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On Entire and Analytic Functions of the Matrices AB and BA

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Abstract

The relation $z^q\chi_{AB}(z)=z^p\chi_{BA}(z)$ holds for the characteristic polynomials of complex matrices $A_{p\times q}$ and $B_{q\times p}$. In the present paper, we generalize this relation for *complex polynomials*, *entire functions*, and some *analytic functions* of AB and BA. Some examples are also provided.

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1. Introduction

One of the fundamental concepts of analysis is a system of n simultaneous first order differential equations. If $n\in\mathbb{N}$, and $y_i=y_i(x)$, $(1\le i\le n)$ are unknown functions of the real variable x, then a system of first order differential equations is of the form

$$\frac{dy_i}{dx}=f_i(x,y_1,\dots,y_n),\quad (1\le i\le n), \tag{1}$$

where $f_1,\dots,f_n$ are single-valued functions continuous in a certain domain. A solution of (1) is an n-tuple $(u_1=u_1(x),\dots,u_n=u_n(x))$ of functions satisfying

$$\frac{du_i}{dx}=f_i(x,u_1,\dots,u_n),\quad (1\le i\le n),$$

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for some interval $x \in I$. Put $Y = (y_1, \dots, y_n)$ and $F(x, Y) = (f_1, \dots, f_n)$, where by assumption, F is continuous in Y and x simultaneously. Then the system (1) may be written as a single vector equation

$$\frac{dY}{dx} = F(x, Y). \tag{2}$$

A system of this kind, in which the independent variable x does not appear in the function F, is said to be autonomous.
A linear system is of the form

$$\frac{dY}{dx} = YA + B, \tag{3}$$

where $A = A(x)$ is an $n \times n$ matrix, $B = B(x)$ is an n row vector function, both continuous on an interval I. If the function B is identically zero, then the system (3) is called homogeneous, otherwise it is said to be nonhomogeneous [1].

The systems of first order differential equations arise quite naturally in many scientific problems. The Volterra's prey-predator problem in biology, and the famous n-body problem of classical mechanics are of this kind [2, 3]. Designing controllers to stabilize systems and achieve desired performance in control theory is also studied by systems of differential equations [4]. Unfortunately, some of these systems cannot be solved in terms of elementary functions, but the method of linearization has a good efficiency in finding their approximate solutions. In fact, there are many physical situations in which a linear approximation is valuable and adequate. Hence, expanding and deepening the methods of solving systems of linear differential equations is very productive.
Consider the system of differential equation (3) in the case that A is a constant matrix. Finding an explicit formula for the solution of this equation involves calculating the eigenvalues of A [5]. Therefore, it is quite valuable to extend the available results related to the characteristic polynomial of matrices.

Let $C_{p\times p}$ and $D_{p\times p}$ be two complex matrices. A classical result in matrix theory states that χ_{CD} and χ_{DC}, the characteristic polynomials of CD and DC are equal. Some proofs of this theorem are given in [6, 7]. A generalization of this theorem is also provided in some mathematical texts, for example [8–11]. According to this generalization, if $A_{p\times q}$ and $B_{q\times p}$ are two complex matrices, then χ_{AB} and χ_{BA} satisfy the equation

$$z^q \chi_{AB}(z) = z^p \chi_{BA}(z). \tag{4}$$

The main objective of the present paper is to provide generalizations of (4) for complex polynomials, entire functions, and some analytic functions. Our method is to use (4) and applying a suitable norm of matrices.

2. Preliminaries

We begin by recalling some definitions and the essential concepts used in this work.

Definition 2.1. ([12]). Let f be a complex-valued function defined on an open set U in the complex plane $\mathbb{C}$. Then f is said to be analytic on U if its derivative exists at every point of U. In particular, f is analytic at a point z if there is an open disc about z on which f is analytic. A function f is said to be an entire function if f is analytic on $\mathbb{C}$.

By virtue of the Taylor's theorem, if a function f is analytic throughout an open disc $|z| < R \neq 0$, then $f(z)$ has the power series representation $f(z) = \Sigma_{k=0}^{+\infty} a_k z^k$ [12].

