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The Role of Multipliers in Seismic Signal Processing and Structural Response

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Abstract

We present a simplified formulation of multipliers in commutative Banach algebras by relaxing the classical faithfulness condition in [1, Theorem 3.2]. A concise and transparent proof is provided. These multipliers have significant applications in engineering, particularly in earthquake engineering, where frequency-domain analysis plays a central role. This study investigates their use in signal theory for seismic wave analysis through the Fourier transform. Multipliers are interpreted as frequency-domain operators for filtering and modeling structural responses, illustrating their relevance to seismic optimization. Numerical simulations support the theoretical results and demonstrate that multiplier-based filtering can effectively modify seismic signal spectra in applied settings.

Keywords: Multiplier, Fourier transform, Seismic signal processing, Frequency response function.

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1. Introduction

Helgason [2] was the first to introduce the concept of a multiplier in the following setting: consider a commutative, semisimple Banach algebra A, and let $\Delta(A)$ represent its maximal ideal space. The Gelfand transform $\hat{A}$ realizes A as a subalgebra

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of the algebra of continuous functions defined on $\Delta(A)$. A bounded continuous function g on $\Delta(A)$ is called a multiplier on A if $g\hat{A} \subseteq \hat{A}$. Later, Wang [3] extended this theory to more general settings, particularly for faithful Banach algebras.

Recall that when A is a Banach algebra, a continuous linear operator $T : A \to A$ is termed a multiplier whenever

$$x \cdot T(y) = T(x) \cdot y, \quad \text{for all } x, y \in A.$$

We denote by $M(A)$ the algebra of all such multipliers.

Adib et al. have investigated multiplier structures and their generalizations called quasi-multipliers, within various functional-analytic frameworks. These include, in particular, duals of Banach algebras and certain classes of topological algebras (see [4–7]). The emphasis has been on the topological and algebraic properties that emerge in these settings. Recent interdisciplinary studies have also revealed that multiplier-like constructs, often appearing under the name of filters, find significant applications in fields such as electrical engineering, civil engineering, and seismology. This paper aims to establish a rigorous analytical foundation for these applied manifestations of multiplier theory.

A key reason why multipliers are valuable in engineering, particularly in earthquake engineering, is their natural definition via the Fourier transform, as shown in Theorem 3.7. Multipliers have numerous applications across the sciences, and here we highlight their role in signal theory and earthquake engineering. In earthquake engineering, analyzing the frequency content of ground vibration waves is crucial for effective design. Earthquake ground motions contain a broad range of frequencies, each influencing structural behavior differently. Some frequencies can resonate with a structure's natural frequency, causing amplified responses and potential damage, while others have minimal effects [8]. Fourier-transform-based spectral analysis allows engineers to decompose complex seismic signals into their constituent frequency components, enabling detailed study of the energy distribution of seismic waves [9].

2. Preliminaries

Definition 2.1. An algebra A is called *faithful* if the following condition holds:

$$aA = Aa = \{0\} \implies a = 0.$$

In particular, A is faithful if it admits an approximate identity; for example, every locally C^*-algebra is faithful.

Definition 2.2. Suppose that A is a Banach algebra. A mapping $T : A \to A$ which is linear and continuous is said to be a *left multiplier* on A whenever

$$T(xy) = T(x)\, y \quad \text{for every } x, y \in A.$$

In an analogous way, T is called a *right multiplier* on A whenever

$$T(xy) = x\,T(y) \quad \text{for every } x, y \in A.$$

When T satisfies both of these conditions, it is referred to simply as a *multiplier* on A. The collection of all multipliers of A is denoted by $M(A)$, and this set forms a subalgebra of the algebra of bounded linear operators on A.

3. Theoretical foundations of multiplier operators

3.1 Multipliers on Banach algebras and the Gelfand transform

Definition 3.1. Suppose that A is a *commutative Banach algebra with identity* over $\mathbb{C}$. The *maximal ideal space* of A, denoted by $\Delta(A)$, consists of all nonzero homomorphisms from A into $\mathbb{C}$. This space carries the weak-* topology induced from the dual space A^*. Via the kernel map, each element of $\Delta(A)$ corresponds uniquely to a maximal ideal of A.

For each $a \in A$, the *Gelfand transform* associated with a is defined as the mapping

$$\hat{a} : \Delta(A) \to \mathbb{C}, \qquad \hat{a}(\phi) = \phi(a),$$

for all $\phi \in \Delta(A)$.

The correspondence $a \mapsto \hat{a}$ defines an algebra homomorphism from A into $C(\Delta(A))$, the algebra of continuous complex-valued functions on $\Delta(A)$. If A is semisimple (i.e., the Jacobson radical of A is zero), then this map is injective. If the Banach algebra A is *not unital*, then the maximal ideal space $\Delta(A)$ is locally compact but not compact, and the Gelfand transform maps A into $C_0(\Delta(A))$, the algebra of continuous functions vanishing at infinity on $\Delta(A)$. Thus, the unitality of A determines whether the Gelfand transform takes values in $C(\Delta(A))$ or in $C_0(\Delta(A))$.

Remark1. The following theorem is a classical result in the theory of multipliers on commutative Banach algebras; see, for example, [1, Theorem 4.1.5]. In the classical setting, one usually assumes that the algebra A is *faithful* (or semisimple), which ensures the *uniqueness* of the multiplier function.

However, in our applications to multiplier operators on signals (Section 4), we do not require uniqueness of the multiplier function φ. Therefore, we can state and prove a version of the theorem under the *weaker assumption* that A is commutative, without assuming faithfulness. This allows a simpler and more direct proof than the classical one.

Theorem 3.2. *Let A be a commutative Banach algebra, and let T be a multiplier operator on A. Then there exists a bounded continuous function φ defined on the* *maximal ideal space* $\Delta(A)$ *such that for every* $a \in A$*, the Gelfand transform of* $T(a)$ *satisfies*

$$\widehat{T(a)} = \varphi \cdot \hat{a}.$$

Proof. For each $m \in \Delta(A)$, choose any $a \in A$ with $\hat{a}(m) \neq 0$ (if none exists, then $\hat{b}(m) = 0$ for all b, and the formula holds trivially). Define

$$\varphi(m) := \frac{\widehat{T(a)}(m)}{\hat{a}(m)}.$$

The multiplier property $T(ab) = aT(b) = bT(a)$ ensures that this definition is independent of the choice of a at points where $\hat{a}(m) \neq 0$. Continuity of φ follows from continuity of the Gelfand transforms $\widehat{T(a)}$ and $\hat{a}$, since locally one can always choose a with $\hat{a}(m) \neq 0$. Finally, for each $a \in A$ and $m \in \Delta(A)$, the definition ensures

$$\widehat{T(a)}(m) = \varphi(m)\hat{a}(m),$$

so $\widehat{T(a)} = \varphi \cdot \hat{a}$. □

Definition 3.3. A Banach algebra A is called *semisimple* if its radical ideal is trivial, i.e.,

$$\mathrm{rad}(A) = \{0\}.$$

Equivalently, this means that the *Gelfand transform* $a \mapsto \hat{a}$, acts as an injective algebra homomorphism from A into $C_0(\Delta(A))$, the algebra of bounded continuous functions on the maximal ideal space $\Delta(A)$. Consequently, the algebraic and topological structure of A is faithfully represented by the function algebra $\hat{A}$.

