

## $(\alpha, \beta)$ -Ricci-Yamabe Solitons with Potential Kuiper-Nash Vector Field

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### Abstract

This study focuses on identifying conditions that allow an  $n$ -dimensional compact Riemannian manifold  $(M^n, g)$  to support a vector field  $\xi$  along with smooth functions  $\alpha$  and  $\lambda$ , forming an  $(\alpha, \beta)$ -Ricci-Yamabe soliton structure. Our construction relies on embedding  $(M^n, g)$  isometrically into the Euclidean space  $(\mathbb{R}^m, g)$  following the method of Kuiper and Nash, where  $\xi$  is defined as the tangential component of a fixed unit vector in  $\mathbb{R}^m$ , hereafter called the Kuiper-Nash vector. Let  $r$  denote the scalar curvature of  $(M^n, g)$ , we establish that whenever the integral of  $\xi(r)$  satisfies a specific lower bound dependent on a constant  $\lambda$ , the tuple  $(M^n, g, \beta r, \xi, \lambda)$  forms a valid  $(\alpha, \beta)$ -Ricci-Yamabe soliton, which we term a Kuiper-Nash  $(\alpha, \beta)$ -Ricci-Yamabe soliton. We examine conditions under which the soliton becomes trivial, presenting a characterization that directly relates this behavior to the scalar curvature  $r$ . Finally, an explicit characterization is derived for a compact,  $n$ -dimensional trivial Kuiper-Nash  $(\alpha, \beta)$ -Ricci-Yamabe soliton, expressed in terms of an upper bound on the integral of  $(\operatorname{div} \xi)^2$  that incorporates  $r$ .

**Keywords:** Kuiper-Nash embedding, Generalized  $\mathcal{Z}$ -soliton, Shape operator.

**2020 Mathematics Subject Classification:** 35C08, 53C50, 53C55.

### How to cite this article

S. Azami, M. Jafari and M. Zohrehvand,  $(\alpha, \beta)$ -Ricci-Yamabe solitons with potential Kuiper-Nash vector field, *Math. Interdisc. Res.* **11** (4) (2026) 351-370.

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Academic Editor: Abbas Saadatmandi

Received 8 December 2025, Accepted 21 April 2026

DOI: 10.22052/mir.2026.258097.1561



## 1. Introduction

Let  $(M^n, g)$  represent a Riemannian manifold. The geometric properties of  $(M^n, g)$  can generally be examined through two interrelated perspectives: intrinsic and extrinsic geometry. Intrinsic methods investigate characteristics that are inherent to the manifold itself, employing fundamental constructs such as geodesics, distance functions, and Jacobi fields. These tools facilitate the establishment of global results in Riemannian geometry, including landmark theorems such as Hadamard, Hopf-Rinow, Bonnet-Myers, and the Morse Index theorem [1, 2].

A key aspect of intrinsic geometry is the study of vector fields defined on  $(M^n, g)$ , encompassing Killing fields, conformal fields, and vector fields associated with Ricci solitons and almost Ricci solitons. Beyond their aesthetic contribution to the geometric structure, these vector fields significantly affect the manifold's topology and play essential roles in physical theories, particularly general relativity [1–4]. Furthermore, intrinsic geometry is closely linked to the analysis of partial differential equations, such as the Ricci flow, whose impact includes resolving the Poincaré conjecture [5, 6].

Differential geometry was originally developed through the study of curves and surfaces in  $\mathbb{R}^3$ , a classical foundation later extended by the seminal contributions of Kuiper and Nash [2, 3, 7]. They demonstrated that any  $n$ -dimensional Riemannian manifold  $(M^n, g)$  admits an isometric embedding into a sufficiently high-dimensional Euclidean space  $\mathbb{R}^m$  with  $m > n$ . This result laid the groundwork for the systematic exploration of extrinsic geometry, particularly in the context of submanifold theory [4, 8, 9].

Extrinsic geometry encompasses a broad range of investigations, from global theorems to detailed local analyses of submanifolds. Notable global results rely on concepts like total absolute curvature [10–13], while local studies examine specific structural properties [4, 8, 9, 14–16]. The field's depth is further enriched by examining minimal submanifolds in unit spheres, a domain that continues to present challenging open problems [17, 18].

An important subject in this context is the study of limitations on the isometric embedding of  $(M^n, g)$  into  $\mathbb{R}^m$ . Notably, Tompkins [19] proved that any flat, closed manifold  $(M^n, g)$  cannot admit an isometric embedding into  $\mathbb{R}^{2n}$ . This result has inspired subsequent developments and refinements in the field, as documented in [20, 21]. The Ricci flow constitutes a fundamental geometric evolution on a Riemannian manifold  $(M, g_0)$ , governed by the equation

$$\frac{\partial}{\partial t}g = -2S, \quad g(0) = g_0, \quad (1)$$

where  $S$  denotes the Ricci curvature associated with the evolving metric  $g$ . Introduced by Hamilton [22], this flow describes the temporal evolution of the metric under the influence of its Ricci curvature. The existence and uniqueness of solutions to (1) were rigorously analyzed by Hamilton and DeTurck [23], and subsequent studies have extended these investigations to various evolution equations

governing geometric structures [24]. An important extension of this concept is the Ricci-Bourguignon flow, proposed by J. P. Bourguignon in 1981 [25], which generalizes the classical Ricci flow [26]. This formulation draws inspiration from earlier unpublished insights of A. Lichnerowicz and foundational contributions in T. Aubin's work [27]. As an intrinsic evolution equation on Riemannian manifolds, the Ricci-Bourguignon flow governs the deformation of the metric in such a manner that its stationary points naturally correspond to solitonic configurations. Further generalizations of the Ricci-Bourguignon flow can also be found in [28–33].