Theorem 2.2. *([12]). If a power series $\Sigma_{k=0}^{+\infty} a_k z^k$ converges when $z = z_1 \neq 0$, then it is absolutely convergent at each point z in the open disc $|z| < R$ where $R = |z_1|$.*

Definition 2.3. For an infinite sequence of $p \times q$ complex matrices $\{R_k\}$, $(k = 0, 1, 2, \dots)$, we denote the ij-entry of $\{R_k\}$ by $r_{ij}^{(k)}$ $(i = 1, \dots, p; j = 1, \dots, q)$. If all pq scalar series

$$\Sigma_{k=0}^{+\infty} r_{ij}^{(k)}$$

converge, then we say the series $\Sigma_{k=0}^{+\infty} R_k$ is convergent, and its sum which is denoted by $\Sigma_{k=0}^{+\infty} R_k$, is defined to be the $p \times q$ complex matrix whose ij-entry is $\Sigma_{k=0}^{+\infty} r_{ij}^{(k)}$.

Definition 2.4. ([8]). The Frobenius norm of the $p \times q$ complex matrix $R = [r_{ij}]$, denoted by $\|R\|_F$, is defined to be the non-negative real number given by the formula

$$\|R\|_F = \sqrt{\Sigma_{i=1}^{p} \Sigma_{j=1}^{q} |r_{ij}|^2},$$

where $|r_{ij}|$ is the Euclidean norm of the entry r_{ij}.

Remark 1. The Frobenius norm satisfies $|r_{ij}| \leq \|R\|_F$ and submultiplicative property [8]. For this reason, the use of this norm is convenient but not essential. In fact, any submultiplicative matrix norm with the mentioned property could be used.

As a result of the comparison test for the series of real numbers [13] we have:

Theorem 2.5. *If $\Sigma_{k=0}^{+\infty} \|R_k\|_F$ converges, then the matrix series $\Sigma_{k=0}^{+\infty} R_k$ is convergent.*

In the following, the set of natural (respectively, complex) numbers is denoted by $\mathbb{N}$ (respectively, $\mathbb{C}$). Let $\mathbb{C}[z]$ be the set of all polynomials with the coefficients in $\mathbb{C}$ and complex variable z. If $P_n(z) = \Sigma_{k=0}^{n} p_k z^k \in \mathbb{C}[z]$ is a polynomial of degree $n \in \mathbb{N} \cup \{0\}$, then P_n evaluated at a square matrix M is defined by $P_n(M) = \Sigma_{k=0}^{n} p_k M^k$, where $M^0 = I_{n \times n}$ and $M^{l+1} = M^l \times M$ for $l \in \mathbb{N} \cup \{0\}$.

Lemma 2.6. *Let $p, q \in \mathbb{N}$, and the complex matrices $A_{p \times q}$, $B_{q \times p}$, $M_{p \times p}$ and $N_{q \times q}$ satisfy $M = AB$ and $N = BA$. If $P_n \in \mathbb{C}[z]$ then*

$$MP_n(M) = P_n(M)M = AP_n(N)B.$$

3. Main results

In this section, all the matrices are assumed to be complex. Let S be an arbitrary square matrix and $a_k \in \mathbb{C}$ for $k \in \mathbb{N} \cup \{0\}$. If $\Sigma_{k=0}^{+\infty} |a_k| \|S\|_F^k$ converges, then Theorem 2.5 and the submultiplicative property of the Frobenius norm imply that the matrix series $\Sigma_{k=0}^{+\infty} a_k S^k$ converges. As a result of this assertion and Lemma 2.6, we have:

Corollary 3.1. *For matrices $A_{p\times q}$, $B_{q\times p}$, let $M = AB$ and $N = BA$ be such that $\Sigma_{k=0}^{+\infty} |a_k| \|M\|_F^k$ and $\Sigma_{k=0}^{+\infty} |a_k| \|N\|_F^k$ converge. Then $\Sigma_{k=0}^{+\infty} a_k M^k$ and $\Sigma_{k=0}^{+\infty} a_k N^k$ converge and*

$$M\left(\Sigma_{k=0}^{+\infty} a_k M^k\right) = \left(\Sigma_{k=0}^{+\infty} a_k M^k\right) M = A\left(\Sigma_{k=0}^{+\infty} a_k N^k\right) B.$$

Theorem 2.2 implies that Taylor's series of an entire function satisfies the desired condition in Corollary 3.1, so we have:

Corollary 3.2. *For matrices $A_{p\times q}$, $B_{q\times p}$ and the entire function $f(z) = \Sigma_{k=0}^{+\infty} a_k z^k$, the matrices $f(AB)$ and $f(BA)$ exist and*

$$ABf(AB) = f(AB)AB = Af(BA)B.$$

In the following, we provide some generalizations of (4).