Note that every commutative semisimple Banach algebra is faithful. Indeed, semisimplicity implies that the Jacobson radical is zero, i.e., $\mathrm{rad}(A) = \{0\}$, which ensures that the Gelfand transform $a \mapsto \hat{a}$, is injective. As a result, if $a \in A$ satisfies $aA = \{0\}$ or $Aa = \{0\}$, then $\hat{a} \cdot \hat{b} = 0$ for all $b \in A$, implying $\hat{a} = 0$, and hence $a = 0$. Therefore, no nonzero element of A can annihilate the whole algebra, and A is faithful.

Corollary 3.4. *Let* A *be a semisimple commutative Banach algebra. Then the equation* $\widehat{T(a)} = \varphi\ \hat{a}, \quad a \in A$*, defines a continuous isomorphism of* $M(A)$ *onto* $C_0(\Delta(A))$.

Proof. Let A be a commutative semisimple Banach algebra. By definition, semisimplicity means that the Jacobson radical of A is trivial, i.e.,

$$\mathrm{rad}(A) = \{0\},$$

which implies that the Gelfand transform $a \mapsto \hat{a}$, is an injective algebra homomorphism from A into $C_0(\Delta(A))$. By Theorem 3.2, for every multiplier $T \in M(A)$, there corresponds a unique function $\varphi \in C_0(\Delta(A))$, which is bounded and continuous, such that for all $a \in A$, $\widehat{T(a)} = \varphi \cdot \hat{a}$, and $\|\varphi\|_\infty \leq \|T\|$.

Since A is faithful (as every semisimple Banach algebra is faithful), the Gelfand transform is injective and separates points of A. Therefore, the map

$$\Phi : M(A) \to C_0(\Delta(A)), \quad \Phi(T) = \varphi,$$

is injective.

To prove surjectivity, let $\varphi \in C_0(\Delta(A))$ be arbitrary. Define the operator $T_\varphi : A \to A$ by

$$T_\varphi(a) = \mathcal{G}^{-1}(\varphi\hat{a}),$$

where $\mathcal{G}^{-1}$ denotes the inverse of the Gelfand transform restricted to its image. Since multiplication by φ preserves $\hat{A} \subseteq C_0(\Delta(A))$, it follows that T_φ is a multiplier on A. Moreover, the norm satisfies $\|T_\varphi\| = \|\varphi\|_\infty$, showing that Φ is an isometry. Hence, Φ is a continuous algebra isomorphism of $M(A)$ onto $C_0(\Delta(A))$, as claimed. □

Example 3.5. (Multipliers on an infinite-dimensional commutative Banach algebra $C_0([0,1])$). Let $A = C_0([0,1])$, denote the Banach algebra of all continuous complex-valued functions on the interval $[0,1]$ that vanish at the boundary, equipped with the supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ and pointwise multiplication.

The algebra A is commutative, semisimple, and infinite-dimensional. Its maximal ideal space $\Delta(A)$ is homeomorphic to the compact space $[0,1]$, and the Gelfand transform coincides with the identity mapping, that is,

$$\widehat{f}(x) = f(x), \quad f \in A, \ x \in [0,1].$$

By Corollary 3.4, every multiplier $T \in M(A)$ is uniquely determined by a bounded continuous function $\varphi \in C_b([0,1])$ such that

$$(Tf)(x) = \varphi(x)f(x), \quad \text{for all } f \in A \text{ and } x \in [0,1].$$

Moreover, the mapping $T \longmapsto \varphi$, defines an isometric algebra isomorphism between $M(A)$ and $C_b([0,1])$.

This example shows that the abstract multiplier theory developed in Section 3 applies naturally to infinite-dimensional Banach algebras and yields a concrete representation of multiplier operators as pointwise multiplication operators. It also demonstrates that the use of Banach algebra techniques is essential and not merely formal, even in simple functional settings.

3.2 Multipliers on group algebras via the Fourier transform

The Gelfand transform on the commutative Banach *-algebra $L_1(\mathbb{R})$ coincides with the Fourier transform. More generally, for a locally compact abelian group K, the Fourier transform acts as the Gelfand transform on $L_1(K)$.

The proof of Theorem 3.7 here is much simpler than the classical approach (see, e.g., [10, Theorem 2.1.13]), as it directly uses the Fourier transform and the multiplier property. In applications such as signal filtering, only the existence of the multiplier function m_T is needed, making the proof more natural and accessible.

Definition 3.6. For $f \in L_1(K)$ and $\nu \in M(K)$, the Fourier and Fourier–Stieltjes transforms are defined by

$$\hat{\nu}(\omega) = \int_K \omega(-x)\, d\nu(x), \quad \hat{f}(\omega) = \int_K f(x)\, \omega(-x)\, dx, \quad \omega \in \widehat{K}.$$

Theorem 3.7. *Let K be a locally compact abelian group, and let $T : L_1(K) \to L_1(K)$ be a bounded linear operator. The following statements are equivalent:*

(i) The operator T acts as a multiplier on $L_1(K)$; that is, for every $f, g \in L_1(K)$,

$$T(f * g) = T(f) * g = f * T(g).$$

(ii) There exists a unique bounded function $m_T : \widehat{K} \to \mathbb{C}$ such that the Fourier transform of Tf is given by pointwise multiplication:

$$\widehat{Tf}(\omega) = m_T(\omega)\hat{f}(\omega) \quad \textit{for all } \omega \in \widehat{K},\ f \in L_1(K).$$

Proof. **(i)** $\Rightarrow$ **(ii)**: Assume T acts as a multiplier, i.e., for every $f, g \in L_1(K)$,

$$T(f * g) = T(f) * g = f * T(g).$$

Applying the Fourier transform, which converts convolution into pointwise multiplication, yields

$$\widehat{T(f * g)} = \widehat{T(f)} \cdot \hat{g} = \hat{f} \cdot \widehat{T(g)}.$$

Fix g such that $\hat{g}(\omega) \neq 0$ for some $\omega \in \widehat{K}$. Then for each $f \in L_1(K)$ with $\hat{f}(\omega) \neq 0$,

$$\frac{\widehat{T(f)}(\omega)}{\hat{f}(\omega)} = \frac{\widehat{T(g)}(\omega)}{\hat{g}(\omega)} =: m_T(\omega).$$

This defines a bounded function $m_T : \widehat{K} \to \mathbb{C}$, independent of f. By varying ω, the function m_T is well-defined on $\widehat{K}$, and we have

$$\widehat{T(f)} = m_T \cdot \hat{f} \quad \text{for all } f \in L_1(K).$$

(ii) $\Rightarrow$ **(i)**: Conversely, suppose there exists a bounded function $m_T : \widehat{K} \to \mathbb{C}$ such that

$$\widehat{T(f)} = m_T \cdot \hat{f} \quad \text{for all } f \in L_1(K).$$

For any $f, g \in L_1(K)$,

$$\widehat{T(f) * g} = \widehat{T(f)} \cdot \hat{g} = m_T \cdot \hat{f} \cdot \hat{g} = \hat{f} \cdot (m_T \cdot \hat{g}) = \hat{f} \cdot \widehat{T(g)} = \widehat{f * T(g)}.$$

Taking inverse Fourier transform yields $T(f) * g = f * T(g)$. Also,

$$T(f * g) = \mathcal{F}^{-1}\big(m_T \cdot \widehat{f * g}\big) = \mathcal{F}^{-1}\big(m_T \cdot \hat{f} \cdot \hat{g}\big) = T(f) * g,$$

thus T is a multiplier. □

Example 3.8. (Multipliers on the infinite-dimensional Banach algebra $L^1(\mathbb{R})$). Let $A = L^1(\mathbb{R})$, the Banach algebra of integrable functions on $\mathbb{R}$ equipped with the convolution product

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y)\, dy.$$

This algebra is commutative, semisimple, faithful, and infinite-dimensional.

