

Topological Indices and Structural Properties of a Graph Transformation

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Abstract

Graph rewriting or graph transformation has existed for many years. The R_k -Construction is obtained by replacing each vertex of a graph G with k independent copies. Each edge uv of G is replaced by edge joining every copy of u to every copy of v . Hence, every vertex in $R_k(G)$ has degree multiplied by k . A topological index is a value associated a molecule's chemical structure. Among various topological indices, degree-based indices form an important and extensively studied class that is fundamental in chemical graph theory and structural graph analysis. In this paper, we study the structural invariants and metrics of the R_k -Construction. Moreover, we examine the behavior of certain topological indices on this transformation and demonstrate that an infinite family of graphs possesses an integer-valued topological index. Furthermore, we correct and generalize several previously reported results in the literature and calculate several topological indices for the graphs generated by this construction.

Keywords: Topological indices, Homogeneous function, Graph transformation.

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1. Introduction

Throughout this paper, $G = (V, E)$ is a simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. We refer to Harary [1] and Zhang [2] for unexplained

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terminology. The degree of a vertex u in G is denoted by $d_G(u)$. The distance between two vertices u and v in G is denoted by $d_G(u, v)$. Many graphs arise from smaller graphs by being reconstructed using specific transformations. One of these transformations is introduced in [3], namely R_k -Construction. This transformation creates k copies for a given graph G , and each vertex is connected to its corresponding vertices in these copies, creating $k^2|E(G)|$ new edges. Topological indices are numerical measures that remain unchanged under graph isomorphism. They play an important role in predicting the chemical properties of molecular structures. Topological indices are generally categorized into several major classes, such as distance-based, degree-based, and others. Within these classes, degree-based topological indices hold significant importance and play a fundamental role in chemical graph theory. The Randić index of a graph G , denoted by $R(G)$, introduced by Milan Randić in 1975, was the first degree-based topological index [4]. $R(G)$ is defined as:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

The atom-bond connectivity index of a graph G ($ABC(G)$), was introduced by Estrada et al. [5]. This index is a modified version of the Randić index. The $ABC(G)$ is defined as:

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u)d_G(v)}}.$$

Furtula et al. proposed a modified version of the ABC index [6], known as the augmented Zagreb index AZI , given by:

$$AZI(G) = \sum_{uv \in E(G)} \left(\frac{d_G(u)d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3.$$

Another variant of the Randić index is the harmonic index [7]. The harmonic index of G , denoted by $H(G)$, is defined as:

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

The reciprocal Randić index of a graph G [8] is defined as:

$$RR(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)d_G(v)}.$$

Two of the oldest and most studied topological indices are the first and second Zagreb indices [9], defined as:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2, \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

The Geometric–arithmetic index of a graph G , introduced by Vukičević et al. [10], is defined as:

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_G(u)d_G(v)}}{d_G(u) + d_G(v)}.$$

Among the numerous degree-based topological indices, the Sombor index has become one of the most widely examined. This was introduced by Gutman [11]. For a graph G , the Sombor index, denoted by $SO(G)$, is defined as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}.$$

Recently, Movahedi et al. introduced a new degree-based topological index called the Diminished Sombor index [12]. For a graph G , the Diminished Sombor index denoted by $DSO(G)$, is defined as:

$$DSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G^2(u) + d_G^2(v)}}{d_G(u) + d_G(v)}.$$

For comprehensive discussions and additional background on topological indices, readers are referred to [5, 13–17] and [18–21]. Although the number of degree-based indices is immense, there has been a comparative lack of emphasis on the study of the reaction of these indices to graph transformations. In this paper, the structural invariants and metrics of the R_k -Construction are examined. Moreover, we explore the behavior of several topological indices on this specific graph construction and demonstrate that an infinite family of graphs admits integer-valued topological indices for the graphs obtained from this construction.

2. Main results

In this section, we first examine the features of R_k -Construction. Then, based on this construction, we derive a general relationship between several topological indices of a graph and the corresponding indices after applying this construction. We then examined the effect of this construction on several other well-known topological indices and determined the values for the resulting graphs.

