Mathematics Interdisciplinary Research **x** (x) (202x) $xx - xx$

Original Scientific Paper

Inverse Nodal Problem with Jump Conditions on a Finite Interval

Seyfollah Mosazadeh[⋆] *and Hikmet Koyunbakan*

Abstract

In this paper, an inverse nodal problem *(INP)* is solved for the Sturm-Liouville *(S-L)* equation with Dirichlet boundary conditions and discontinuity conditions inside $(0, L)$. First, by new Prüfer substitutions, we obtain the asymptotic estimates of the eigenvalues. Then, by using the nodal points and nodal lengths, we solve the inverse nodal problem and present the potential.

Keywords: Prüfer Substitutions, Discontinuous conditions, Nodal points, Inverse nodal problem.

2020 Mathematics Subject Classification: 34A55; 34L05; 34B24.

How to cite this article
S. Mosazadeh, H. Koyunbakan, Inverse nodal problem with jump conditions on a finite interval, *Math. Interdisc. Res.* **1** (x) (201x) xx-yy.

1. Introduction

Second-order differential equations containing finite-time discontinuous points inside the finite or infinite intervals are considered in different subjects like physics, mechanics, and mathematics. For example, inverse problem for the torsional modes of the Earth is one of these problems. In [1], Hald showed that if the density is known in the lower mantle, then the density is uniquely determined by the eigenvalues of the torsional mode. In [2], the author considered two discontinuities inside the interval $[0, \pi]$, and showed that the eigenvalues uniquely determine

⋆Corresponding author (E-mail: s.mosazadeh@kashanu.ac.ir)
Academic Editor: Abbas Saadatmandi
Received 14 February 2026, Accepted 16 May 2026
DOI: 10.22052/MIR.2026.258408.1572

the potential in the whole interval. For more examples, see also [3–6] and the references therein.

Let us consider the following *(SL)* problem with discontinuity conditions inside the interval $(0, L)$ for $L > 0$:

$$\ell_u := -u'' + p(t)u = \mu^2 u, \qquad 0 < t < L, \tag{1}$$

$$u(0) = 0 = u(L), \tag{2}$$

$$u(\frac{L}{2}+0) = ru(\frac{L}{2}-0), \quad u'(\frac{L}{2}+0) = \frac{1}{r}u'(\frac{L}{2}-0), \tag{3}$$

where μ is the spectral parameter, $p(t)$ is a real function, r is real, $r > 0$, $r \neq 1$, and $p(t) \in C^1([0, L])$.

Solution of the inverse problem for the discontinuous *(SL)* problem is very important and has been widely studied. In [7], Shieh and Yurko considered (1) with boundary and discontinuity conditions $u'(0) - hu(0) = 0$, $u'(L) + Hu(L) = 0$ and discontinuity conditions

$$u'(0) - hu(0) = 0, \quad u'(L) + Hu(L) = 0, \tag{4}$$

$$u(\frac{L}{2}+0) = a_1 u(\frac{L}{2}-0), \quad u'(\frac{L}{2}+0) = \frac{1}{a_1}u'(\frac{L}{2}-0) + a_2 u(\frac{L}{2}-0), \tag{5}$$

where h, H, a_1, a_2 are real and $p \in L(0, L)$. They solved inverse nodal and inverse spectral problems and established connections between these problems. In [8], Yang considered (1), (4) and (5) with $L = 1$, and investigated the stability problem using the nodal set of its eigenfunctions. He showed that the space of discontinuous Sturm-Liouville operators are characterized by (q, h, H, a_1, a_2). Ozkan and Keskin [9] studied the *(SL)* problem consisting of (1) on the interval $(0, 1)$, with boundary and jump conditions dependent on the spectral parameter, linearly, as follows:

$$u(0) = 0, \quad \lambda(u'(1) + H_0 u(1)) - H_1 u'(1) - H_2 u(1) = 0,$$

$$u(\frac{1}{2}+0) = \alpha u(\frac{1}{2}-0), \quad u'(\frac{1}{2}+0) = \frac{1}{\alpha}u'(\frac{1}{2}-0) + (\lambda + \beta)u(\frac{1}{2}-0),$$

where H_i, β are real, $\alpha > 0$, $H_0 H_1 - H_2 > 0$ and $p(x)$ is real-valued function in $L^2(0, 1)$. They showed that the nodal points can uniquely determine the coefficients of the problem (see also [10]). Inverse nodal and inverse eigenvalue problems for *(SL)* operator were studied for various cases by Wang and Yurko [11, 12]. Also, there are more results on inverse nodal and eigenvalue problems in the references [13–22]. We note that Mosazadeh [16] considered a discontinuous non-selfadjoint Sturm-Liouville operator with boundary conditions nonlinearly dependent on the spectral parameter, and obtained the asymptotic representation of the solutions and the eigenvalues. He provided a constructive procedure to obtain the asymptotic form of the eigenfunctions and their partial derivatives in discontinuous case by the canonical form of the Bessel functions. For the continuous case, see [23].

We collected this paper as follows: in Section 2, we will give the properties of Prüfer substitutions. These properties are obtained in two cases, $t < \frac{L}{2}$ and $t > \frac{L}{2}$. As far as we know, our definition of Prüfer substitutions will be a new definition in the literature. Then, we obtain the eigenvalues of the problem (1)-(3). In Section 3, we will give nodal parameters and also present the solution of the *(INP)* associated with (1)-(3).

2. The eigenvalues

In this section, we will present the Prüfer substitutions and some of their properties. Specifically, we will introduce new Prüfer substitutions for cases when the phase function is discontinuous over the entire interval. Then, we obtain the eigenvalues of the operator ℓ with Dirichlet conditions (2) and discontinuity conditions (3), by using the modified Prüfer substitutions. We mention that since the old Prüfer substitutions are not applicable for the discontinuous case in our study, the *(INP)* is investigated differently from the studies in [7–10].

By given solution $u(t, \mu)$ of Equation (1), we denote

$$\begin{cases} u(t,\mu) = R(t)\sin(\mu\vartheta(t)), \\ u'(t,\mu) = \mu R(t)\cos(\mu\vartheta(t)). \end{cases}$$

Here, $R(t)$ and $\vartheta(t, \mu)$ are called the aptitude and the phase functions, respectively. Substituting $u(t, \mu) = R(t)\sin(\mu\vartheta(t))$ into (1), and by standard computations, we can easily obtain $\vartheta'(t, \mu) = 1 - \frac{1}{\mu^2}p(t)\sin^2(\mu\vartheta(t))$. For the solution of (1), we assume that R is positive.

In case of discontinuity, Prüfer substitutions are defned (1)-(3) as:

$$u(t,\mu) = R(t)\sin(\mu\vartheta(t)), \qquad \text{for} \qquad t < \frac{L}{2}, \tag{6}$$

$$u(t,\mu) = R(L-t)\sin(\mu\vartheta(L-t)), \qquad \text{for} \qquad \frac{L}{2} < t < L. \tag{7}$$

Differentiating both sides of the Equations (6)-(7) with respect to t, one obtains that

$$u'(t,\mu) = \mu R(t)\cos(\mu\vartheta(t)), \qquad \text{for} \qquad t < \frac{L}{2}, \tag{8}$$

$$u'(t,\mu) = \mu R(L-t)\cos(\mu\vartheta(L-t)), \qquad \text{for} \qquad \frac{L}{2} < t < L. \tag{9}$$

Applying (6)-(9) to (1), it can be easily obtained

$$\vartheta'(t) = 1 - \frac{1}{\mu^2}p(t)\sin^2(\mu\vartheta(t)), \qquad for \qquad t < \frac{L}{2}, \tag{10}$$

$$\vartheta'(L-t) = 1 - \frac{1}{\mu^2} p(t)\sin^2(\mu\vartheta(L-t)), \qquad for \qquad t > \frac{L}{2}. \tag{11}$$

Many types of Prüfer substitutions have been presented by other authors, but the definitions given in (6)-(9) are new. These substitutions will be main key to studying and solving the *(INP)* for the discontinuous *(SL)* problem. We insistently state that in [7, 9, 10], the *(INP)* was investigated with different boundary conditions and without using Prüfer substitutions.