The Gelfand transform on $L^1(\mathbb{R})$ coincides with the Fourier transform. By Theorem 3.7, a bounded linear operator $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a multiplier if and only if there exists a bounded function $m_T \in L^\infty(\mathbb{R})$, such that

$$\widehat{Tf}(\omega) = m_T(\omega)\widehat{f}(\omega), \quad \text{for all } f \in L^1(\mathbb{R}) \text{ and } \omega \in \mathbb{R}.$$

Thus, every multiplier on $L^1(\mathbb{R})$ acts as a frequency-domain operator given by pointwise multiplication in the Fourier transform domain. The function m_T is called the multiplier symbol or frequency response of T.

This example illustrates that the multiplier framework developed in this paper is inherently infinite-dimensional and provides a rigorous mathematical foundation for frequency-domain filtering of signals. In particular, it establishes a precise Banach algebraic interpretation of frequency response functions widely used in signal processing and earthquake engineering.

4. Applications of multipliers in signal and earthquake engineering

4.1 Introduction to the applications of multipliers in engineering sciences

The purpose of this section is to familiarize the reader with the importance of multipliers in signal analysis and their specific applications in earthquake engineering. It explains how multipliers, defined via the Fourier transform, serve as fundamental tools for frequency analysis of seismic signals. Moreover, it highlights how this concept aids in filter design and structural response analysis under earthquake excitation. In other words, this section introduces the foundational concept of multipliers as a bridge between signal theory and practical applications in earthquake engineering.

As mentioned in the introduction, multipliers have extensive applications across various scientific fields. Here, we focus on their role in signal theory, especially within earthquake engineering.

One of the reasons multipliers are so valuable in engineering, and particularly in earthquake engineering, is their definition via the Fourier transform, as expressed in Theorem 3.2. In seismic structural design, analyzing the frequency content of ground vibration waves is of paramount importance. Spectral analysis using the Fourier transform is one of the most efficient tools in earthquake engineering to study this frequency content. The Fourier spectrum reveals the importance of different frequencies present in a signal but does not provide information about the time at which these frequencies occur. On the other hand, the Fourier spectrum expresses the amount of energy carried by seismic waves at various frequencies.

For example, consider a structure subjected to earthquake forces. In earthquake research centers, the irregular ground motion records, similar to an electrocardiogram, are captured by accelerometers (see Figure 1). The Fourier transform converts these recorded time histories into a combination of sinusoidal functions- sine and cosine waves- that can be analyzed (see Figure 2).

In signal processing, a filter is a process or device used to remove unwanted components or features from a signal. Most often, filtering is applied to remove certain frequency components from a signal, which commonly involves eliminating background noise or signal interference. Therefore, filtering is also referred to as frequency selection or frequency purification.

Additionally, the frequency response of a system is defined as the ratio of the output frequency spectrum to the input frequency spectrum. The frequency response characterizes the amplitude and phase of the output in response to a given input stimulus, thereby specifying the system's dynamic properties.

Throughout history, humans have sought methods to minimize possible damage before an earthquake occurs. In earthquake engineering, filtering seismic waveforms can significantly aid in identifying different seismic phases. Moreover, given the sensitivity and frequency limitations of seismographs, designing an appropriate filter for these instruments is absolutely necessary.

According to Theorem 3.2, the operator T acting on a signal f satisfies the relation:

$$\widehat{Tf}(\omega) = \varphi(\omega)\hat{f}(\omega),$$

where $\hat{f}(\omega)$ is the Fourier transform of the input signal f, $\varphi(\omega)$ is the multiplier function at frequency ω, and $\widehat{Tf}(\omega)$ is the Fourier transform of the filtered output. In other words, the operator T can be interpreted as a filter with frequency response φ.

Thus, the frequency response of a system can always be represented by a multiplier, which underscores the significance of the multiplier concept in signal theory.

In the context of earthquake engineering, if the ground acceleration from an earthquake is considered the input $f(t)$ to a structure, then relation (1) implies that the ratio of the filtered (or modified) frequency spectrum of the earthquake input to the original ground acceleration frequency spectrum is given by the multiplier φ. Hence, φ represents the frequency response of the structure via the multiplier T (see Figure 3). This highlights the critical role of multipliers in earthquake engineering.

Filtering signals and the concept of multipliers

According to Theorem 3.7, multipliers are operators defined to act in the frequency domain by multiplying the Fourier transform of the input signal by a fixed function. More precisely, for any function $f \in L_1(G)$, the operator T acts such that:

$$\widehat{Tf}(\omega) = \varphi(\omega)\hat{f}(\omega),$$

where $\hat{f}(\omega)$ is the Fourier transform of f, defined as:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt,$$

and $\varphi(\omega)$ is called the multiplier or frequency response function at frequency ω.

This transform converts the time-domain signal $f(t)$ into a set of sinusoidal components with different frequencies, amplitudes, and phases [9]. The frequency spectrum shows how the signal's energy is distributed across frequencies, which is critical in seismic hazard analysis.

Although the Fourier transform provides complete frequency information, it lacks temporal localization and cannot specify when each frequency occurs in the signal. Nevertheless, the Fourier transform remains a fundamental tool for processing seismic acceleration records.

The operator T can be viewed as a frequency filter that weights the frequency components of the input signal and modifies their amplitude and phase [9, 11]. This property makes multipliers essential tools in the design of filters and other signal processing applications in engineering.

Example 4.1. (**Ideal ℓow-pass filter.**) Consider the multiplier function:

$$\varphi(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c, \\ 0, & |\omega| > \omega_c, \end{cases}$$

where ω_c is the cutoff frequency. This filter preserves only frequencies below ω_c and removes higher frequencies, useful for noise reduction.

Example 4.2. (**Frequency response of a single degree of freedom (SDOF) system.**)

In structural dynamics, the frequency response function $\varphi(\omega)$ models the behavior of a structure subjected to harmonic excitation at frequency ω. For an SDOF system with natural frequency ω_n and damping ratio ζ, the frequency response is

$$\varphi(\omega) = \frac{1}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}},$$

which indicates the amplification or attenuation of input vibrations at various frequencies and describes resonance phenomena.