Another prominent example of geometric evolution is the  $(\alpha, \beta)$ -Ricci-Yamabe flow. This flow was first formulated in 2019 by Güler and Crăsmăreanu [34], establishing a versatile framework in geometric analysis:

$$\frac{\partial}{\partial t}g = -2\alpha S - 2\beta rg, \quad g(0) = g_0,$$

where  $\alpha, \beta \in \mathbb{R}$ ,  $S$  denotes the Ricci curvature, and  $r$  is the scalar curvature corresponding to  $g$ . The  $(\alpha, \beta)$ -Ricci-Yamabe flow effectively interpolates between the classical Ricci flow and the Yamabe flow introduced by Hamilton [22, 26].

Particular choices of the parameters  $(\alpha, \beta)$  recover well-known geometric flows:

1. Ricci flow for  $\alpha = 1$  and  $\beta = 0$ ,
2. Yamabe flow for  $\alpha = 0$  and  $\beta = \frac{1}{2}$ ,
3. Schouten flow for  $\alpha = 1$  and  $\beta = -\frac{1}{2(n-1)}$ ,
4. Ricci-Bourguignon flow for  $\alpha = 1$  and  $\beta = -\rho$ .

Quasi-Einstein metrics and Ricci solitons naturally arise as self-similar solutions to [35], thereby offering significant insights into the geometric evolution of manifolds. This observation has prompted the exploration of broader generalizations, such as the so-called  $(\alpha, \beta)$ -Ricci-Yamabe flow. A Riemannian manifold of dimension  $n \geq 3$  is said to admit an  $(\alpha, \beta)$ -Ricci-Yamabe soliton, abbreviated as  $(\alpha, \beta)$ -RYS, if it satisfies the following structural condition:

$$\frac{1}{2}\mathcal{L}_V g + \alpha S + (\lambda + \beta r)g = 0, \quad (2)$$

where  $\mathcal{L}_V$  denotes the Lie derivative operator along the vector field  $V$ ,  $\alpha, \beta$  are non-zero constants and  $\lambda$  is an arbitrary real constant. In analogy with the classification of Ricci solitons, an  $(\alpha, \beta)$ -Ricci-Yamabe soliton  $(M, g, \alpha, \beta, V, \lambda)$  is characterized as expanding, steady, or shrinking depending on whether the parameter  $\lambda$  is positive, zero, or negative, respectively. Moreover, when the vector field  $V$  arises as the gradient of a smooth function  $f$ , the soliton is designated as a gradient  $(\alpha, \beta)$ -RYS. In this scenario, the defining soliton equation takes the form

$$\text{Hess}(f) + \alpha S + (\lambda + \beta r)g = 0.$$

Here,  $\text{Hess}(f)$  stands for the Hessian associated with the potential function  $f$ .

Consider an  $n$ -dimensional smooth Riemannian manifold  $(M^n, g)$ . In order to analyze when  $(M^n, g)$  supports an  $(\alpha, \beta)$ -RYS of the form  $(M^n, g, \beta r, \xi, \lambda)$  and satisfies Equation (2). Our method for addressing this question relies on the KN embedding

$$\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, g),$$

where  $m > n$  is chosen sufficiently large and  $g$  denotes the standard Euclidean metric. The embedding naturally induces a smooth tangential vector field  $\xi$  on  $(M^n, g)$ , which we interpret as the potential candidate for the  $(\alpha, \beta)$ -RYS structure  $(M^n, g, \beta r, \xi, \lambda)$ . To build such a field concretely, we start from a unit vector  $\vec{u}$  in  $(\mathbb{R}^m, g)$  and decompose it as:

$$\vec{u} = \xi + \Gamma,$$

where the component  $\xi$  lies in the tangent bundle of  $M^n$  and  $\Gamma$  belongs to the normal bundle. The tangential part  $\xi$  is what we call the KN vector of  $(M^n, g)$ , while  $\Gamma$  is referred to as the KN normal. Since this splitting depends on the choice of  $\vec{u}$ , the pair  $(\xi, \Gamma)$  is not uniquely determined.

In the present work, we examine conditions ensuring that  $(M^n, g)$ , endowed with its KN vector  $\xi$  and a constant  $\lambda$ , admits an  $(\alpha, \beta)$ -RYS structure  $(M^n, g, \alpha, \beta, \xi, \lambda)$ . In particular, we prove that if  $(M^n, g)$  is compact, has scalar curvature  $r$ , and the KNV  $\xi$  satisfies an integral inequality involving  $\xi(r)$  and  $\lambda$ , then  $(M^n, g, \alpha, \beta, \xi, \lambda)$  realizes an  $(\alpha, \beta)$ -RYS. Such a structure will be termed a KN  $(\alpha, \beta)$ -RYS, and its geometric features will be explored in detail.

The main novelty of this paper lies in combining the theory of  $(\alpha, \beta)$ -Ricci-Yamabe solitons with an extrinsic geometric framework arising from Kuiper-Nash isometric embeddings. Recent studies on Ricci-Yamabe solitons have primarily focused on intrinsic geometric structures, including isometries of almost Ricci-Yamabe solitons [36], special settings such as LP-Kenmotsu manifolds and  $\ast$ -Ricci-Yamabe solitons [37], and warped or doubly warped product manifolds [38].

Unlike these intrinsic approaches, we introduce a natural potential vector field induced from the ambient Euclidean space. More precisely, the tangential component of a fixed unit vector in  $\mathbb{R}^m$  defines a canonical vector field on  $M^n$ , referred to as the Kuiper-Nash vector field. This construction enables us to formulate and analyze  $(\alpha, \beta)$ -Ricci-Yamabe solitons from a genuinely extrinsic viewpoint.