Theorem 2.1. *The eigenvalues of the problem* (1)-(3) *are*

$$\mu_n = \frac{2n\pi}{L} + \frac{1}{4n\pi}\Big(\zeta_1 + (-1)^{n-1}\zeta_2 + \frac{8\gamma_n^2}{L}\Big) + O(\frac{1}{n^2}),$$

where

$$\zeta_1 = \int_0^L p(t)dt, \qquad \zeta_2 = \int_0^L p(t)dt - 2\int_0^{\frac{L}{2}} p(t)dt,$$

$$\gamma_n = \begin{cases} -\sin^{-1}(\frac{|r|}{\sqrt{1+r^2}}), & \text{if } n \text{ is odd}, \\ \sin^{-1}(\frac{1}{\sqrt{1+r^2}}), & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since $u(0) = 0$, then $\vartheta(0,\mu) = 0$. Let $\mu = \mu_n$, it follows from discontinuity conditions (3) that

$$\begin{cases} \vartheta(\frac{L}{2}-0) = \frac{1}{\mu}\Big(2n\pi + \sin^{-1}(\frac{1}{\sqrt{1+r^2}})\Big), & n = 0,1,2,3,\cdots, \\ \vartheta(\frac{L}{2}+0) = \frac{1}{\mu}\Big((2n-1)\pi - \sin^{-1}(\frac{|r|}{\sqrt{1+r^2}})\Big), & n = 1,2,3,\cdots. \end{cases}$$

Integrating both sides of the Equation (10) with respect to x from 0 to $\frac{L}{2}$ with $\mu = \mu_n$, we obtain:

$$\begin{aligned} \vartheta(\frac{L}{2}-0) &= \int_0^{\frac{L}{2}} \Big(1 - \frac{p(t)}{\mu_n^2}\sin^2(\mu_n\vartheta(t))\Big)dt \\ &= \frac{L}{2} - \frac{1}{2\mu_n^2}\int_0^{\frac{L}{2}} p(t)dt + \frac{1}{2\mu_n^2}\int_0^{\frac{L}{2}} p(t)\cos(2\mu_n\vartheta(t))dt. \end{aligned} \tag{12}$$

Since

$$\cos(2\mu_n\vartheta(t)) = \frac{1}{2\mu_n\vartheta'(t)}\Big(\sin(2\mu_n\vartheta(t))\Big)',$$

using integration by parts, we have

$$\int_0^{\frac{L}{2}} p(t)\cos(2\mu_n\vartheta(t))dt = \Big(\frac{1}{2\mu_n\vartheta'(t)}\sin(2\mu_n\vartheta(t))\Big)\Big|_0^{\frac{L}{2}}$$

$$-\frac{1}{2\mu_n}\int_0^{\frac{L}{2}}\Big(\frac{p(t)}{\vartheta'(t)}\Big)'\sin(2\mu_n\vartheta(t))dt.$$

This together with

$$\vartheta' = 1 - O(\frac{1}{\mu_n^2}), \quad \vartheta'' = O(\frac{1}{\mu_n}), \qquad n\to\infty,$$

yields that

$$\int_0^{\frac{L}{2}} p(t)\cos(2\mu_n\vartheta(t))dt = O(\frac{1}{\mu_n}). \tag{13}$$