2.1. R_k -Construction

In [3], for a graph G and integer $k \geq 2$, a construction denoted by $R_k(G)$ was introduced as follows: Let $V(G) = \{v_1, v_2, \dots, v_n\}$, and consider k copies of G as $V(G^i) = \{v_1^i, v_2^i, \dots, v_n^i\}$, and $E(G^i) = \{v_r^i v_s^i \mid v_r v_s \in E(G)\}$, $i = \{1, 2, \dots, k\}$. Then $V(R_k(G)) = \bigcup_{i=1}^k V(G^i)$, and

$$E(R_k(G)) = \bigcup_{i=1}^k E(G^i) \bigcup \{v_r^i v_s^j \mid i \neq j, v_r v_s \in E(G)\}.$$

For example, let G be a graph as shown in Figure 1. For $k = 3$, the graph $R_3(G)$ is shown in Figure 2. The following results are obtained immediately.



Figure 1: Graph G .

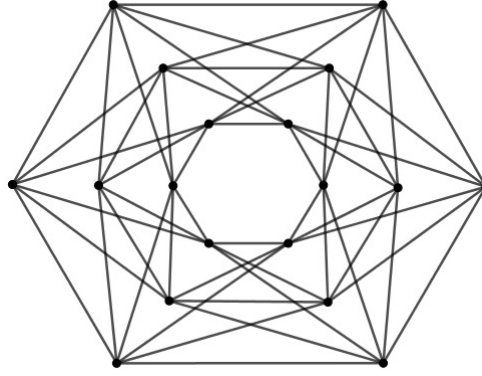


Figure 2: Graph $R_3(G)$.

Proposition 2.1. For any $k \geq 2$, and every $v_j \in V(G)$, $d_{R_k(G)}(v_j^i) = kd_G(v_j)$.

Proposition 2.2. Let $G = (V, E)$ be a graph. Then, for any $k \geq 2$,

$$|V(R_k(G))| = k|V(G)|, \quad |E(R_k(G))| = k^2|E(G)|.$$

Proposition 2.3. The R_k -Construction of a graph G contains isolated vertices if and only if G has isolated vertices.

Corollary 2.4. For a given graph G , $R_k(G)$ is connected if and only if G is connected.

Proof. Each copy G^i isomorphic to G and edges between copies are introduced only corresponding to edges of G , any path in G induced paths between corresponding vertices in $R_k(G)$. Hence connectivity is preserved. $\square$

Theorem 2.5. Let G be a triangle-free graph. Then, for every integer $k \geq 2$, $R_k(G)$ is also triangle-free.

Proof. Assume, to the contrary, that $R_k(G)$ contains a triangle. That is, there exist three distinct vertices a , b , and c in $V(R_k(G))$ such that ab , bc , $ca \in E(R_k(G))$. We distinguish three cases according to how many of these three vertices lie in the same copy of G :

1. All three vertices belong to the same copy G^i , $1 \leq i \leq k$. Because G^i is isomorphic to G (by definition of the construction), the three edges ab , bc , and ca correspond to edges of G between the same three vertices. Hence G would contain a triangle, contradicting the hypothesis that G is triangle-free.
2. Exactly two of the vertices lie in one copy, and the third lies in a different copy. Without loss of generality, suppose that a and b belong to G^j , while c lies in G^m with $j \neq m$ and $1 \leq j, m \leq k$. Since ab is an edge of G^j , the corresponding vertices of G , which we denote again by a and b , are adjacent in G . The edge ac connects a vertex of G^j with a vertex of G^m . By the definition of $R_k(G)$, such an edge exists if and only if the original vertices a and c are adjacent in G . Similarly, the edge bc exists if and only if b and c are adjacent in G . Thus, in G we have the three vertices a , b , and c with $ab \in E(G)$ and also $ac, bc \in E(G)$. Consequently, a , b , and c form a triangle in G , again a contradiction.
3. The three vertices a , b , and c lie in three pairwise distinct copies. Let $a \in V(G^i)$, $b \in V(G^j)$, $c \in V(G^k)$ with i, j, k all distinct. For each of the three edges ab , bc , and ca the same rule applies: an edge between a vertex of copy p and a vertex of copy q ($p \neq q$) exists if and only if the corresponding original vertices are adjacent in G . Therefore, from $ab \in E(R_k(G))$ we obtain that a and b are adjacent in G ; from $bc \in E(R_k(G))$ we obtain that b and c are adjacent in G ; and from $ac \in E(R_k(G))$ we obtain that a and c are adjacent in G . Hence the three vertices a, b, c are pairwise adjacent, i.e., they form a triangle in G , contradicting the hypothesis.