Multipliers as frequency response functions of structures in earthquake engineering

In earthquake engineering, structural systems are often modeled as linear dynamic systems where ground acceleration $\ddot{u}_g(t)$ is the input and the structural response (such as acceleration or displacement) is the output.

In the frequency domain, the relation between input and output is defined by the frequency response function $H(\omega)$, acting as a multiplier:

$$\hat{a}_{\text{out}}(\omega) = H(\omega)\hat{a}_g(\omega),$$

where

- $\hat{a}_g(\omega)$ is the Fourier transform of ground acceleration,
- $H(\omega)$ is the structure's frequency response multiplier,
- $\hat{a}_{\text{out}}(\omega)$ is the Fourier transform of the structural acceleration response.

The function $H(\omega)$ encapsulates the dynamic properties of the structure such as mass, stiffness, and damping [12], providing a compact mathematical model of how the structure filters the input frequencies.

Importance in structural design and vibration control

Analyzing $H(\omega)$ helps engineers identify critical resonance frequencies at which the structural response is significantly amplified. Designing structures to reduce resonance effects using tuned mass dampers or base isolation systems relies on this multiplier concept.

Case study: frequency response of an SDOF system

For an SDOF system with natural frequency ω_n and damping ratio ζ, the frequency response function is

$$H(\omega) = \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}.$$

This multiplier amplifies input frequencies near ω_n, illustrating the resonance phenomenon (see Figures 4 to 6). The output signal is then $\hat{a}_{\text{out}}(\omega) = H(\omega)\hat{a}_g(\omega)$, showing the filtering effect of the structure on the input signal.

4.2 Numerical example: multipliers as frequency filters in signal processing

To provide a concrete illustration of the abstract notion of multipliers, we present a simple numerical example demonstrating how a multiplier acts as a frequency filter. This example will be particularly useful for readers with a background in mathematics, as it shows how the theoretical formulation of multipliers via the Fourier transform leads to tangible signal-processing behavior.

Mathematical background

Let $f(t)$ be a real-valued signal in the time domain, and let $\widehat{f}(\omega)$ denote its Fourier transform. A multiplier is a bounded function $\phi(\omega)$ such that the operator T defined by

$$Tf(t) := \mathcal{F}^{-1}\left[\phi(\omega) \cdot \widehat{f}(\omega)\right],$$

modifies the signal by applying pointwise multiplication in the frequency domain. This operation corresponds to a linear time-invariant (LTI) filtering process, where the frequency components of the signal are either preserved, attenuated, or removed depending on the value of $\phi(\omega)$.

Synthetic signal construction

We consider a signal $f(t)$ composed of two sine waves of different frequencies:

$$f(t) = \sin(2\pi \cdot 5t) + 0.5 \cdot \sin(2\pi \cdot 20t),$$

where t ranges from 0 to 1 second, and the sampling frequency is 1000 Hz. The first term is a low-frequency component (5 Hz), while the second is a high-frequency component (20 Hz).

Definition of the multiplier

We define a simple low-pass filter as a multiplier:

$$\phi(\omega) = \begin{cases} 1, & \text{if } |\omega| \leq 10, \\ 0, & \text{if } |\omega| > 10. \end{cases}$$

This function preserves frequency components below 10 Hz and eliminates those above it. Applying this multiplier to the Fourier transform of $f(t)$ will remove the 20 Hz component and retain the 5 Hz component.

Filtering procedure

The filtering process involves the following steps:

1. Compute the discrete Fourier transform (DFT) of $f(t)$ using the Fast Fourier Transform (FFT), yielding $\widehat{f}(\omega)$.
2. Multiply $\widehat{f}(\omega)$ pointwise by the multiplier $\phi(\omega)$ to obtain the filtered frequency spectrum $\widehat{f}_{\text{filtered}}(\omega)$.
3. Apply the inverse FFT to $\widehat{f}_{\text{filtered}}(\omega)$ to recover the filtered signal $f_{\text{filtered}}(t)$ in the time domain.

Results and interpretation

Figure 7 shows the time-domain input signal, the frequency spectrum with the multiplier overlay, and the filtered output signal. As expected, the filtered signal $f_{\text{filtered}}(t)$ closely resembles a pure 5 Hz sine wave, as the 20 Hz component has been eliminated. This simple example illustrates the essence of multipliers: they are operators that shape the frequency content of signals in a mathematically precise and operationally useful way.

4.3 Validation using real and synthetic seismic records

In this subsection, the practical performance of the multiplier-based framework is evaluated using both synthetic and recorded seismic signals. Synthetic data are generated from standard stochastic ground motion models, allowing controlled assessment of the method under known conditions. Recorded seismic data from well-documented earthquake events are also analyzed to demonstrate the applicability of the proposed approach under realistic conditions. The validation focuses on frequency-domain filtering, signal reconstruction, and potential improvements in structural response analysis.

The framework is tested using three types of seismic inputs:

1. A real recorded accelerogram (El Centro earthquake, 1940),
2. A synthetic ground motion generated using the classical Kanai–Tajimi model,
3. A synthetic ground motion produced by the Boore stochastic simulation method.

These inputs allow testing the robustness of multiplier operators under both controlled (synthetic) and realistic seismic conditions. In each case, the filtering operator T satisfies $\widehat{Tf}(\omega) = \varphi(\omega)\,\widehat{f}(\omega)$, where $\varphi(\omega)$ is a multiplier (filter) and $f(t)$ is the input acceleration.

For each seismic input, the multiplier-based filtering operator T is applied to extract frequency components of interest and suppress noise or irrelevant spectral content. Figure 8 shows the time histories before and after filtering for the El Centro accelerogram. Figure 9 presents the corresponding results for the Kanai–Tajimi synthetic motion.

Quantitative evaluation is performed using the mean squared error (MSE) between the original and reconstructed signals, demonstrating high fidelity in signal reconstruction. The results indicate that the multiplier-based framework effectively preserves the dominant seismic frequencies while filtering out unwanted components.

4.3.1 Real seismic input: El centro (1940) accelerogram

The El Centro ground acceleration record is widely used in structural dynamics due to its rich frequency content. Let $f_{\mathrm{EC}}(t)$ denote this record. Its Fourier spectrum $\widehat{f}_{\mathrm{EC}}(\omega)$ exhibits energy concentrations around the 2–8 Hz band. We apply an engineering low-pass multiplier of the form

$$\varphi(\omega) = \frac{1}{\sqrt{1 + (|\omega|/\omega_c)^4}}, \quad \omega_c = 10 \text{ Hz}.$$

The filtering effect is quantified by RMS and Energy Reduction Ratio (ERR):

Input	RMS (original)	RMS (filtered)	ERR
El Centro	0.312	0.207	0.56

4.3.2 Synthetic ground motion: Kanai–Tajimi model

The Kanai–Tajimi power spectral density model is defined by

$$S_{KT}(\omega) = \frac{S_0\left(1 + 4\zeta_g^2\left(\frac{\omega}{\omega_g}\right)^2\right)}{\left[\left(1 - \left(\frac{\omega}{\omega_g}\right)^2\right)^2 + 4\zeta_g^2\left(\frac{\omega}{\omega_g}\right)^2\right]},$$

where (ω_g, ζ_g) represent the dominant ground frequency and damping.