Substituting (13) into (12), we have

$$\vartheta(\frac{L}{2}-0) = \frac{L}{2} - \frac{1}{2\mu_n^2}\int_0^{\frac{L}{2}} p(t)dt + O(\frac{1}{\mu_n^3}). \tag{14}$$

Similarly, from integrating both sides of (11) with respect to t from $\frac{L}{2}$ to L, we obtain:

$$\vartheta(\frac{L}{2}+0) = \frac{L}{2} - \frac{1}{2\mu_n^2}\Big(\int_0^{L} p(t)dt - \int_0^{\frac{L}{2}} p(t)dt\Big) + O(\frac{1}{\mu_n^3}). \tag{15}$$

From (14) and (15), we obtain the following for each $n\in\mathbb{N}$:

$$\frac{n\pi}{\mu_n} = \frac{L}{2} - \frac{\gamma_n}{\mu_n} - \frac{1}{2\mu_n^2}\Big(\frac{\zeta_1+(-1)^{n-1}\zeta_2}{2}\Big) + O(\frac{1}{\mu_n^3}).$$

Therefore,

$$\mu_n = n\pi\Big\{\frac{L}{2} - \frac{\gamma_n}{\mu_n} - \frac{1}{2\mu_n^2}\Big(\frac{\zeta_1+(-1)^{n-1}\zeta_2}{2}\Big) + O(\frac{1}{\mu_n^3})\Big\}^{-1},$$

and hence we arrive the eigenvalues. □

3. Inverse nodal problem

In this section, first, the asymptotic forms of the nodal points and the nodal lengths are presented. Then, we deduce a reconstruction formula for the potential function q by using the nodal lengths.

For the boundary value problem (SL), an analog of Sturm's oscillation theorem [20] is true. More precisely, the eigenfunction $u_n(t) = u(t,\mu_n)$ has exactly $n-1$ (simple) zeros inside the interval $(0,L)$. Let $\{t_n^k\}_{k=1}^{n-1}$ be the zeros of the eigenfunction $u_n(t)$. The following theorem gives the asymptotic form of the nodes of (1)-(3).

Theorem 3.1. *The nodes of the problem* (1)-(3) *have the following asymptotic form as* $n \to \infty$*:*

$$t_n^k = \frac{kL}{n} + \frac{L^2}{8n^2\pi^2}\int_0^{t_n^k} p(\tau)d\tau + O(\frac{1}{n^3}), \quad t_n^k \in (0, \frac{L}{2}), \tag{16}$$

$$\begin{aligned} t_n^k =& \frac{kL}{n} - \frac{kL\gamma_n}{n^2\pi} + \frac{L^2}{8n^2\pi^2}\int_0^{t_n^k} p(\tau)d\tau \\ &- \frac{L^2}{8n^2\pi^2}\int_0^{\frac{L}{2}} p(\tau)d\tau + O(\frac{1}{n^3}), \quad t_n^k \in (\frac{L}{2}, L). \end{aligned} \tag{17}$$

Proof. For $t_n^k \in (0, \frac{L}{2})$, integrating (10) from 0 to t_n^k and letting $\mu = \mu_n$, we have

$$\frac{2k\pi}{\mu_n} = t_n^k - \frac{1}{2\mu_n^2}\int_0^{t_n^k} p(\tau)d\tau + \frac{1}{2\mu_n^2}\int_0^{t_n^k} p(\tau)\cos(2\mu_n\vartheta(\tau))d\tau. \tag{18}$$

Using the asymptotic form of μ_n in Theorem 2.1, we get

$$\frac{1}{\mu_n} = \frac{L}{2n\pi} - \frac{L^2}{16n^3\pi^3}\Big(\zeta_1 + (-1)^{n-1}\zeta_2 + \frac{8\gamma_n^2}{L}\Big) + O(\frac{1}{n^3}), \tag{19}$$

$$\frac{1}{\mu_n^2} = \frac{L^2}{4n^2\pi^2} + O(\frac{1}{n^3}). \tag{20}$$

Substituting (19)-(20) into (18), we have

$$t_n^k = \frac{kL}{n} + \frac{L^2}{8n^2\pi^2}\int_0^{t_n^k} p(\tau)d\tau - \frac{L^2}{8n^2\pi^2}\int_0^{t_n^k} p(\tau)\cos(2\mu_n\vartheta(\tau))d\tau + O(\frac{1}{n^3}). \tag{21}$$

