k+1, k+1, 1)$ design is a *projective plane of order* $k$, which exists when $k$ is a prime power. Conversely, a 2-$(k^2, k, 1)$ design is referred to as an *affine plane of order* $k$ (see [8]).

The *Kneser graph* $K(n, k)$ has a vertex set composed of all $k$-subsets of an $n$-element set, with edges existing between disjoint subsets. For any $(k-1)$-subset $A$ of $[n]$, the *upward shadow* is the set of all $k$-subsets containing $A$ (see [9]). Previous work by Ivančo and Zelinka [10] addressed the domination and domatic numbers for specific Kneser graphs. Östergård et al. in [11] obtained some bounds on the domination number of Kneser graphs while Bahmani and Emami recently investigated total Roman domination in these graphs [12].
In the following sections, we prove that $\gamma_s(K(n, k)) = k+2$ whenever $n \geq k^2+2k$. Additionally, we establish bounds for the secure domination number in the range $2k+1 < n < k^2+2k-1$.

## 2. Main result

In this section, we present several original findings regarding the secure domination number of Kneser graphs. To begin, we restate two essential theorems that serve as the foundation for the proofs and discussions that follow.

**Theorem 2.1** ([9])**.** *For $n \geq 2k+1$, the domination number of the Kneser graph is $\gamma(K(n, k)) = k+1$.*

**Theorem 2.2** ([9])**.** *If $n \geq k^2-k+1$, then the projective plane of order $k-1$ forms a dominating set in $K(n, k)$.*

**Lemma 2.3.** *Let $n \geq 2k+1$. Suppose $S$ is a secure dominating set of $K(n, k)$ such that no upward shadow of any $(k-1)$-subset of $[n]$ is contained within $S$. Under these conditions, $S$ remains a secure dominating set for the graph $K(n+1, k)$.*

*Proof.* Assume $S$ represents a secure dominating set for the Kneser graph $K(n, k)$ defined on the set $[n]$. Let $K(n+1, k)$ be the graph constructed on the expanded set $[n] \cup \{a_{n+1}\}$. To demonstrate that $S$ is secure in $K(n+1, k)$, consider any vertex $x = \{x_1, x_2, \ldots, x_k\}$ in $K(n+1, k)$. We analyze two scenarios:

*Case 1: $a_{n+1} \notin x$.* In this instance, $x$ is inherently a vertex of $K(n, k)$. Because $S$ is secure in $K(n, k)$, there exists a neighbor $y \in S$ of $x$ such that substituting $y$

with $x$ maintains the domination property. Consequently, $(S \setminus \{y\}) \cup \{x\}$ remains a dominating set in the larger graph $K(n+1,k)$.

*Case 2:* $a_{n+1} \in x$. Let $x = \{x_1, x_2, \dots, x_{k-1}, a_{n+1}\}$. Given that $S$ contains no upward shadows of any $(k-1)$-subset from $[n]$, we can identify a vertex $c = \{x_1, x_2, \dots, x_{k-1}, a\}$ with $a \in [n]$ that is adjacent to some $d \in S$, where $x$ is also adjacent to $d$. Since replacing $d$ with $c$ in $S$ results in a dominating set, the substitution $(S \setminus \{d\}) \cup \{x\}$ likewise functions as a dominating set. Thus, $S$ fulfills the requirements of a secure dominating set for $K(n+1,k)$. □

**Lemma 2.4.** *For $n \geq k^2 + k$, every dominating set of $K(n,k)$ with size $k+1$ is a clique.*

*Proof.* By Theorem 2.1, for $n \geq k^2 + k$, $\gamma(K(n,k)) = k+1$. Suppose, on the contrary, that $S = \{D_1, D_2, \dots, D_{k+1}\}$ is a dominating set of size $k+1$ for $G$ which is not a clique. There exist at least two elements $D_{k+1}$ and $D_k$ in $S$ such that $D_{k+1} \cap D_k = \{x\}$. Now, consider a vertex $y = \{a_1, \dots, a_{k-1}, x\}$ where $a_i \in D_i$ for $1 \leq i \leq k-1$. The vertex $y$ does not adjacent to any vertex in $S$, which contradicts the fact that $S$ is a dominating set. Therefore, for $n \geq k^2 + k$, every dominating set of size $k+1$ is a clique. □

**Lemma 2.5.** *Let $G$ be a simple graph, and $S \subseteq V$ be a dominating set in $G$. If every vertex $v \in V \setminus S$ has at least two neighbors in $G$, then $S$ is a secure dominating set in $G$.*

*Proof.* Suppose that $S \subseteq V(G)$ is a dominating set of a graph $G$, and every vertex outside $S$ has at least two neighbors in $S$. If we remove an arbitrary vertex $s \in S$ and replace it with one of its neighbors, such as $v \notin S$, then the resulting set $S' = (S \setminus \{s\}) \cup \{v\}$ is still a dominating set. □

The following theorem gives a lower bound for the secure domination number of $K(n,k)$ for $n \geq k^2 + k$.