Theorem 3.3. *(Generalization for complex polynomials) Let $P_n \in \mathbb{C}[z]$ and $A_{p\times q}$, $B_{q\times p}$, be two arbitrary matrices. Then,*

$$z^q \chi_{ABP_n(AB)}(z) = z^p \chi_{BAP_n(BA)}(z).$$

In addition, if $P_n(0) = 0$ then

$$z^q \chi_{P_n(AB)}(z) = z^p \chi_{P_n(BA)}(z).$$

Proof. Put $M = AB$ and $N = BA$. As a result of Lemma 2.6 and (4) we have,

$$\begin{aligned} z^q \chi_{ABP_n(AB)}(z) &= z^q \chi_{MP_n(M)}(z) \\ &= z^q \chi_{(AP_n(N))B}(z) \\ &= z^p \chi_{B(AP_n(N))}(z) \\ &= z^p \chi_{(BA)P_n(BA)}(z). \end{aligned}$$

□

Theorem 3.4. *(Generalization for entire functions) Let $A_{p\times q}$ and $B_{q\times p}$ be two arbitrary matrices. The following statements are true:*

(1) If f is an entire function, then $z^q \chi_{ABf(AB)}(z) = z^p \chi_{BAf(BA)}(z)$.

(2) If g is an entire function and $g(0) = 0$, then $z^q \chi_{g(AB)}(z) = z^p \chi_{g(BA)}(z)$.

Furthermore, the two previous statements are equivalent.

Proof. Item (1) is a consequence of Corollary 3.2 and Theorem 3.3. If (1) holds and the entire function g satisfies $g(0)=0$, then there exists an entire function f such that $g(z)=zf(z)$. Therefore,

$$z^q\chi_{g(AB)}(z)=z^q\chi_{ABf(AB)}(z)=z^p\chi_{BAf(BA)}(z)=z^p\chi_{g(BA)}(z),$$

and hence (1) implies (2). Conversely, if (2) holds and f be an entire function, then $g(z)=zf(z)$ is an entire function satisfies $g(0)=0$ and

$$z^q\chi_{ABf(AB)}(z)=z^q\chi_{g(AB)}(z)=z^p\chi_{g(BA)}(z)=z^p\chi_{BAf(BA)}(z).$$

Therefore (2) implies (1). □

Example 3.5. Consider the entire function $f(z)=e^z$. Also, let $A=\begin{bmatrix}-i & 0\\ 0 & -i\\ -i & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & i & 0\\ 2i & 0 & 2i\end{bmatrix}$. Then $AB=\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 2\\ 0 & 1 & 0\end{bmatrix}$ and $BA=\begin{bmatrix}0 & 1\\ 4 & 0\end{bmatrix}$. In addition, for $H=\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}$ we have $(AB)^{2k}=2^{2k-1}H, (AB)^{2k+1}=2^{2k}AB$, $(k=1,2,\dots)$. So, a direct calculation shows that $f(AB)=\begin{bmatrix}\cosh^2 1 & \frac{1}{2}\sinh 2 & \sinh^2 1\\ \sinh 2 & \cosh 2 & \sinh 2\\ \sinh^2 1 & \frac{1}{2}\sinh 2 & \cosh^2 1\end{bmatrix}$, and

$$ABf(AB)=f(AB)AB=\begin{bmatrix}\sinh 2 & \cosh 2 & \sinh 2\\ 2\cosh 2 & 2\sinh 2 & 2\cosh 2\\ \sinh 2 & \cosh 2 & \sinh 2\end{bmatrix}. \tag{5}$$

Similarly,

$$(BA)^k=\begin{cases}2^{k-1}BA & k=1,3,5,\dots,\\ 2^kI & k=0,2,4,\dots,\end{cases}$$

and a calculation shows that $f(BA)=\begin{bmatrix}\cosh 2 & \frac{1}{2}\sinh 2\\ 2\sinh 2 & \cosh 2\end{bmatrix}$, and

$$Af(BA)B=\begin{bmatrix}\sinh 2 & \cosh 2 & \sinh 2\\ 2\cosh 2 & 2\sinh 2 & 2\cosh 2\\ \sinh 2 & \cosh 2 & \sinh 2\end{bmatrix}. \tag{6}$$

Now, (5) and (6) demonstrate Corollary 3.2.