All possible configurations lead to a contradiction. Therefore, our initial assumption was false, and $R_k(G)$ cannot contain a triangle. $\square$

Theorem 2.6. Suppose G is a graph with diameter greater than two. Then, for every integer $k \geq 2$,

$$\text{diam}(R_k(G)) = \text{diam}(G).$$

Proof. We prove the equality by establishing two inequalities. First, we show that $\text{diam}(R_k(G)) \leq \text{diam}(G)$. Take any two vertices a_i and b_j in $R_k(G)$, corresponding to the vertices a and b in G . Since G is connected, there exists a shortest path of length $l = d_G(a, b) \leq \text{diam}(G)$ between a and b in G . Suppose this path is given by:

$$a = v_0, v_1, \dots, v_l = b.$$

For each vertex v_t in this path, choose an arbitrary copy v_{ti_t} (for $t = 0, \dots, l$) in such a way that $i_0 = i$ and $i_l = j$. By definition of the $R_k(G)$, whenever $v_t v_{t+1}$ is

an edge in G , the vertices v_{ti_t} and $v_{t+1i_{t+1}}$ are adjacent in $R_k(G)$ (regardless of whether $i_t = i_{t+1}$ or not) because edges between any copies of adjacent original vertices are always present. Hence, we obtain a path in $R_k(G)$ of length l from a_i to b_j . Consequently,

$$d_{R_k(G)}(a_i, b_j) \leq \text{diam}(G).$$

Since a_i and b_j were arbitrary, the diameter cannot exceed $\text{diam}(G)$; i.e.

$$\text{diam}(R_k(G)) \leq \text{diam}(G). \quad (1)$$

Now, let u and v in $V(G)$ be two vertices that realise the diameter of G :

$$d_G(u, v) = \text{diam}(G).$$

Consider the vertices u_1 and v_1 in $V(R_k(G))$ from the set of vertices corresponding to u and v . We claim that $d_{R_k(G)}(u_1, v_1) \geq d_G(u, v)$. Indeed, let P be any shortest path in $R_k(G)$ from u_1 to v_1 . Write its vertex sequence as:

$$u_1 = x_{0p_0}, x_{1p_1}, \dots, x_{mp_m} = v_1.$$

where $x_t \in V(G)$ and $p_t \in \{1, \dots, k\}$. By the definition of the $R_k(G)$, for every consecutive pair $(x_{tp_t}, x_{t+1p_{t+1}})$, we have either $p_t = p_{t+1}$ and $x_tx_{t+1} \in E(G)$, or $p_t \neq p_{t+1}$ and $x_tx_{t+1} \in E(G)$ (the edge exists exactly when the original vertices are adjacent). Thus, the sequence $x_0, x_1, \dots, x_m$ is a walk in G from u to v . Any walk from u to v has length at least $d_G(u, v)$, because the shortest path distance is the minimum length over all walks. Therefore $d_G(u, v) \leq m$. Since P was an arbitrary shortest path, we obtain:

$$d_{R_k(G)}(u_1, v_1) \geq d_G(u, v) = \text{diam}(G).$$

Hence, the diameter of $R_k(G)$ is at least $\text{diam}(G)$. Therefore,

$$\text{diam}(R_k(G)) \geq \text{diam}(G). \quad (2)$$

The result follows from inequalities 1 and 2. For completeness, when $k = 1$, we have $R_1(G) = G$. So the statement trivially holds. $\square$

Corollary 2.7. *The structure of $R_k(G)$ preserves the graph distances and does not alter the graph metrics.*

2.2. Topological indices of $R_k(G)$

Now, we compute several degree-based topological indices for $R_k(G)$. For indices whose defining bivariate function is homogeneous, we apply Theorem 2.8 to obtain a closed-form expression in terms of the original index of G ; this includes the Sombor, Randić, reciprocal Randić, Diminished Sombor, harmonic, second

Zagreb, and Geometric-arithmetic indices. For indices that do not satisfy the homogeneity condition, the atom-bond connectivity (*ABC*) and augmented Zagreb (*AZI*) indices, the general formula does not apply, and we perform a direct calculation using the explicit degree information. Let $f(x, y)$ be a function. If $\alpha > 0$ is a constant and the function satisfies $f(\alpha x, \alpha y) = \alpha^t f(x, y)$, then f is called homogeneous of degree t .