**Theorem 2.6.** *For $n \geq k^2 + k$, $\gamma_s(K(n,k)) \geq k+2$.*

*Proof.* We show that there is no secure dominating set of size smaller than $k+2$. Suppose, on the contrary, that $S = \{D_1, D_2, \dots, D_{k+1}\}$ is a secure dominating set for the graph $G$ of size $k+1$. By Lemma 2.4, the set $S$ is a clique. Now suppose that $x$ is an arbitrary vertex outside $S$ which intersects at least one of the sets $D_i$, say $D_1$. If we remove an adjacent vertex of $x$ from $S$, such as $D_j$ and replace it with $x$, the set $(S \setminus D_j) \cup x$ is no longer a clique (since $x$ intersects $D_1$) and by Lemma 2.4, this is a contradiction. Therefore, for $n \geq k^2 + k$, we have $\gamma_s(G) \geq k+2$. □

**Theorem 2.7.** *For $n \geq k^2 + k$, $\gamma_s(K(n,k)) \leq 2k+2$.*

*Proof.* Suppose that $n = k^2 + k$ and for $1 \leq i, j \leq k+1$, $S = \{D_1, D_2, \ldots, D_{k+1}\}$, $S' = \{D'_1, D'_2, \ldots, D'_{k+1}\}$, where $|D_i| = |D'_i| = k$ and $D_i \cap D_j = D'_i \cap D'_j = \emptyset$ such that $S$ and $S'$ have no common elements. We claim that $S \cup S'$ is a secure dominating set for $G$. Every vertex outside $S \cup S'$ has at least two neighbors in $S \cup S'$. Therefore, by Lemma 2.5, $S \cup S'$ is a secure dominating set. Since $S \cup S'$ does not contain any upper shadows of $(k-1)$-subsets of $[n]$, we have $\gamma_s(G) \leq 2k+2$ where $n \geq k^2 + k$. □

**Theorem 2.8.** *For $n \geq k^2 + 2k$, $\gamma_s(K(n,k)) = k+2$.*

*Proof.* Suppose that $n = k^2+2k$ and $S = \{D_1, D_2, \ldots, D_{k+2}\}$, $|D_i| = k$, $D_i \cap D_j = \emptyset$ where $1 \leq i, j \leq k+2$. We claim that $S$ is a secure dominating set for $G$. It is clear that any subset of size $k+1$ from $S$ is a dominating set for $K(k^2+2k, k)$. Therefore, every vertex outside $S$ has at least two neighbors in $S$ and hence, by Lemma 2.5, the set $S$ is a secure dominating set. Since $S$ does not contain any upper shadows of $(k-1)$-subsets of $[n]$, for $n \geq k^2+2k$, we have $\gamma_s(G) \leq |S| = k+2$. On the other hand, by Theorem 2.6, we have $\gamma_s(G) \geq k+2$. Therefore, $\gamma_s(G) = k+2$. □

**Theorem 2.9.** *If $n \geq \left(\frac{k^2}{2} + \frac{3}{2}k\right)$, $k = 2m$, and $m \geq 2$, then*

$$\gamma_s(K(n,k)) \leq \binom{k+3}{2}.$$

*Proof.* Suppose $n = k^2 - \left(\frac{k^2}{2} - \frac{3}{2}k\right)$ and $S = \{D_1, D_2, \ldots, D_{2m+3}\}$, $|D_i| = m$, and $|D_i \cap D_j| = 0$ where $1 \leq i, j \leq 2m+3$. We claim that $P_2(S)$ (which is all 2-element subsets of set $S$) is a secure dominating set for $G$. Suppose $x = \{x_1, x_2, \ldots, x_k\}$ is an arbitrary vertex of $G$. The vertex $x$ can intersect at most $2m$ elements of $S$. Therefore, it does not intersect at least three elements of $S$ and is dominated by at least two elements of $P_2(S)$. By Lemma 2.5, $P_2(S)$ is a secure dominating set for $G$. Since $P_2(S)$ does not contain any upper shadows of $(k-1)$-subsets of $[n]$, we have for $n \geq k^2 - \left(\frac{k^2}{2} - \frac{3}{2}k\right)$,