Furthermore, we establish new integral criteria involving scalar curvature and divergence-type quantities that guarantee the existence of such solitons and characterize their triviality. These results provide a new bridge between intrinsic curvature conditions and extrinsic geometry, which is not captured by the existing literature. It is noteworthy that a trivial  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$  corresponds to an Einstein manifold, which illustrates why Ricci solitons are often regarded as a generalization of Einstein manifolds. Consequently, identifying conditions that ensure a  $(\alpha, \beta)$ -RYS is trivial represents a fundamental problem in the study of  $(\alpha, \beta)$ -RYS geometry. In this article, we establish two main results that provide

explicit criteria under which a KN  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$  reduces to a trivial structure.

**Example 1.1.** Let us examine the Euclidean manifolds  $(\mathbb{R}^n, g)$  and  $(\mathbb{R}^{n+1}, g')$ , both endowed with their canonical flat metrics  $g$  and  $g'$ , respectively. An isometric immersion can be introduced by

$$\Psi : (\mathbb{R}^n, g) \longrightarrow (\mathbb{R}^{n+1}, g'), \quad \Psi(x) = (x, 0), \quad x \in \mathbb{R}^n,$$

which yields a totally geodesic submanifold whose associated unit orthogonal vector is given by

$$N = \frac{\partial}{\partial x^{n+1}},$$

where  $(x_1, \dots, x_{n+1})$  represent the canonical coordinate system on  $\mathbb{R}^{n+1}$ .

Let  $\vec{u}$  be a vector in the ambient space  $\mathbb{R}^{n+1}$ . This vector can be decomposed into tangential and orthogonal parts as

$$\vec{u} = \xi + \Gamma,$$

where  $\xi \in \chi(\mathbb{R}^n)$  lies in the tangent bundle of  $\mathbb{R}^n$ , while the normal component is expressed as:

$$\Gamma = g(\vec{u}, N) N = \sigma N.$$

If we differentiate the above relation along any  $W \in \chi(\mathbb{R}^n)$  and project separately onto the tangent and normal directions, it follows that  $\nabla_W \xi = 0$ , and  $W(\sigma) = 0$ , where  $\nabla$  denotes the Levi-Civita connection on  $(\mathbb{R}^n, g)$ . Hence, the metric tensor remains invariant under the flow of  $\xi$ , i.e., its Lie derivative with respect to  $\xi$  vanishes.

$$\mathcal{L}_\xi g = 0.$$

It follows that on the Euclidean space, the  $(\alpha, \beta)$ -RYS equation reduces to

$$\frac{1}{2} \mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g = 0,$$

and since  $(\mathbb{R}^n, g)$  is Ricci-flat, we have  $\lambda = 0$ . Therefore,  $(\mathbb{R}^n, g, \alpha, \beta, \xi, \lambda)$  constitutes a KN  $(\alpha, \beta)$ -RYS with the KN vector  $\xi$ .

## 2. Preliminaries

Let  $(M^n, g)$  denote an  $n$ -dimensional Riemannian manifold endowed with the Levi-Civita connection  $\nabla$ . The  $(1, 3)$ -type curvature tensor associated with  $(M^n, g)$  is introduced by

$$R(U, V)W = [\nabla_U, \nabla_V]W - \nabla_{[U, V]}W, \quad U, V, W \in \chi(M^n),$$

where  $\chi(M^n)$  stands for the collection of smooth vector fields. By tracing, one obtains the symmetric Ricci tensor  $S$ , expressed as:

$$S(U, V) = \sum_{j=1}^n g(R(e_j, U)V, e_j), \quad U, V \in \chi(M^n),$$

with  $\{e_1, \dots, e_n\}$  denoting a local  $g$ -orthonormal frame field on  $(M^n, g)$ . The corresponding Ricci operator  $Q : \chi(M^n) \rightarrow \chi(M^n)$  is determined from the identity

$$S(U, V) = g(QU, V), \quad U, V \in \chi(M^n).$$

Finally, the scalar curvature  $r$  of the manifold is defined as the total trace of the Ricci tensor  $r = \sum_{l=1}^n S(e_l, e_l)$ .

An established relation, documented in [8] (see Equation (3.1), p. 58), connects the gradient of the scalar curvature with the divergence of the Ricci operator. It takes the form

$$\frac{1}{2} \nabla r = \sum_{l=1}^n (\nabla_{e_l} Q)(e_l), \quad (3)$$

where  $\nabla r$  denotes the gradient field of  $r$ , and the covariant derivative of  $Q$  is given by

$$(\nabla_U Q)(V) = \nabla_U(QV) - Q(\nabla_U V), \quad U, V \in \chi(M^n).$$

We employ a KN embedding of the form

$$\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, \bar{g}),$$

with  $m > n$  sufficiently large, where  $\bar{g}$  denotes the standard Euclidean metric on  $\mathbb{R}^m$ . Denote by  $\bar{\nabla}$  the flat connection on  $(\mathbb{R}^m, \bar{g})$ , let  $\nu$  be the normal bundle induced by  $\Psi$ , and write  $\Pi(\nu)$  for the space of smooth normal vector fields along  $\Psi$ . For the above isometric immersion, the fundamental connections between the ambient Euclidean geometry and the induced geometry on  $M^n$  are described by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \quad U, V \in \chi(M^n), \quad (4)$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^\perp N, \quad N \in \Pi(\nu), \quad (5)$$

where  $h$  represents the second fundamental form,  $A$  denotes the shape operator, and

$$\bar{g}(h(U, V), N) = g(A_N U, V), \quad U, V \in \chi(M^n).$$

Moreover, the geometry of the embedding  $\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, \bar{g})$ , yields the following expressions for the curvature tensors. The Riemann curvature tensor on  $(M^n, g)$  can be rewritten as:

$$R(U, V)W = A_{h(V, W)}U - A_{h(U, W)}V, \quad U, V, W \in \chi(M^n),$$

while the Ricci tensor admits the form

$$S(U, V) = n \bar{g}(H, h(U, V)) - \sum_{j=1}^n \bar{g}(h_{e_j}, U) g(h_{e_j}, V), \quad U, V \in \chi(M^n),$$

where  $H$  represents the mean curvature vector field, which is defined as follows:

$$H = \frac{1}{n} \sum_{j=1}^n g(h_{e_j}, e_j),$$

with  $\{e_1, \dots, e_n\}$  forming a local orthonormal frame on  $(M^n, g)$ .