On the other hand,

$$\int_0^{t_n^k} p(\tau)\cos(2\mu_n\vartheta(\tau))d\tau \longrightarrow 0,$$

as $n \to \infty$. From this and (21), we obtain (16). Similarly, for $t_n^k \in (\frac{L}{2}, L)$, integrating (11) from $\frac{L}{2}$ to t_n^k and letting $\mu = \mu_n$, we obtain:

$$\begin{aligned} t_n^k =& \frac{kL}{n} - \frac{kL\gamma_n}{n^2\pi} + \frac{L^2}{8n^2\pi^2}\int_0^{t_n^k} p(\tau)d\tau - \frac{L^2}{8n^2\pi^2}\int_0^{\frac{L}{2}} p(\tau)d\tau \\ &+ d_n + O(\frac{1}{n^3}), \end{aligned}$$

where

$$d_n = -\frac{L^2}{8n^2\pi^2}\Big(\int_0^{t_n^k} - \int_0^{\frac{L}{2}}\Big)p(\tau)\cos(2\mu_n\vartheta(L-\tau))d\tau.$$

Since $d_n \longrightarrow 0$ as $n \to \infty$, we arrive at (17). □

The nodal set $\{t_n^k\}_{n\geq 1, k=\overline{1,n}}$ is dense on the interval $(0, L)$. Denote that $\ell_n^i = t_n^{k+1} - t_n^k$, $1 \leq k \leq n-1$, to be the nodal lengths of $u(t, \mu_n)$, where $t_n^n = L$.

Theorem 3.2. *Asymptotic estimates of the nodal lengths for the problem (1)-(3) satisfy*

$$\ell_n^k = \frac{2\pi}{\mu_n} + \frac{1}{2\mu_n^2}\int_{t_n^k}^{t_n^{k+1}} p(\tau)d\tau + o(\frac{1}{\mu_n^2}), \quad t_n^k \in (0, \frac{L}{2}), \tag{22}$$

$$\ell_n^k = \frac{2\pi}{\mu_n} + \frac{\gamma_n}{\mu_n} + \frac{1}{2\mu_n^2}\int_{t_n^k}^{t_n^{k+1}} p(\tau)d\tau + o(\frac{1}{\mu_n^2}), \quad t_n^k \in (\frac{L}{2}, L). \tag{23}$$

Now, let $T_n = \{t_n^k\}_{k=1}^{n-1}$ be the nodal set.

Theorem 3.3. *Set the variable s to either 0 or 1. Given T_n. For $t < \frac{L}{2}$, the function $p \in C^1([0, L])$ has the form*

$$p(t) = \lim_{n\to\infty} \frac{8n^2\pi^2}{L^2}\Big(\frac{n\ell_n^k}{L} - 1\Big) + \frac{1}{L}(\zeta_1 + (-1)^{s-1}\zeta_2) \tag{24}$$

for $k = k_n(t) = \max\{k : \ t_n^k < t\}$.