Furthermore, a simple calculation shows that $BAf(BA)=\begin{bmatrix}2\sinh 2 & \cosh 2\\ 4\cosh 2 & 2\sinh 2\end{bmatrix}$, and

$$z^2\chi_{ABf(AB)}(z)=z^3\chi_{BAf(BA)}(z)=z^3(z^2-4z\sinh 2-4)$$

which justifies the first part of Theorem 3.4. For the second part of this Theorem, let's set $g(z) = e^z - 1$. In this case, we have:

$$g(AB) = \begin{bmatrix} \sinh^2 1 & \frac{1}{2}\sinh 2 & \sinh^2 1 \\ \sinh 2 & 2\sinh^2 1 & \sinh 2 \\ \sinh^2 1 & \frac{1}{2}\sinh 2 & \sinh^2 1 \end{bmatrix}, g(BA) = \begin{bmatrix} 2\sinh^2 1 & \frac{1}{2}\sinh 2 \\ 2\sinh 2 & 2\sinh^2 1 \end{bmatrix}.$$

and

$$z^2\chi_{g(AB)}(z) = z^3\chi_{g(BA)}(z) = z^3(z^2 - 4z\sinh^2 1 - 4\sinh^2 1),$$

So, the second part of Theorem 3.4 is confirmed.
Now if $P_1(z) = g(z) = z + i$, then $P_1(0) = g(0) \neq 0$, and

$$g(AB) = \begin{bmatrix} i & 1 & 0 \\ 2 & i & 2 \\ 0 & 1 & i \end{bmatrix}, g(BA) = \begin{bmatrix} i & 1 \\ 4 & i \end{bmatrix},$$

but

$$z^2\chi_{g(AB)}(z) = z^2(z-i)(z-i-2)(z-i+2), \quad z^3\chi_{g(BA)}(z) = z^3(z-i-2)(z-i+2).$$

Hence $z^2\chi_{g(AB)}(z) \neq z^3\chi_{g(BA)}(z)$. Thus, the hypothesis $P_n(0) = 0$ (respectively, $g(0) = 0$) is essential in the second part of Theorem 3.3 (respectively, Theorem 3.4).

Remark 2. A generalization of Theorem 3.4 could be established for some analytic functions that are not essentially entire.

Theorem 3.6. *If f is an analytic function on an open disc $|z| < R \neq 0$, and $\| AB \|_F < R, \| BA \|_F < R$, then the same results as stated in Theorem 3.4 hold.*

Example 3.7. Let $f(z) = \Sigma_{k=0}^{+\infty} z^k$ whenever $|z| < 1$, $A = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, and $B = \begin{bmatrix} a & 0 & 0 \end{bmatrix}$ for $a \in \mathbb{C}, |a| < 1$. A simple calculation shows that $(AB)^k = \begin{bmatrix} a^{2k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $k \in \mathbb{N}$, $f(AB) = \begin{bmatrix} \frac{1}{1-a^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and

$$ABf(AB) = \begin{bmatrix} \frac{a^2}{1-a^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, $BAf(BA) = \Sigma_{k=1}^{+\infty} a^{2k} = \frac{a^2}{1-a^2}$ and

$$z\chi_{ABf(AB)}(z) = z^3\chi_{BAf(BA)}(z) = z^3\left(z - \frac{a^2}{1-a^2}\right),$$

and the first part of Theorem 3.4 is confirmed. For the second part, let $g(z) = \Sigma_{k=1}^{+\infty} z^k$ whenever $|z| < 1$, then we have:

$$g(AB) = \begin{bmatrix} \frac{a^2}{1-a^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, g(BA) = \Sigma_{k=1}^{+\infty} a^{2k} = \frac{a^2}{1-a^2}.$$

Therefore,

$$z\chi_{g(AB)}(z) = z^3 \chi_{g(BA)}(z) = z^3 \left(z - \frac{a^2}{1-a^2} \right),$$

which justifies the second part of 3.4.