Theorem 2.8. *Let $T(G)$ be a degree-based topological index of a graph G , defined as follows:*

$$T(G) = \sum_{uv \in E(G)} f(d_G(u), d_G(v)).$$

If f be a homogeneous function of degree t , then $T(R_k(G)) = k^{t+2}T(G)$.

Proof. By [Proposition 2.1](#), we know that for every vertex $v_j^i \in V(R_k(G))$, $d_{R_k(G)}(v_j^i) = kd_G(v_j)$, for $i = 1, 2, \dots, k$, and $j = 1, 2, \dots, n$. Conversely, for every edge uv in G , there exist exactly k^2 corresponding edges in $R_k(G)$. Therefore, for each edge uv , the total contribution of its corresponding edges in $R_k(G)$ is given by:

$$\sum_{i=1}^k \sum_{j=1}^k f(d_{R_k(G)}(u_i), d_{R_k(G)}(v_j)) = k^2 f(kd_G(u), kd_G(v)).$$

Because f is homogeneous of degree t , we have

$$f(kd_G(u), kd_G(v)) = k^t f(d_G(u), d_G(v)).$$

Therefore,

$$k^2 (kd_G(u), kd_G(v)) = k^{t+2} f(d_G(u), d_G(v)).$$

By summing over all edges,

$$T(R_k(G)) = \sum_{uv \in E(G)} k^{t+2} f(d_G(u), d_G(v)) = k^{t+2} T(G).$$

□

Corollary 2.9. *For each $l \geq 1$ and $k \geq 2$,*

$$T(R_k^l(G)) = k^{l(t+2)} T(G).$$

Proof. As defined in [\[3\]](#), we know that $R_k^l(G) = R_k(R_k^{l-1}(G))$, hence the claim follows. □

In the next three corollaries, we compute the $SO(G)$, $R(G)$, and $RR(G)$ for the graphs obtained from this construction, and we will correct the corresponding results reported in [\[3\]](#). In all cases, if we prove for $l = 1$, then the results follow by induction on l .

Corollary 2.10. For each $l \geq 1$, and $k \geq 2$,

$$SO(R_k^l(G)) = k^{3l}SO(G).$$

Proof. We assume that $l = 1$. Consider $f(x, y) = \sqrt{x^2 + y^2}$. For every $\alpha > 0$, we obtain:

$$f(\alpha x, \alpha y) = \sqrt{(\alpha x)^2 + (\alpha y)^2} = \alpha \sqrt{x^2 + y^2} = \alpha^1 f(x, y).$$

Therefore, f is a homogeneous function of degree 1. According to [Theorem 2.8](#),

$$SO(R_k(G)) = k^{1+2}SO(G) = k^3SO(G).$$

□

We know that k^3 is an integer; therefore, if $SO(G)$ is an integer, then $SO(R_k(G))$ is also an integer.

Corollary 2.11. For each $l \geq 1$ and $k \geq 2$,

$$R(R_k^l(G)) = k^l R(G).$$

Proof. We assume that $l = 1$. Consider $f(x, y) = \frac{1}{\sqrt{xy}}$. For every $\alpha > 0$, we obtain:

$$f(\alpha x, \alpha y) = \frac{1}{\sqrt{\alpha^2 xy}} = \alpha^{-1} f(x, y).$$

Therefore, f is a homogeneous function of degree -1 . According to [Theorem 2.8](#),

$$R(R_k(G)) = k R(G).$$

□

Corollary 2.12. For each $l \geq 1$ and $k \geq 2$,

$$RR(R_k^l(G)) = k^{3l}RR(G).$$

Proof. We assume that $l = 1$. If we consider $f(x, y) = \sqrt{xy}$, then for every $\alpha > 0$, we have:

$$f(\alpha x, \alpha y) = \sqrt{\alpha^2 xy} = \alpha \sqrt{xy} = \alpha^1 f(x, y).$$

Hence, f is a homogeneous function of degree 1, and according to [Theorem 2.8](#),

$$RR(R_k(G)) = k^3 RR(G).$$

□

Corollary 2.13. For each $k \geq 2$, $DSO(R_k(G)) = k^2 DSO(G)$.