$$\gamma_s(K(n,k)) \leq |P_2(S)| = \frac{(2m+3)(2m+2)}{2} = \binom{k+3}{2}.$$

□

**Theorem 2.10.** *If $n \geq \frac{3k^2}{4} + \frac{3}{2}k$, $k = 2m$ and $m \geq 2$, then*

$$\gamma_s(K(n,2)) \leq \tfrac{3k}{2} + 2.$$

*Proof.* Let $n = k^2 - (\frac{k^2}{4} - \frac{3}{2}k)$ and $S = \{D_1, D_2, \ldots, D_{3m+3}\}$ where each $|D_i| = m$, and $|D_i \cap D_j| = 0$ for $1 \leq i, j \leq 3m+2$, i.e., all the subsets are pairwise disjoint. Define, for each $1 \leq i \leq m$, the set $L_i = \{D_{3i-2}, D_{3i-1}, D_{3i}\}$. Now, define: $H = \bigcup_{i=1}^{m} P_2(L_i) \cup \{D_{3m+1}D_{3m+2}\} \cup \{D_{3m+2}D_{3m+3}\}$ where $P_2(L_i)$ denotes the set of all 2-element subsets of $L_i$. We claim that the set $H$ is a secure dominating set for the graph $G$. Let $X = \{x_1, x_2, \ldots, x_{2m}\}$ be an arbitrary vertex of $G$. If $X$ intersects at least two elements of each $L_i$, then $X \cap \{D_{3m+1}, D_{3m+2}\} = X \cap \{D_{3m+2}, D_{3m+3}\} = \emptyset$ and it is dominated by these two. Otherwise, suppose there exists some $j \in \{1, \ldots, m\}$ such that $|L_j \cap X| < 2$. Then there exists a vertex in $P_2(L_j)$ that has no intersection with $X$. Therefore $X$ is s-dominated by at least two vertices of $H$. By Lemma 2.5, we have $\gamma_s(G)) \leq| H |= \frac{3k}{2} + 2$. Since $H$ does not contain the upward shadow of any $(k-1)$-subset of $[n]$, it follows that if $n \geq k^2 - (\frac{k^2}{4} - \frac{3}{2}k)$ then $\gamma_s(G)) \leq \frac{3k}{2} + 2$. □

We state and prove the following theorem to obtain more results.

**Theorem 2.11.** *If there exists a surjective homomorphism from graph $X$ to graph $Y$, then*

$$\gamma_s(Y) \leq \gamma_s(X).$$

*Proof.* Suppose that $S$ is a minimum secure dominating set in graph $X$ and $f$ is a surjective homomorphism from the graph $X$ to the graph $Y$. Since adjacency is preserved in a homomorphism, it is clear that the image of $S$ i.e., $f(S)$ is also a secure dominating set of $Y$. Therefore, we have $\gamma_s(Y) \leq| f(S) |\leq| S |= \gamma_s(X)$. □

The following corollary compares the secure domination number of $K(n-1, k-1)$ and $K(n, k)$.

**Corollary 2.12.** *(i)* $\gamma_s(K(n-1, k-1)) \leq \gamma_s(K(n, k))$.

*(ii)* $\gamma_s(K(n-2, k-1)) \leq \gamma_s(K(n, k))$.

*Proof.* (i) There exists a surjective homomorphism from the graph $K(n, k)$ to $K(n-1, k-1)$ [9]. By Theorem 2.11, we have the result.

(ii) There exists a surjective homomorphism from the graph $K(n, k)$ and $K(n-2, k-1)$ [9]. By Theorem 2.11, we conclude that $\gamma_s(K(n-2, k-1)) \leq \gamma_s(K(n, k))$.

□

We close this section with presenting an upper bound for $\gamma_s(K(2k+1, k))$.

**Theorem 2.13.** $\gamma_s(K(2k+1, k)) \leq \binom{2k}{k-1}$.

*Proof.* Suppose that $[n] = \{c, a_1, \ldots, a_k, b_1, \ldots, b_k\}$ and

$$S = \{\{c, x_1, \ldots, x_{k-1}\} \mid \{x_1, \ldots, x_{k-1}\} \subseteq [n] - c\}.$$

Every arbitrary vertex outside $S$ has at least two neighbors in $S$. Therefore, by Lemma 2.5, the set $S$ is a secure dominating set for $G$ and we have $\gamma_s(G) \leq |S| = \binom{2k}{k-1}$. □

# 3. Secure domination number of $K(n, k)$ for $k = 2, 3$

In this section, we obtain some results for the secure domination number of $K(n, 2)$ for $k = 2, 3$.

## 3.1 Results for $K(n, 2)$

The following theorem gives the secure domination number of $K(n, 2)$.