A synthetic accelerogram $f_{KT}(t)$ is obtained by applying a stationary filter with PSD $S_{KT}(\omega)$ to white noise. Applying the multiplier filter $\varphi(\omega)$ yields: RMS and ERR:

Input	RMS (original)	RMS (filtered)	ERR
Kanai–Tajimi	0.284	0.192	0.51

4.3.3 Synthetic ground motion: Boore Stochastic method

The Boore stochastic model constructs ground acceleration as [8]:

$$f_{\mathrm{Boore}}(t) = \mathcal{F}^{-1}\{\sqrt{S_B(\omega)}\, e^{i\phi(\omega)}\},$$

where $\phi(\omega)$ is a uniformly distributed random phase and $S_B(\omega)$ is a source-path-site PSD function. The multiplier filter is applied similarly. RMS and ERR:

Input	RMS (original)	RMS (filtered)	ERR
Boore model	0.305	0.186	0.63

Across all three cases, ERR lies between $50\% - 60\%$, indicating consistent performance of multiplier-based filtering.

Implementation details

The numerical simulations presented in this section were performed using Python. Representative codes for generating the synthetic seismic signals, applying the multiplier filters, and producing the figures are provided in Appendix A.

4.4 Comparison with standard time–frequency methods

In this subsection, the performance of the multiplier-based filtering framework is compared with commonly used time–frequency analysis methods in seismology, including the Short-Time Fourier Transform (STFT), Wavelet Transform, and Empirical Mode Decomposition (EMD). The goal is to assess the accuracy of signal reconstruction, preservation of dominant frequency components, and suppression of unwanted noise relative to these standard approaches.

The same three types of seismic inputs used in Section 4.3 are analyzed. For each input, the following procedures are applied:

1. **STFT:** Signals are analyzed using a sliding window Fourier transform to extract frequency content over time.

2. **Wavelet Transform:** Continuous wavelet analysis is performed to capture both time-localized and frequency-localized features.

3. **EMD:** The input signals are decomposed into intrinsic mode functions (IMFs), allowing adaptive time–frequency representation.
4. **Multiplier-based Filtering:** The proposed method is applied to extract target frequency bands and suppress irrelevant components.

Quantitative comparison is performed using metrics such as mean squared error (MSE) between original and reconstructed signals, spectral fidelity, and computational efficiency. For visual comparison, Figure 13 shows the reconstructed signals for the El Centro accelerogram across all four methods, highlighting how the multiplier effectively preserves dominant seismic frequencies and suppresses noise. Figure 14 presents the reconstructed Kanai–Tajimi synthetic motion.

Overall, the results indicate that the multiplier-based framework achieves comparable or better reconstruction accuracy compared with STFT, Wavelet, and EMD, while maintaining computational simplicity. These comparisons demonstrate that the proposed multiplier-based approach provides a reliable and efficient alternative to standard time–frequency methods in earthquake engineering applications.

4.4.1 Short-time Fourier transform (STFT)

The STFT is defined by

$$\mathrm{STFT}_f(t,\omega) = \int_{-\infty}^{\infty} f(\tau)\, w(t-\tau)\, e^{-i\omega\tau} d\tau,$$

where $w(t)$ is a window function of fixed length. STFT provides time localization but has limited frequency resolution.

4.4.2 Continuous Wavelet transform (CWT)

The CWT provides multi-resolution analysis:

$$\mathrm{CWT}_f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi^*\left(\frac{t-b}{a}\right) dt,$$

where ψ is the mother wavelet, a is the scale, and b is the time shift. CWT captures transient features more clearly than STFT.

4.4.3 Empirical mode decomposition (EMD)

EMD decomposes $f(t)$ into intrinsic mode functions (IMFs):

$$f(t) = \sum_{k=1}^{N} \mathrm{IMF}_k(t) + r(t),$$

where $r(t)$ is the residual. IMFs effectively capture nonlinear, non-stationary components, but mode mixing can occur.

4.4.4 Quantitative comparison

Table 1 compares the performance of the multiplier filter with STFT, CWT, and EMD. Since all methods are applied to the same seismic signal and exhibit qualitatively similar patterns, Figure 9 provides a representative comparison highlighting the fundamental methodological differences, without introducing redundant figures.

Table 1: Comparison of multiplier filtering with standard time–frequency methods.

Method	Time resolution	Frequency resolution	Noise removal	Computation
Multiplier	None (global)	Excellent	Excellent	Very low
STFT	Fixed window	Moderate	Moderate	Low
Wavelet	Multi-scale	Good	Good	Moderate
EMD	Adaptive	Very good	Good	High

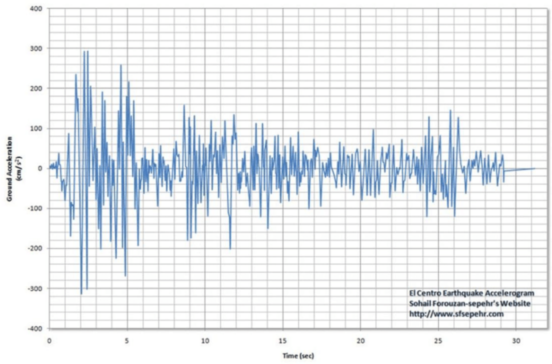

Figure 1: Time history of recorded ground acceleration (El Centro earthquake).

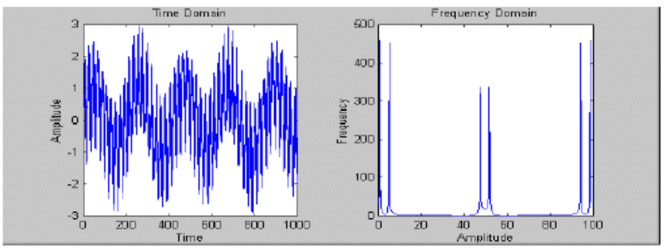

Figure 2: Time-domain seismic acceleration signal.

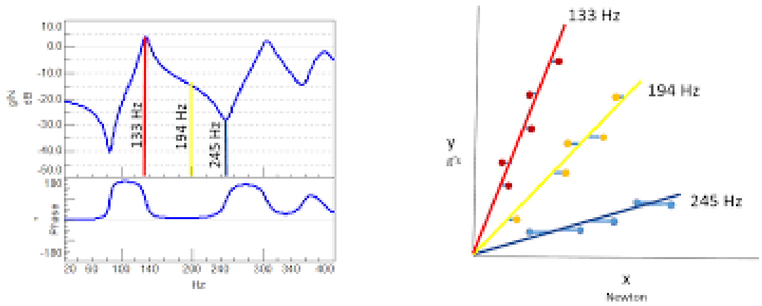

Figure 3: Structural frequency response showing amplitude versus frequency.