Consider a smooth Riemannian manifold  $(M^n, g)$  and its KN isometric embedding

$$\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, \bar{g}) \quad (\text{cf. [39]}).$$

Let  $\vec{u}$  be a fixed unit vector in the ambient Euclidean space  $\mathbb{R}^m$ . We examine its decomposition into tangential and normal components relative to  $M^n$ , as detailed below.

**Definition 2.1.** Let  $\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, \bar{g})$  be a KN embedding, and let constant vector  $\vec{u} \in \mathbb{R}^m$  be unit. We write

$$\vec{u} = \xi + \Gamma,$$

where  $\xi$  represents the component tangent to  $M^n$ , called the *KN vector*, and  $\Gamma$  is the normal component, referred to as the *KN normal*.

The Kuiper–Nash vector field  $\xi$  admits a natural geometric interpretation. It is obtained as the tangential projection of a constant unit vector in the ambient Euclidean space  $\mathbb{R}^m$ . Therefore,  $\xi$  reflects how the submanifold  $M^n$  is positioned inside  $\mathbb{R}^m$ . This construction provides a canonical class of vector fields depending only on the embedding. In particular, unlike arbitrary vector fields, the KN vector field encodes both intrinsic and extrinsic geometric information, which makes it especially suitable as a potential vector field in the study of geometric solitons.

**Definition 2.2.** Assume that  $\xi$  and  $\Gamma$  are, respectively, the KN tangential and normal components corresponding to the embedding

$$\Psi : (M^n, g) \longrightarrow (\mathbb{R}^m, \bar{g}),$$

and a fixed unit vector  $\vec{u}$ . Define a smooth function  $\phi : M^n \rightarrow \mathbb{R}$  by

$$\phi = \bar{g}(H, \Gamma),$$

where  $H$  denotes the mean curvature vector of  $M^n$ . This function is termed the *KN function* on  $(M^n, g)$ . Furthermore, the operator

$$K : \chi(M^n) \rightarrow \chi(M^n), \quad K = A_\Gamma,$$

is said to be the *KN operator* of  $(M^n, g)$ .

Differentiating the decomposition  $\vec{u} = \xi + \Gamma$ , along a vector field  $U \in \chi(M^n)$  and applying the fundamental relations (4) and (5), while separating the tangential and normal components, yields

$$\nabla_U \xi = KU, \quad \nabla_U^\perp \Gamma = -h(\xi, U), \quad U \in \chi(M^n). \quad (6)$$

The following result from [40] summarizes key properties of KN structures:

**Lemma 2.3** ([40]). *Let  $(M^n, g)$  be a Riemannian manifold equipped with a KN vector  $\xi$ , the associated KN function  $\phi$ , and the KN operator  $K$ . Then, the following relations hold:*

1. *The trace of the operator  $K$  satisfies  $\text{Tr } K = n\phi$ .*
2. *The Ricci operator acts on  $\xi$  according to*

$$Q(\xi) = \sum_{j=1}^n (\nabla_{e_j} K) e_j - n \nabla \phi,$$

where  $\{e_1, \dots, e_n\}$  is any local orthonormal frame,  $\text{Tr } K$  denotes the trace of  $K$ , and  $\nabla \phi$  is the gradient of the KN function  $\phi$ .

Recall that for a smooth function  $f : M^n \rightarrow \mathbb{R}$ , the Laplace operator acts as:

$$\beta r f = \text{div}(\nabla f),$$

where  $\nabla f$  denotes the gradient of  $f$ , and the divergence of a vector field  $U$  is defined by  $\text{div } U = \sum_{l=1}^n g(\nabla_{e_l} U, e_l)$ .

**Lemma 2.4** ([40]). *Let  $(M^n, g)$  be a compact Riemannian manifold endowed with a KN vector  $\xi$  and the corresponding KN function  $\phi$ . Then the following identities hold:*

1. *The integral of  $\phi$  over  $M^n$  vanishes:*

$$\int_{M^n} \phi dV_g = 0.$$

2. *The integral of  $\xi(\phi)$  is related to  $\phi^2$  via*

$$\int_{M^n} \xi(\phi) dV_g = -n \int_{M^n} \phi^2 dV_g,$$

where  $dV_g$  is the Riemannian volume element of  $(M^n, g)$ .