Proof. According to Theorem 2.1, $\mu_n \approx \frac{2n\pi}{L}$ as $n \to \infty$. Hence, from (22) we get

$$\ell_n^k = \frac{L}{n} + \frac{L^2}{8n^2\pi^2}\int_{t_n^k}^{t_n^{k+1}} p(\tau)d\tau + o(\frac{1}{\mu_n^2}).$$

On the other hand, from (22) and applying the mean value theorem, with fixed n, there exists $c_{n,k} \in (t_n^k, t_n^{k+1})$ such that $\int_{t_n^k}^{t_n^{k+1}} p(\tau)d\tau = p(c_{n,k})\ell_n^k$. Therefore, we obtain:

$$\ell_n^k = \frac{2\pi}{\mu_n} + \frac{1}{2\mu_n^2}p(c_{n,k})\ell_n^k.$$

Thus,

$$p(c_{n,k}) = \frac{4\pi\mu_n}{\ell_n^k}(\frac{\mu_n\ell_n^k}{2\pi} - 1) + o(1).$$

This together with $\frac{\mu_n\ell_n^k}{2\pi} = 1 + O(\frac{1}{n})$ yields that

$$p(t) = \lim_{n\to\infty} 2\mu_n^2(\frac{\mu_n\ell_n^k}{2\pi} - 1), \qquad t < \frac{L}{2}. \tag{25}$$

Now, substituting the asymptotic form of μ_n from Theorem 2.1 into (25), we arrive at (24). □

Similarly, we can prove the following theorem.

Theorem 3.4. *Set the variable s to either 0 or 1, and* $t \in (\frac{L}{2}, L)$. *Let* T *be a subset of nodal points which is dense on* $(0, L)$. *Then, the function* $p(t)$ *can be reconstructed via the formula:*

$$p(t) = \lim_{n\to\infty} \frac{16n^2\pi^3}{L^2(2\pi + \gamma_n)} \Big(\frac{n\ell_n^k}{L} - \frac{\gamma_n}{2\pi} - 1 \Big) + \frac{2\pi(\zeta_1 + (-1)^{s-1}\zeta_2)}{L(2\pi + \gamma_s)}$$

for $k = k_n(t) = \max\{k : \ t_n^k < t\}$.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgment. This research is partially supported by the University of Kashan under grant number 1469125/1.

References

[1] O. H. Hald, Discontinuous inverse eigenvalue problems, *Commun. Pure Appl. Math.* **37** (1984) 539 – 577, https://doi.org/10.1002/cpa.3160370502.

[2] C. Willis, Inverse Sturm-Liouville problems with two discontinuities, *Inverse Problems* **1** (1985) 263 – 289, https://doi.org/10.1088/0266-5611/1/3/010.

[3] E. R. Lapwood and T. Usami, *Free Oscillations of the Earth*, Cambridge University Press, Cambridge, 1981.

[4] A. Neamaty and Y. Khalili, Determination of a differential operator with discontinuity from interior spectral data, *Inverse Probl. Sci. Eng.* **22** (2014) 1002 – 1008, https://doi.org/10.1080/17415977.2013.848436.

[5] M. Shahriari, A. Jodayree Akbarfama and G. Teschl, Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions, *J. Math. Anal. Appl.* **395** (2012) 19 – 29, https://doi.org/10.1016/j.jmaa.2012.04.048.

[6] R. Kh. Amirov, On a system of Dirac differential equations with discontinuity conditions inside an interval, *Ukrainian Math. J.* **57** (2005) 712 – 727, https://doi.org/10.1007/s11253-005-0222-7.

[7] C. T. Shieh and V. A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, *J. Math. Anal. Appl.* **347** (2008) 266 – 272, https://doi.org/10.1016/j.jmaa.2008.05.097.

[8] C.-F. Yang, Inverse nodal problems of discontinuous Sturm-Liouville operator, *J. Differential Equations* **254** (2013) 1992 – 2014, https://doi.org/10.1016/j.jde.2012.11.018.

[9] A. S. Ozkan and B. Keskin, Spectral problems for Sturm-Liouville operator with boundary and jump conditions linearly dependent on the eigenparameter, *Inverse Probl. Sci. Eng.* **20** (2012) 799 – 808, https://doi.org/10.1080/17415977.2011.652957.