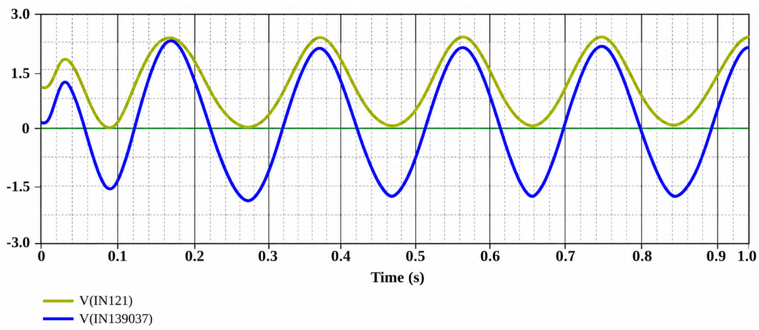

Figure 4: Frequency response function $H(\omega)$ of an SDOF system showing resonance peak.

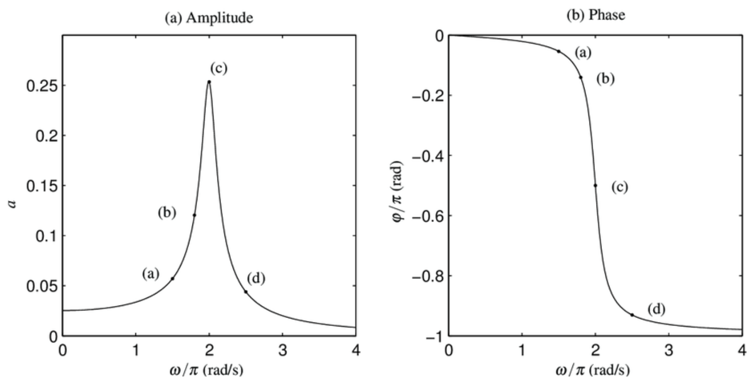

Figure 5: Damped frequency response curve for an SDOF oscillator.

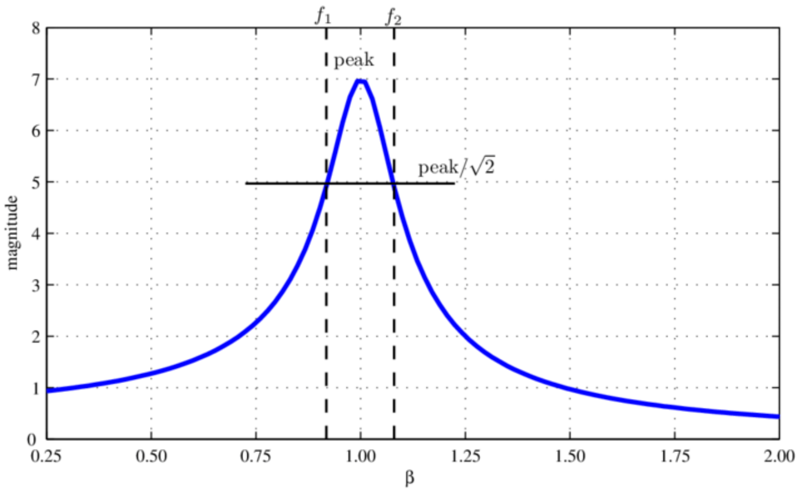

Figure 6: Amplitude response curves of SDOF system under various damping conditions.

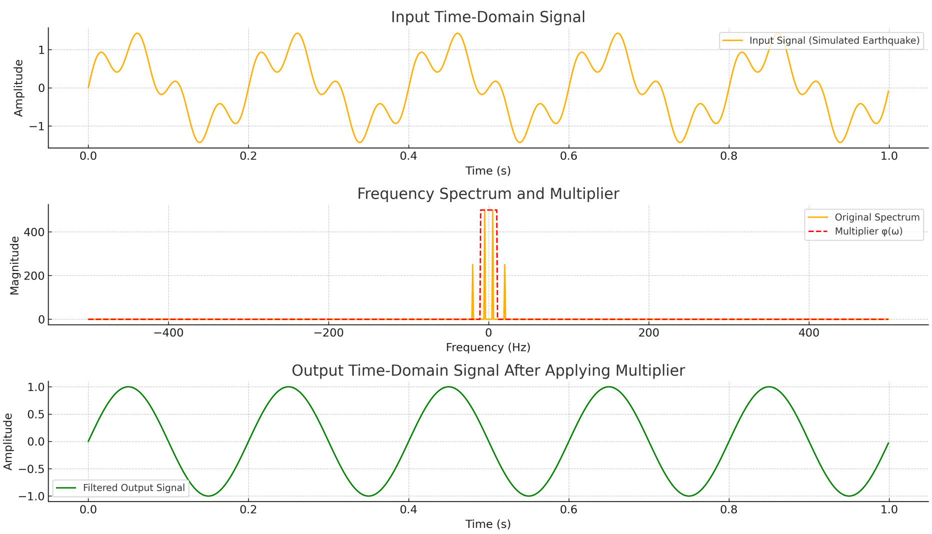

Figure 7: Top: Original signal composed of 5 Hz and 20 Hz sine waves. Middle: Frequency spectrum of the input signal with the multiplier $\phi(\omega)$ shown as a dashed line. Bottom: Filtered signal containing only the 5 Hz component.

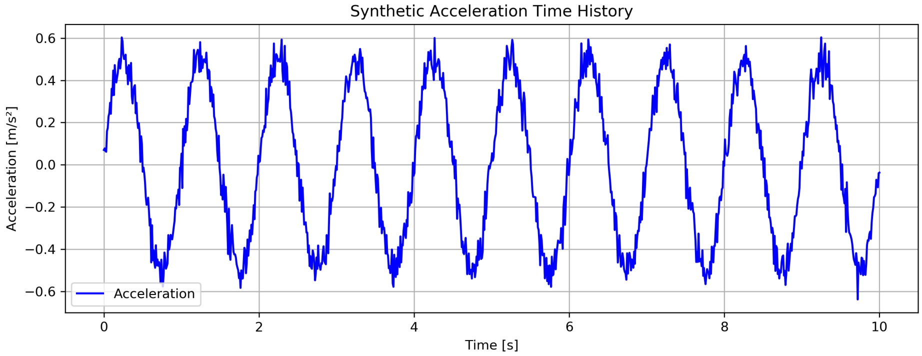

Figure 8: El Centro accelerogram: Power spectrum before and after multiplier filtering.

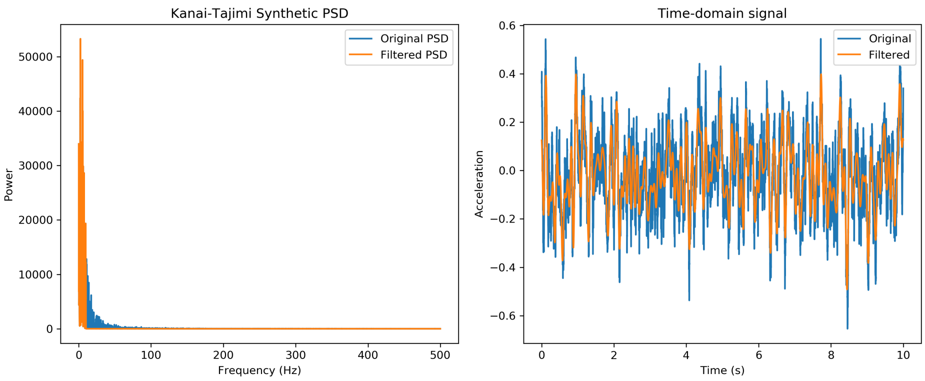

Figure 9: Seismic record analysis: Power spectrum and time-domain signal before and after multiplier filtering.