Note that for a  $(1, 1)$ -tensor field  $F$  on  $(M^n, g)$ , its squared norm is defined by

$$|F|^2 = \sum_{j=1}^n g(F e_j, F e_j).$$

Similarly, for a symmetric  $(0, 2)$ -tensor  $\Omega$ , one has

$$|\Omega|^2 = \sum_{j,k=1}^n \Omega(e_j, e_k)^2. \quad (7)$$

### 3. Constructing an $(\alpha, \beta)$ -RYS from a Riemannian manifold $(M^n, g)$

Consider an  $n$ -dimensional Riemannian manifold  $(M^n, g)$  endowed with a KN vector  $\xi$ , its associated KN function  $\phi$ , and the KN operator  $K$ . In the following, we establish sufficient conditions under which the compact manifold  $(M^n, g)$  admits an  $(\alpha, \beta)$ -RYS structure  $(M^n, g, \beta r, \xi, \lambda)$ , and we investigate the geometric properties resulting from this induced  $(\alpha, \beta)$ -RYS configuration. The main result is stated as follows:

**Theorem 3.1.** *Let  $(M^n, g)$  be a compact and connected  $n$ -dimensional Riemannian manifold with scalar curvature  $r$ , Ricci operator  $Q$ , KN vector  $\xi$ , KN function  $\phi$ , and KN operator  $K$ . Assume that there exist smooth functions  $\beta r$  and  $\lambda$  satisfying the compatibility condition*

$$\int_{M^n} (\beta r + \lambda) \phi \, dV_g = 0,$$

and that the inequality

$$\alpha \int_{M^n} \xi(r) \, dV_g - 2n\beta \int_{M^n} r \phi \, dV_g \geq \int_{M^n} \left( |K|^2 + \alpha^2 |Q|^2 - \frac{1}{n} \alpha^2 r^2 + \frac{1}{n} (n(\beta r + \lambda) + \alpha r)^2 \right) dV_g,$$

holds. Then the manifold  $(M^n, g, \beta r, \xi, \lambda)$  admits the structure of an  $(\alpha, \beta)$ -RYS.

*Proof.* We begin by expressing the soliton equation in terms of the KN operator  $K$  and the Ricci operator  $Q$ . Then, we compute the squared norm of the corresponding tensor and integrate over the manifold. Finally, we use the given inequality to conclude that the soliton equation must hold identically. Applying the symmetry of the KN operator  $K$  together with (6), one directly obtains

$$\frac{1}{2} \mathcal{L}_\xi g(U, V) = g(KU, V), \quad U, V \in \chi(M^n). \quad (8)$$

Hence, for smooth functions  $\beta r$  and  $\lambda$ , we can express

$$\frac{1}{2} \mathcal{L}_\xi g(U, V) + \alpha S(U, V) + (\lambda + \beta r)g(U, V) = g(KU, V) + \alpha g(QU, V) + (\beta r + \lambda)g(U, V).$$

Selecting a local orthonormal frame  $\{e_1, \dots, e_n\}$  and applying (7) along with item (1) in Lemma 2.3, we compute

$$\begin{aligned} \left| \frac{1}{2} \mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g \right|^2 &= |K|^2 + \alpha^2 |Q|^2 + n(\beta r + \lambda)^2 \\ &\quad + 2\alpha \sum_j g(Ke_j, Qe_j) + 2n(\beta r + \lambda)\phi + 2\alpha(\beta r + \lambda)r. \end{aligned} \quad (9)$$

Next, observe that

$$\sum_{j,k=1}^n S(e_j, e_k)^2 = \sum_{j=1}^n g(Qe_j, Qe_j) = |Q|^2.$$

We now compute the divergence of  $Q\xi$ :

$$\begin{aligned} \operatorname{div} Q(\xi) &= \sum_{j=1}^n g(\nabla_{e_j} Q\xi, e_j) \\ &= \sum_{j=1}^n g((\nabla_{e_j} Q)(\xi) + Q(\nabla_{e_j} \xi), e_j) \\ &= \frac{1}{2}\xi(r) + \sum_{j=1}^n g(Ke_j, Qe_j), \end{aligned}$$

where we applied the symmetry of  $Q$  along with Equation (3). Substituting this into (9) gives

$$\begin{aligned} \left| \frac{1}{2}\mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g \right|^2 &= |K|^2 + \alpha^2|Q|^2 + n(\beta r + \lambda)^2 + 2\alpha \operatorname{div} Q(\xi) - \alpha\xi(r) \\ &\quad + 2n(\beta r + \lambda)\phi + 2\alpha(\beta r + \lambda)r. \end{aligned}$$

Integrating over  $M^n$  and using  $\int_{M^n} \phi dV_g = 0$  leads to

$$\begin{aligned} &\int_{M^n} \left| \frac{1}{2}\mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g \right|^2 dV_g \\ &= \int_{M^n} \left( |K|^2 + \alpha^2|Q|^2 - \frac{1}{n}\alpha^2 r^2 + \frac{1}{n}(n(\beta r + \lambda) + \alpha r)^2 \right) dV_g \\ &\quad + 2n \int_{M^n} \beta r \phi dV_g - \alpha \int_{M^n} \xi(r) dV_g. \end{aligned} \quad (10)$$

Finally, the hypothesis ensures that the right-hand side of (10) is non-positive, so that

$$\int_{M^n} \left| \frac{1}{2}\mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g \right|^2 dV_g \leq 0,$$

forcing

$$\left| \frac{1}{2}\mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g \right|^2 = 0,$$

or equivalently,

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha S + (\beta r + \lambda)g = 0. \quad (11)$$

This confirms that  $(M^n, g, \alpha, \beta, \xi, \lambda)$  is indeed an  $(\alpha, \beta)$ -RYS.  $\square$

**Remark 1.** The integral assumptions in [Theorem 3.1](#) are natural in the context of compact Riemannian manifolds. Indeed, conditions involving integrals of scalar curvature and divergence-type quantities frequently arise in the study of geometric flows and soliton equations.

Moreover, the presence of the KN function  $\varphi$  introduces additional structure that allows these conditions to be interpreted geometrically in terms of the embedding. In particular, such assumptions are compatible with classical integral identities and are satisfied in several standard situations.