Corollary 3.8. *Let g be an entire function (respectively, an analytic function on an open disc $|z| < R \neq 0$) such that $g(0) = 0$. Let A and B be $p \times q$ and $q \times p$ matrices (respectively, such that $\| AB \|_F < R, \| BA \|_F < R$). Then the following statements are true:*

(1) $g(AB)$ and $g(BA)$ have the same non-zero eigenvalues, counting multiplicity.

(2) $det(I_{p\times p} \pm g(AB)) = det(I_{q\times q} \pm g(BA))$.

Proof. Item (1) is a consequence of Theorems 2.2, 3.4 and 3.6. Furthermore, the product $\epsilon g, \epsilon \in \mathbb{C}$ has the same properties of g as stated in this corollary. So, Theorems 3.4 and 3.6(2) imply that:

$$z^q \det(zI_{p\times p} - \epsilon g(AB)) = z^q \chi_{\epsilon g(AB)}(z) = z^p \chi_{\epsilon g(BA)}(z) = z^p \det(zI_{q\times q} - \epsilon g(BA)).$$

Now, $z = 1$ and $\epsilon = \pm 1$ imply (2). $\square$

The function $g(z) = z + i$ satisfies $g(0) \neq 0$, and Example 3.5 shows that $g(0) = 0$ is essential in the hypotheses of Corollary 3.8.

4. Conclusion

In this paper, we first generalize the relation $z^q \chi_{AB}(z) = z^p \chi_{BA}(z)$ to polynomials, then to entire functions, and finally to some analytic functions. Similar arguments show that the relation

$$tr(CD) = tr(DC), \tag{7}$$

has also generalizations as stated for characteristic polynomials, and the theorems in the present paper involve corresponding results for traces. Let $p, q \in \mathbb{N}$ and $P_n \in \mathbb{C}[z]$. Then, as stated in Lemma 2.6, the complex matrices $A_{p\times q}$, $B_{q\times p}$ satisfy

$$ABP_n(AB) = P_n(AB)AB = AP_n(BA)B.$$

Therefore, (7) implies that

$$tr(ABP_n(AB)) = tr(AP_n(BA)B) = tr(P_n(BA)BA) = tr(BAP_n(BA)). \quad (8)$$

Thus, the following two equivalent statements are direct consequences of Theorem 2.5 and (8):

(1) If f is an entire function, then $tr(ABf(AB)) = tr(BAf(BA))$.

(2) If g is an entire function and $g(0) = 0$, then $tr(g(AB)) = tr(g(BA))$.

Furthermore,

(3) If g is an analytic function on an open disc $|z| < R \neq 0$ such that $g(0) = 0$ and $\| (AB) \|_F < R, \| (BA) \|_F < R)$, then $tr(g(AB)) = tr(g(BA))$.

Examples 3.5 and 3.7 confirm these assertions.
In the following, we briefly summarize the main contributions and possible directions for future work.
Let $A = [a_{ij}]$ and $B = [b_{kl}]$ be rectangular matrices of orders $p \times q$ and $r \times s$ respectively. A matrix of order $pr \times qs$ represented in a block form

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix},$$

is called the Kronecker product (or tensor product) of A and B. In addition, for rectangular matrices $C = [c_{ij}]$ and $D = [d_{ij}]$ of order $p \times q$, the Hadamard product (or Schur product) $A \odot B$ is a $p \times q$ matrix with elements given by $A \odot B = [c_{ij}d_{ij}]$ [11]. Further research could explore similar possible generalizations for Kronecker product or Hadamard product of matrices.
On the other hand, because of applications of eigenvalues of matrices in the theory of differential equations, the research in the context of systems of differential equations may be considered as another possible direction for future work. For instance, the study of equilibria plays a central role in the theory of systems of differential equations and their applications [5]. Let g be an entire function (respectively, an analytic function on an open disc $|z| < R \neq 0$) such that $g(0) = 0$. Let A, B be $p \times q$ and $q \times p$ matrices (respectively, such that $\| AB \|_F < R, \| BA \|_F < R$). Let $Y = (y_1, \dots, y_p)$ and $Z = (z_1, \dots, z_q)$ be unknown functions of a single independent variable x. Investigating the relationship between the solutions (also stable equilibrium points) of the following systems of differential equations

$$\frac{dY}{dx} = YAB, \ \frac{dZ}{dx} = ZBA,$$

may be another continuation of this research.

Conflicts of Interest. The author declare that he has no conflicts of interest regarding the publication of this article.

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