Proof. If we consider $f(x, y) = \frac{\sqrt{x^2+y^2}}{x+y}$, then for every $\alpha > 0$, we have,

$$f(\alpha x, \alpha y) = \frac{\sqrt{\alpha^2 x^2 + \alpha^2 y^2}}{\alpha x + \alpha y}.$$

Hence, f is a homogeneous function of degree 0, and according to [Theorem 2.8](#),

$$DSO(R_k(G)) = k^2 DSO(G).$$

□

Corollary 2.14. *For each $k \geq 2$, $H(R_k(G)) = kH(G)$.*

Proof. If we consider $f(x, y) = \frac{2}{x+y}$, then for every $\alpha > 0$, we have:

$$f(\alpha x, \alpha y) = \frac{2}{\alpha x + \alpha y} = \alpha^{-1} f(x, y).$$

Hence, f is a homogeneous function of degree -1 , and according to [Theorem 2.8](#),

$$H(R_k(G)) = kH(G).$$

□

Corollary 2.15. *For each $k \geq 2$, $M_2(R_k(G)) = k^4 M_2(G)$.*

Proof. If we consider $f(x, y) = xy$, then for every $\alpha > 0$, we have:

$$f(\alpha x, \alpha y) = \alpha^2 xy = \alpha^2 f(x, y).$$

Hence, f is a homogeneous function of degree 2, and according to [Theorem 2.8](#),

$$M_2(R_k(G)) = k^4 M_2(G).$$

□

Corollary 2.16. *For each $k \geq 2$, $GA(R_k(G)) = k^2 GA(G)$.*

Proof. If we consider $f(x, y) = \frac{2\sqrt{xy}}{x+y}$, then for every $\alpha > 0$, we have:

$$f(\alpha x, \alpha y) = \frac{2\sqrt{\alpha x \alpha y}}{\alpha x + \alpha y} = \frac{2\sqrt{xy}}{x + y} = f(x, y).$$

Hence, f is a homogeneous function of degree 0, and according to [Theorem 2.8](#),

$$GA(R_k(G)) = k^2 GA(G).$$

□

In the following theorems, we determine the values of the atom-bond connectivity and augmented Zagreb indices of the graph $R_k(G)$ in terms of degrees of the vertices of G .

Theorem 2.17. *For each $k \geq 2$,*

$$ABC(R_k(G)) = \sum_{uv \in E} k \sqrt{\frac{k(d_G(u) + d_G(v)) - 2}{d_G(u)d_G(v)}}.$$

Proof. It was established that for every edge $uv \in E$, there exist exactly k^2 corresponding edges in $R_k(G)$, and the degree of each vertex is k times its degree in the original graph G . Also, according to Proposition 2.1 for every $v \in V(G)$, $d_{R_k(G)}(v) = kd_G(v)$. Therefore, the contribution of each corresponding edge to $ABC(R_k(G))$ is given by:

$$\begin{aligned} \sqrt{\frac{(kd_G(u) + kd_G(v)) - 2}{(kd_G(u))(kd_G(v))}} &= \sqrt{\frac{(kd_G(u) + kd_G(v)) - 2}{k^2 d_G(u) d_G(v)}} \\ &= \frac{1}{k} \sqrt{\frac{(kd_G(u) + kd_G(v)) - 2}{d_G(u) d_G(v)}}. \end{aligned}$$

By summing over all k^2 corresponding edges, we obtain:

$$k^2 \frac{1}{k} \sqrt{\frac{kd_G(u) + kd_G(v) - 2}{d_G(u)d_G(v)}} = k \sqrt{\frac{kd_G(u) + kd_G(v) - 2}{d_G(u)d_G(v)}}.$$

Finally, summing over all edges yields

$$ABC(R_k(G)) = \sum_{uv \in E} k \sqrt{\frac{k(d_G(u) + d_G(v)) - 2}{d_G(u)d_G(v)}}.$$