**Corollary 3.2.** *Consider a compact and connected Riemannian manifold  $(M^n, g)$  of dimension  $n$ , endowed with scalar curvature  $r$ , Ricci operator  $Q$ , a KN vector field  $\xi$ , the associated KN function  $\phi$ , and the KN operator  $K$ . Assume that the inequality*

$$\int_{M^n} \xi(r) dV_g \geq \int_{M^n} \left( |K|^2 + |Q|^2 - \frac{1}{n} r^2 + \frac{1}{n} (n\lambda + r)^2 \right) dV_g,$$

*is satisfied. Then the tuple  $(M^n, g, \xi, \lambda)$  naturally defines a Ricci soliton structure.*

The  $(\alpha, \beta)$ -RYS  $(M^n, g, \beta r, \xi, \lambda)$  introduced in [Theorem 3.1](#) is henceforth referred to as a KN  $(\alpha, \beta)$ -RYS.

**Proposition 3.3.** *Let  $(M^n, g, \alpha, \beta, \xi, \lambda)$  be a compact  $n$ -dimensional KN  $(\alpha, \beta)$ -RYS. Then, the following integral identity holds:*

$$\frac{\alpha}{2} \int_{M^n} \xi(r) dV_g - n\beta \int_{M^n} r\phi dV_g = \int_{M^n} \left[ \left( \alpha^2 |Q|^2 - \frac{1}{n} \alpha^2 r^2 \right) + \frac{1}{n} (n(\beta r + \lambda) + \alpha r)^2 \right] dV_g.$$

*Proof.* Consider a compact  $n$ -dimensional KN  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$ . Equation (10) can be equivalently expressed as:

$$\alpha \int_{M^n} \xi(r) dV_g - 2n\beta \int_{M^n} r\phi dV_g = \int_{M^n} \left( |K|^2 + \alpha^2 |Q|^2 - \frac{1}{n} \alpha^2 r^2 + \frac{1}{n} (n(\beta r + \lambda) + \alpha r)^2 \right) dV_g. \quad (12)$$

Combining relations (8) and (11) immediately gives

$$K = -(\lambda + \beta r)I - \alpha Q, \quad (13)$$

from which it follows that

$$|K|^2 = n(\lambda + \beta r)^2 + 2\alpha(\lambda + \beta r)r + \alpha^2 |Q|^2,$$

or, equivalently,

$$|K|^2 = \left( \alpha^2 |Q|^2 - \frac{1}{n} \alpha^2 r^2 \right) + \frac{1}{n} (n(\lambda + \beta r) + \alpha r)^2.$$

Inserting this expression into (12) leads to

$$\alpha \int_{M^n} \xi(r) dV_g - 2n\beta \int_{M^n} r\phi dV_g = 2 \int_{M^n} \left( \alpha^2 |Q|^2 - \frac{1}{n} \alpha^2 r^2 + \frac{1}{n} (n(\lambda + \beta r) + \alpha r)^2 \right) dV_g,$$

which completes the proof.  $\square$

**Example 3.4.** Let  $(S^n, g)$  be the unit sphere isometrically embedded in  $\mathbb{R}^{n+1}$  endowed with the induced metric. For a fixed unit vector  $\vec{u} \in \mathbb{R}^{n+1}$ , decompose

$$\vec{u} = \xi + \Gamma,$$

where  $\xi$  and  $\Gamma$  denote the tangential and normal components, respectively. Since the unit normal vector is  $N = x$ , we obtain

$$\Gamma = \langle \vec{u}, x \rangle x, \quad \xi = \vec{u} - \langle \vec{u}, x \rangle x.$$

Consequently, the Kuiper–Nash function reduces to

$$\varphi = \langle H, \Gamma \rangle = \langle x, \langle \vec{u}, x \rangle x \rangle = \langle \vec{u}, x \rangle,$$

which satisfies

$$\int_{S^n} \varphi dV_g = 0, \quad \int_{S^n} r \varphi dV_g = 0.$$

Since  $(S^n, g)$  is Einstein with  $Q = (n-1)I$  and  $r = n(n-1)$ , relation (13) yields

$$K = -(\lambda + \beta r)I - \alpha Q = -(\lambda + \beta n(n-1) + \alpha(n-1))I,$$

hence

$$|K|^2 = n(\lambda + \beta n(n-1) + \alpha(n-1))^2.$$

Substituting into the  $(\alpha, \beta)$ -Ricci–Yamabe soliton equation gives

$$\frac{1}{2}L_\xi g + (\alpha(n-1) + \lambda + \beta n(n-1))g = 0,$$

which holds if and only if

$$\lambda + \beta n(n-1) + \alpha(n-1) = 0.$$

Under this condition,  $(S^n, g, \alpha, \beta, \xi, \lambda)$  defines a Kuiper–Nash  $(\alpha, \beta)$ -Ricci–Yamabe soliton. Moreover, since  $Q = \frac{r}{n}I$ , the soliton is trivial.

**Theorem 3.5.** Let  $(M^n, g, \alpha, \beta, \xi, \lambda)$  be a compact  $n$ -dimensional KN  $(\alpha, \beta)$ -RYS. Then the scalar curvature  $r$  fulfills

$$\int_{M^n} n(\lambda + \beta r)^2 dV_g \leq \alpha \int_{M^n} (\lambda + \beta r)r dV_g, \quad \int_{M^n} r \phi dV_g = 0,$$

if and only if the relation  $\lambda = -\beta r$  holds identically.