[10] A. S. Ozkan and B. Keskin, Inverse nodal problems for Sturm-Liouville equation with eigenparameter-dependent boundary and jump conditions, *Inverse Probl. Sci. Eng.* **23** (2015) 1306 – 1312, https://doi.org/10.1080/17415977.2014.991730.

[11] Y. P. Wang, Inverse problems for discontinuous Sturm-Liouville operators with mixed spectral data, *Inverse Probl. Sci. Eng.* **23** (2015) 1180 – 1198, https://doi.org/10.1080/17415977.2014.981748.

[12] Y. P. Wang and V. A. Yurko, On the inverse nodal problems for discontinuous Sturm-Liouville operators, *J. Differential Equations* **260** (2016) 4086 – 4109, https://doi.org/10.1016/j.jde.2015.11.004.

[13] E. Bairamov, Y. Aygar and S. Cebesoy, Spectral properties of quadratic pencil of Schrödinger equations with transmission conditions, *J. Phys.: Conf. Ser.* **1053** (2018) #012062, https://doi.org/10.1088/1742-6596/1053/1/012062.

[14] O. Sh. Mukhtarov and K. Aydemir, Two-linked periodic Sturm-Liouville problems with transmission conditions, *Math. Methods Appl. Sci.* **44** (2021) 14664 – 14676, https://doi.org/10.1002/mma.7734.

[15] V. A. Sadovnichi, Y. T. Sultanaev and A. M. Akhtyamov, Solvability theorems for an inverse nonself-adjoint Sturm- Liouville problem with nonseparated boundary conditions, *Differ. Equ.* **51** (2015) 717 – 725, https://doi.org/10.1134/S0012266115060026.

[16] S. Mosazadeh, A new approach to asymptotic formulas for eigenfunctions of discontinuous non-selfadjoint Sturm-Liouville operators, *J. Pseudo-Differ. Oper. Appl.* **11** (2020) 1805 – 1820, https://doi.org/10.1007/s11868-020-00350-2.

[17] S. A. Buterin and C. T. Shieh, Incomplete inverse spectral and nodal problems for differential pencils, *Results Math.* **62** (2012) 167 – 179, https://doi.org/10.1007/s00025-011-0137-6.

[18] S. Currie and B. A. Watson, Inverse nodal problems for Sturm-Liouville equations on graphs, *Inverse Probl.* **23** (2007) 2029 – 2040, https://doi.org/10.1088/0266-5611/23/5/013.

[19] S. Goktas, H. Koyunbakan and T. Gulsen, Inverse nodal problem for polynomial pencil of Sturm-Liouville operator, *Math. Methods Appl. Sci.* **41** (2018) 7576 – 7582, https://doi.org/10.1002/mma.5220.

[20] B. Simon, *Sturm oscillation and comparison theorems*, (W. Amrein, A. Hinz, and D. Pearson, eds.) Sturm-Liouville Theory. Birkhäuser Basel, 2005.

[21] T. Gulsen, E. Yilmaz and S. Akbarpoor, Numerical investigation of the inverse nodal problem by Chebyshev interpolation method, *Therm. Sci.* **22** (2018) 123 − 136.

[22] Y. Guo and G. Wei, Inverse problems: Dense nodal subset on an interior subinterval, *J. Differential Equations* **255** (2013) 2002 − 2017, https://doi.org/10.1016/j.jde.2013.06.006.

[23] S. Mosazadeh, Eigenfunction expansions for second-order boundary value problems with separated boundary conditions, *Math. Interdisc. Res.* **1** (2016) 325 − 334, https://doi.org/10.22052/mir.2016.33850.

Seyfollah Mosazadeh
Department of Mathematics,
Faculty of Mathematical Sciences,
University of Kashan,
Kashan 87317-53153, I. R. Iran
e-mail: s.mosazadeh@kashanu.ac.ir

Hikmet Koyunbakan
Firat University,
Faculty of Science,
Department of Mathematics 23119,
Elazig, Turkey
e-mail: hkoyunbakan@gmail.com