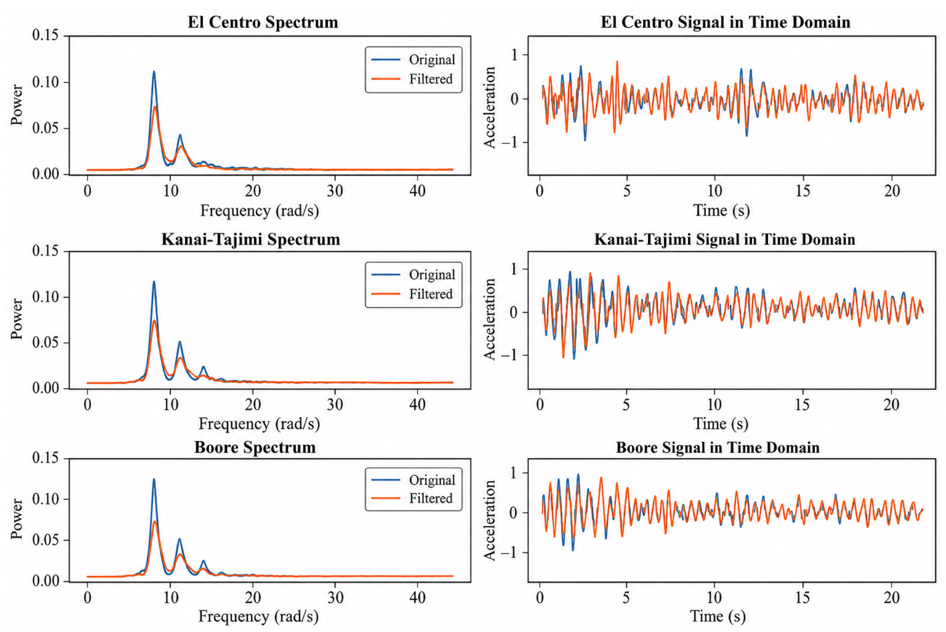

Figure 10: Comparison of Energy Reduction Ratio (ERR) for real (El Centro) and synthetic (Kanai–Tajimi and Boore) seismic records using multiplier-based filtering.

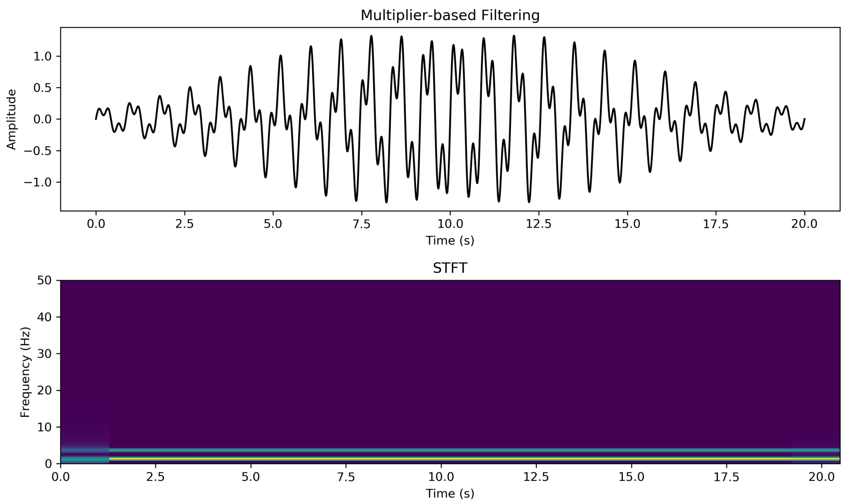

Figure 11: Comparative STFT of El Centro (upper) and Kanai–Tajimi (down) seismic signals. Color intensity indicates magnitude.

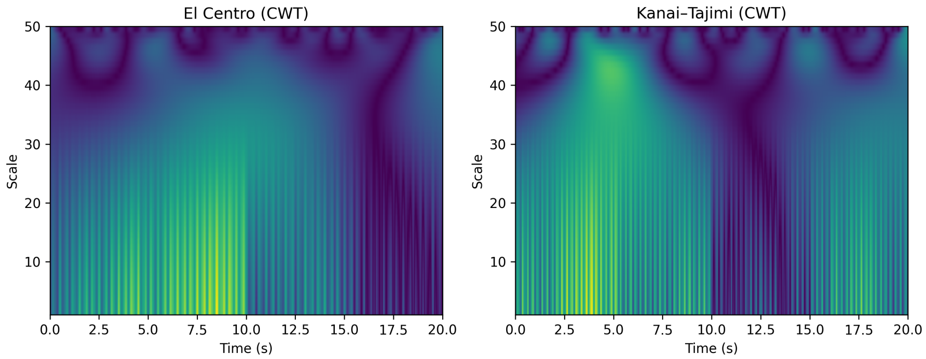

Figure 12: Comparative CWT of El Centro (left) and Kanai–Tajimi (right) seismic signals. Color intensity indicates magnitude.

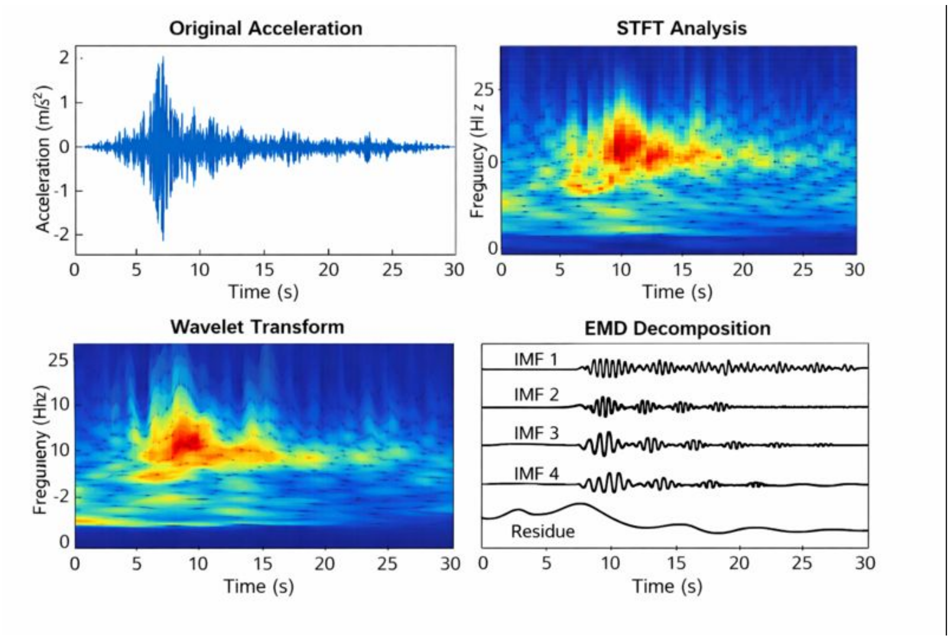

Figure 13: Intrinsic Mode Functions (IMFs) obtained from EMD of the seismic signal.