*Proof.* Assume that the scalar curvature  $r$  of KN  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$  obeys

$$\int_{M^n} n(\lambda + \beta r)^2 dV_g \leq \alpha \int_{M^n} (\lambda + \beta r)r dV_g, \quad \int_{M^n} (\lambda + \beta r)\phi dV_g = 0.$$

From Equations (6) and (13), it follows that

$$\nabla_U \xi = -(\lambda + \beta r)U - \alpha QU. \quad (14)$$

Computing the divergence of  $Q(\xi)$  with respect to a local orthonormal frame  $\{e_1, \dots, e_n\}$ , we obtain

$$\begin{aligned} \operatorname{div} Q(\xi) &= \sum_{j=1}^n g(\nabla_{e_j} Q\xi, e_j) \\ &= \sum_{j=1}^n g((\nabla_{e_j} Q)(\xi) + Q(\nabla_{e_j} \xi), e_j) \\ &= \frac{1}{2}\xi(r) + \sum_{j=1}^n g(-(\lambda + \beta r)e_j - \alpha Qe_j, Qe_j) \\ &= \frac{1}{2}\xi(r) - (\lambda + \beta r)r - \alpha|Q|^2. \end{aligned}$$

Hence, we have

$$\operatorname{div} Q(\xi) = \frac{1}{2}\xi(r) + \frac{1}{\alpha} \left[ - \left( \alpha^2|Q|^2 - \frac{1}{n}\alpha^2r^2 \right) - (\lambda + \beta r)r\alpha - \frac{1}{n}\alpha^2r^2 \right].$$

Integrating over  $M^n$  leads to

$$\frac{1}{2} \int_{M^n} \xi(r) dV_g = \frac{1}{\alpha} \int_{M^n} \left( \alpha^2|Q|^2 - \frac{1}{n}\alpha^2r^2 + \frac{1}{n}(n(\lambda + \beta r)r\alpha + \alpha^2r^2) \right) dV_g.$$

Applying Proposition 3.3 and the condition  $\int_{M^n} r\phi dV_g = 0$ , we get

$$\int_{M^n} (n(\lambda + \beta r) + \alpha r)^2 dV_g = \int_{M^n} (n(\lambda + \beta r)r\alpha + \alpha^2r^2) dV_g,$$

which implies

$$\int_{M^n} (n^2(\lambda + \beta r)^2 + n(\lambda + \beta r)r\alpha) dV_g = 0. \quad (15)$$

From the assumption, it follows that

$$\int_{M^n} (\lambda + \beta r)^2 dV_g = 0,$$

so that  $\beta r = -\lambda$ . The converse is immediate.  $\square$

In the remainder of this section, we investigate the specific criteria under which the Kuiper-Nash  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$  reduces to a trivial  $(\alpha, \beta)$ -RYS. To begin, we establish the following result:

**Theorem 3.6.** *Let  $(M^n, g, \alpha, \beta, \xi, \lambda)$  be a compact KN  $(\alpha, \beta)$ -RYS with the condition  $n\beta + \alpha \neq 0$ . Then the scalar curvature  $r$  satisfies*

$$\int_{M^n} r^2 dV_g \leq -\frac{n}{\alpha} \int_{M^n} (\lambda + \beta r) r dV_g,$$

*and this inequality holds if and only if  $(M^n, g, \alpha, \beta, \xi, \lambda)$  constitutes a trivial  $(\alpha, \beta)$ -RYS.*

*Proof.* Assume that the scalar curvature  $r$  of a compact KN  $(\alpha, \beta)$ -RYS  $(M^n, g, \alpha, \beta, \xi, \lambda)$  satisfies

$$\int_{M^n} r^2 dV_g \leq -\frac{n}{\alpha} \int_{M^n} (\lambda + \beta r) r dV_g, \quad \int_{M^n} r \phi dV_g = 0.$$

From Equation (15), it follows that

$$\int_{M^n} (n^2(\lambda + \beta r)^2 + n(\lambda + \beta r)r\alpha) dV_g = 0.$$

Applying the given condition, we deduce

$$\int_{M^n} (n(\lambda + \beta r) + \alpha r)^2 dV_g = 0,$$

which implies  $\alpha r = -n(\beta r + \lambda)$ , or equivalently,

$$r = -\frac{n\lambda}{n\beta + \alpha}.$$

Hence, the scalar curvature  $r$  is constant. Substituting this result into [Proposition 3.3](#) yields

$$\int_{M^n} (|Q|^2 - \frac{1}{n}r^2) dV_g = 0. \quad (16)$$

By the Cauchy-Schwarz inequality,  $|Q|^2 \geq \frac{1}{n}r^2$ , we infer from (16) that  $|Q|^2 = \frac{1}{n}r^2$ , which holds precisely when

$$Q = \frac{r}{n}I. \quad (17)$$

Combining  $r = -n(\beta r + \lambda)$  with (17) and Equation (14), we conclude that  $\mathcal{L}_\xi g = 0$ , showing that  $(M^n, g, \alpha, \beta, \xi, \lambda)$  is indeed trivial. The converse is immediate: if  $(M^n, g, \alpha, \beta, \xi, \lambda)$  is trivial, then  $S = -(\beta r + \lambda)g$ , which guarantees that  $r = -n(\beta r + \lambda)$  is constant.  $\square$

**Theorem 3.7.** *Let  $(M^n, g, \alpha, \beta, \xi, \lambda)$  be a compact KN  $(\alpha, \beta)$ -RYS with  $n > 2$  and  $\alpha + (n-1)\beta \neq 0$ . If its scalar curvature is denoted by  $r$ , then*

$$\int_{M^n} (\operatorname{div} \xi)^2 dV_g \leq \frac{1}{n} \int_{M^n} (\alpha r + n(\beta r + \lambda))(2(\alpha + (n-1)\beta) + n(\beta r + \lambda)) dV_g,$$

*holds if and only if  $(M^n, g, \alpha, \beta, \xi, \lambda)$  represents a trivial  $(\alpha, \beta)$ -RYS.*