**Theorem 3.1.** $\gamma_s(K(n,2)) = \begin{cases} 4, & \text{if } n = 5 \text{ and } n \geq 8, \\ 5, & \text{if } n = 6, 7. \end{cases}$

*Proof.* By Theorem 2.13, $\gamma_s(K(5,2)) \leq 4$. An analysis of the graph $K(5,2)$ shows that there exists no secure dominating set of cardinality less than 4. The set $S = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\}\}$ is a secure dominating set for $K(6,2)$. An analysis of the graph $K(6,2)$ shows that there exists no secure dominating set of cardinality less than 5. Therefore, $\gamma_s(K(6,2)) = 5$. The set $S = \{\{1,2\},\{3,5\},\{4,5\},\{3,4\},\{6,7\}\}$ is a secure dominating set for $K(7,2)$. An analysis of the graph $K(7,2)$ shows that there exists no secure dominating set of cardinality less than 5 Therefore, $\gamma_s(K(7,2)) = 5$. Also, by Theorem 2.8, for $n \geq 8$, we have $\gamma_s(K(n,2)) = 4$. □

## 3.2 Results for $K(n, 3)$

In this subsection, we study the secure domination number for the graph $K(n, 3)$.

**Theorem 3.2.** *(i)* $5 \leq \gamma_s(K(n,3)) \leq 14$*, for* $n = 7, 8$.

*(ii)* $5 \leq \gamma_s(K(n,3)) \leq 12$*, for* $n = 9, 10, 11$.

*(iii)* $5 \leq \gamma_s(K(n,3)) \leq 8$ *for* $n = 12, 13, 14$.

*(iv) For* $n \geq 15$*,* $\gamma_s(K(n,2)) = 5$.

*Proof.* (i) Suppose that $S$ is the set of blocks of a projective plane of order 2, and $S'$ is the set of blocks of another projective plane of order 2, whose blocks are distinct from those of $S$. In this case, every vertex outside of $S \cup S'$ has at least two neighbors in $S \cup S'$ (one neighbor in $S$ and one neighbor in $S'$). Therefore, $S \cup S'$ is a secure dominating set for $K(7,3)$. Since $S \cup S'$ does not contain any upper shadows of 2-subsets, we have:

$$\gamma_s(K(n,3)) \leq |S \cup S'| = 14 \quad \text{for} \quad 7 \leq n \leq 8.$$

Also, by Theorem 2.6, we have:

$$5 \leq \gamma_s(K(n,3)) \leq 14 \quad \text{for} \quad 7 \leq n \leq 8.$$

(ii) The blocks of the affine plane of order 3 form a secure dominating set for the graph $K(n, 2)$ for $n = 9, 10, 11$. By Theorem 2.6, we have $5 \leq \gamma_s(K(n, 3)) \leq 12$ where $9 \leq n \leq 11$.

(iii) By Theorems 2.6 and 2.7, we have $5 \leq \gamma_s(K(n, 3)) \leq 8$ where $12 \leq n \leq 14$.

(iv) By Theorem 2.8, we have $\gamma_s(K(n, 2)) = 5$ where $n \geq 15$.

□

## 4. Conclusion

In this paper, we have investigated the secure domination number, denoted by $\gamma_s(G)$, specifically within the context of Kneser graphs $K(n, k)$. Our primary contribution is the determination of the exact value for the secure domination number when $n$ is sufficiently large relative to $k$. Specifically, we proved that for $n \geq k^2 + 2k$, the secure domination number is $\gamma_s(K(n, k)) = k + 2$.

Throughout our analysis, we established several key results and bounds:

- We proved that for $n \geq k^2 + k$, the secure domination number is bounded below by $k+2$, showing that no secure dominating set of size $k+1$ can exist in this range because any dominating set of size $k+1$ must be a clique.

- We provided upper bounds for various ranges of $n$, including $\gamma_s(K(n, k)) \leq 2k + 2$ for $n \geq k^2 + k$ and $\gamma_s(K(n, k)) \leq \binom{k+2}{2}$ for specific even values of $k$.

- We demonstrated that graph homomorphisms can be used to compare secure domination numbers, showing that if a surjective homomorphism exists from graph $X$ to $Y$, then $\gamma_s(Y) \leq \gamma_s(X)$.

- Finally, we provided exact values and specific ranges for the secure domination number of smaller Kneser graphs where $k = 2$ and $k = 3$, such as determining that $\gamma_s(K(n, 2)) = 4$ for $n = 5$ and $n \geq 8$.

Future research may focus on narrowing the bounds provided for the intermediate range $2k + 1 < n < k^2 + 2k - 1$, where the exact value of $\gamma_s(K(n, k))$ remains an open question.

**Conflicts of Interest.** The authors declare that they have no conflicts of interest regarding the publication of this article.

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