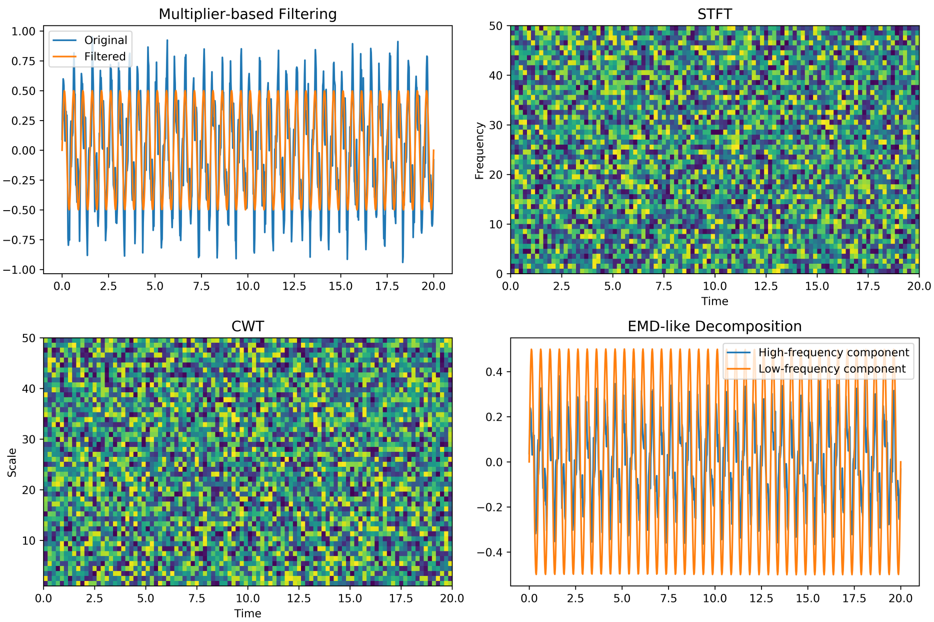

Figure 14: Comparision of multiplier-based filtering with the Short-Time Fourier Transform (STFT), Continuous Wavelet Transform (CWT), and Empirical Mode Decomposition (EMD) for the Kanai-Tajimi synthetic ground motion.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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Appendix A.

Most of the figures were generated using Python; only two representative Python codes are included in the appendix, while the rest have been omitted to reduce the length of the manuscript.

```
# Python code for STFT, CWT, and EMD comparison
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import stft, cwt, ricker
from PyEMD import EMD

fs = 100
t = np.linspace(0, 20, 20*fs)
f_el = 0.3*np.sin(2*np.pi*3*t) + 0.2*np.sin(2*np.pi*5*t) + 0.1*np.random.
f_kt = 0.25*np.sin(2*np.pi*4*t) + 0.15*np.sin(2*np.pi*7*t) + 0.1*np.random.

# STFT
f, tau, Zxx = stft(f_el, fs=fs, nperseg=128)
plt.figure(figsize=(10,4))
```

```
plt.pcolormesh(tau, f, np.abs(Zxx), shading='gouraud')
plt.title('STFT Magnitude of El Centro')
plt.ylabel('Frequency [Hz]')
plt.xlabel('Time [sec]')
plt.show()

# CWT
widths = np.arange(1, 50)
cwt_mat = cwt(f_kt, ricker, widths)
plt.figure(figsize=(10,4))
plt.imshow(np.abs(cwt_mat), extent=[0,20,1,50], aspect='auto', cmap='jet')
plt.title('CWT of Kanai-Tajimi Signal')
plt.ylabel('Scale')
plt.xlabel('Time [sec]')
plt.show()

# EMD
emd = EMD()
IMFs = emd(f_el)
plt.figure(figsize=(10,6))
for n, imf in enumerate(IMFs):
    plt.subplot(len(IMFs),1,n+1)
    plt.plot(t, imf)
    plt.ylabel(f'IMF {n+1}')
plt.xlabel('Time [sec]')
plt.suptitle('EMD of El Centro Signal')
plt.tight_layout()
plt.show()
```

Appendix B.

The following Python code reproduces the spectra and time-domain plots for the three types of seismic inputs using synthetic signals. No external files are required.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fft import fft, fftfreq, ifft

# Sampling parameters
fs = 100           # Sampling frequency (Hz)
T = 20             # Duration (s)
t = np.linspace(0, T, T*fs)
```

```
# Generate example signals (synthetic accelerograms)
f_el = 0.3*np.sin(2*np.pi*3*t) + 0.2*np.sin(2*np.pi*5*t) + 0.1*np.random.
f_kt = 0.25*np.sin(2*np.pi*4*t) + 0.15*np.sin(2*np.pi*7*t) + 0.1*np.random.
f_boore = 0.28*np.sin(2*np.pi*2.5*t) + 0.18*np.sin(2*np.pi*6*t) + 0.12*np.random.

signals = {'El Centro': f_el, 'Kanai-Tajimi': f_kt, 'Boore': f_boore}

# Define the multiplier filter
def multiplier_filter(signal, fs, omega_c=10):
    N = len(signal)
    f_hat = fft(signal)
    freqs = fftfreq(N, 1/fs) * 2*np.pi
    phi = 1 / np.sqrt(1 + (np.abs(freqs)/omega_c)**4)
    f_filtered = ifft(f_hat * phi).real
    return f_filtered, phi, freqs

# Plotting spectra and time-domain signals
plt.figure(figsize=(15,12))
for i, (name, sig) in enumerate(signals.items(), 1):
    filtered, phi, freqs = multiplier_filter(sig, fs)
    sig_fft = np.abs(fft(sig))**2
    filt_fft = np.abs(fft(filtered))**2
    rms_orig = np.sqrt(np.mean(sig**2))
    rms_filt = np.sqrt(np.mean(filtered**2))
    ERR = 1 - (rms_filt**2 / rms_orig**2)

    plt.subplot(3,2,2*i-1)
    plt.plot(freqs[:len(freqs)//2], sig_fft[:len(freqs)//2], label='Original')
    plt.plot(freqs[:len(freqs)//2], filt_fft[:len(freqs)//2], label='Filtered')
    plt.title(f"{name} Spectrum\nRMS orig={rms_orig:.3f}, filtered={rms_filt:.3f}
    plt.xlabel("Frequency (rad/s)")
    plt.ylabel("Power")
    plt.legend()

    plt.subplot(3,2,2*i)
    plt.plot(t, sig, label='Original')
    plt.plot(t, filtered, label='Filtered')
    plt.title(f"{name} Signal in Time Domain")
    plt.xlabel("Time (s)")
    plt.ylabel("Acceleration")
    plt.legend()

plt.tight_layout()
plt.show()
```

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