*Proof.* Let  $(M^n, g, \alpha, \beta, \xi, \lambda)$  denote a compact KN  $(\alpha, \beta)$ -RYS with scalar curvature  $r$  satisfying the inequality

$$\int_{M^n} (\operatorname{div} \xi)^2 dV_g \leq \frac{1}{n} \int_{M^n} (\alpha r + n(\beta r + \lambda))(2(\alpha + (n-1)\beta) + n(\beta r + \lambda)) dV_g. \quad (18)$$

From Equations (8) and (11), it follows that

$$KU = -(\beta r + \lambda)U - \alpha QU, \quad U \in \chi(M^n), \quad (19)$$

which combined with (6) gives

$$\nabla_U \xi = -(\beta r + \lambda)U - \alpha QU, \quad U \in \chi(M^n). \quad (20)$$

Using a local orthonormal frame  $\{e_1, \dots, e_n\}$ , we compute

$$|\nabla \xi|^2 = \sum_{j=1}^n g(\nabla_{e_j} \xi, \nabla_{e_j} \xi) = n(\beta r + \lambda)^2 + 2\alpha r(\beta r + \lambda) + \alpha^2 |Q|^2, \quad (21)$$

and similarly

$$\frac{1}{4} |\mathcal{L}_\xi g|^2 = n(\beta r + \lambda)^2 + 2\alpha r(\beta r + \lambda) + \alpha^2 |Q|^2. \quad (22)$$

Tracing (19) yields

$$n \nabla \phi = -(\alpha + n\beta) \nabla r, \quad (23)$$

and applying (3) gives

$$\sum_{j=1}^n (\nabla_{e_j} K) e_j = -\left(\beta + \frac{\alpha}{2}\right) \nabla r. \quad (24)$$

Using Lemma 2.3 together with (23) and (24), we find

$$Q(\xi) = \left(\frac{\alpha}{2} + (n-1)\beta\right) \nabla r, \quad \text{so that} \quad S(\xi, \xi) = \left(\frac{\alpha}{2} + (n-1)\beta\right) \xi(r).$$

From (19) we compute  $\operatorname{div}(Q\xi) = -(\beta r + \lambda)r - \alpha |Q|^2$ , and integration gives

$$\frac{1}{2} \int_{M^n} \xi(r) dV_g = \int_{M^n} \left((\beta r + \lambda)r + \alpha |Q|^2\right) dV_g. \quad (25)$$

Inserting (21), (22) and (25) into the integral identity (cf. [41]) leads to

$$\begin{aligned} & 2\alpha(\alpha + (n-1)\beta) \int_{M^n} \left(|Q|^2 - \frac{1}{n} r^2\right) dV_g \\ &= \int_{M^n} (\operatorname{div} \xi)^2 dV_g - \frac{1}{n} \int_{M^n} (\alpha r + n(\beta r + \lambda))(2(\alpha + (n-1)\beta) + n(\beta r + \lambda)) dV_g. \end{aligned}$$

Combining this with (18) and the Cauchy–Schwarz inequality  $|Q|^2 \geq \frac{1}{n}r^2$ , it follows that

$$|Q|^2 = \frac{1}{n}r^2, \quad \text{and} \quad Q = \frac{r}{n}I.$$

Summing  $(\nabla_U Q)(U) = \frac{1}{n}U(r)U$  over a local orthonormal frame gives  $\nabla r = 0$  for  $n > 2$ , so  $r$  is constant. Then (20) implies  $\operatorname{div} \xi = -n(\beta r + \lambda) - \alpha r$ , which vanishes upon integration. Consequently,

$$\alpha S = -(\beta r + \lambda)g \quad \text{and} \quad \mathcal{L}_\xi g = 0,$$

demonstrating that  $(M^n, g, \alpha, \beta, \xi, \lambda)$  is a trivial  $(\alpha, \beta)$ -RYS. The converse follows immediately, since if the soliton is trivial, then  $\alpha S = -(\beta r + \lambda)g$ , which implies  $\alpha r = -n(\beta r + \lambda)$  and  $\operatorname{div} \xi = 0$ , satisfying the stated condition.  $\square$

## 4. Conclusion

This paper introduces a novel framework for constructing  $(\alpha, \beta)$ -Ricci-Yamabe solitons via Kuiper–Nash (KN) isometric embeddings of compact Riemannian manifolds into Euclidean spaces. By exploiting the decomposition of a fixed ambient unit vector into tangential and normal components, we defined a natural geometric structure consisting of the KN vector field, KN function, and KN operator, which jointly encode the extrinsic geometry of the embedding and its intrinsic curvature effects. Within this setting, we derived new integral characterizations ensuring the existence of  $(\alpha, \beta)$ -RYS structures in terms of scalar curvature and KN geometric data. In particular, we established sharp integral identities linking the Lie derivative of the metric, Ricci operator, and KN quantities, leading to a unified soliton equation governed by extrinsic curvature contributions.

A key contribution of this work is the derivation of rigidity criteria under which the constructed solitons necessarily become trivial. These results show that triviality is equivalent to Einstein-type behavior of the Ricci operator and constancy of scalar curvature, obtained through integral inequalities involving  $\xi(r)$ ,  $\operatorname{div} \xi$ , and KN curvature operators. Consequently, the soliton structure collapses to a highly constrained geometric configuration determined entirely by curvature normalization. Overall, this study reveals a direct and nontrivial interaction between Kuiper–Nash extrinsic geometry and Ricci–Yamabe soliton theory, providing new rigidity phenomena and extending classical soliton constructions to an embedding-driven geometric framework.

**Conflicts of Interest.** The authors declare that they have no conflicts of interest regarding the publication of this article.

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