□

Theorem 2.18. *Let G be a graph such that for all $u \in V(G)$, $d_G(u) > 1$. Then, for each $k \geq 2$,*

$$AZI(R_k(G)) = \sum_{uv \in E} \frac{k^8 (d_G(u)d_G(v))^3}{(kd_G(u) + kd_G(v) - 2)^3}.$$

Proof. As mentioned in the proof of Theorem 2.17, the contribution of each corresponding edge to $AZI(R_k(G))$ is given by:

$$\left(\frac{(kd_G(u))(kd_G(v))}{kd_G(u) + kd_G(v) - 2} \right)^3 = \frac{k^6 (d_G(u)d_G(v))^3}{(kd_G(u) + kd_G(v) - 2)^3}.$$

Since there are exactly k^2 edges corresponding to the original edges, the total contribution of these edges to $AZI(R_k(G))$ is equal to

$$k^2 \frac{k^6(d_G(u)d_G(v))^3}{(k(d_G(u) + d_G(v)) - 2)^3} = \frac{k^8(d_G(u)d_G(v))^3}{(kd_G(u) + d_G(v) - 2)^3}.$$

This completes the proof. $\square$

3. Numerical examples

In this section, we compute the indices of some simple graphs and their R_k -constructions. Then, we verified the correctness of the derived formula by comparing it with the directly computed values.

Example 3.1. Let $G=K_{3,4}$. Then, $|V(G)| = 7$, $|E(G)| = 12$, and

$$\begin{aligned} SO(G) &= 12\sqrt{3^2 + 4^2} = 60, & R(G) &= 12(3 \times 4)^{\frac{-1}{2}} = 2\sqrt{3}, \\ RR(G) &= 24\sqrt{3}, & DSO(G) &= 12\frac{\sqrt{25}}{7} = \frac{60}{7}, \\ H(G) &= 12 \times \frac{2}{7} = \frac{24}{7}, & M_2(G) &= 12(3 \times 4) = 144, \\ GA(G) &= 12\frac{4\sqrt{3}}{7} = \frac{48\sqrt{3}}{7}. \end{aligned}$$

If we let $k = 3$, then $|V(R_3(G))| = 21$, $|E(R_3(G))| = 108$, and

$$\begin{aligned} SO(R_3(G)) &= 108 \times 15 = 1620, & R(R_3(G)) &= 108(9 \times 12)^{\frac{-1}{2}} = 6\sqrt{3}, \\ RR(R_3(G)) &= 108 \times 6\sqrt{3} = 648\sqrt{3}, & DSO(R_3(G)) &= 108 \times \frac{5}{7} = \frac{540}{7}, \\ H(R_3(G)) &= 108 \times \frac{2}{21} = \frac{72}{7}, & M_2(R_3(G)) &= 108(9 \times 12) = 11664, \\ GA(R_3(G)) &= 108 \times \frac{4\sqrt{3}}{7} = \frac{432\sqrt{3}}{7}. \end{aligned}$$

Table 1 compares the topological indices of $R_3(G)$ obtained by direct calculation with those obtained using the formula.

As shown in **Table 1**, these values confirm our results.

Example 3.2. Let $G=C_6$. Then, $|V(G)| = 6$, $|E(G)| = 6$, and

$$\begin{aligned} SO(G) &= 6\sqrt{2^2 + 2^2} = 6\sqrt{8} = 12\sqrt{2}, & R(G) &= 6(2 \times 2)^{\frac{-1}{2}} = 3, \\ RR(G) &= 6\sqrt{4} = 12, & DSO(G) &= 6 \times \frac{\sqrt{8}}{4} = 3\sqrt{2}, \end{aligned}$$

Table 1: Comparison of computations.

Index	Direct calculation	Formula
SO	1620	$k^3 SO(G) = 1620$
R	$6\sqrt{3}$	$kR(G) = 6\sqrt{3}$
RR	$648\sqrt{3}$	$k^3 RR(G) = 648\sqrt{3}$
DSO	$\frac{540}{7}$	$k^2 DSO(G) = \frac{540}{7}$
H	$\frac{72}{7}$	$kH(G) = \frac{72}{7}$
M_2	11664	$k^4 M_2(G) = 11664$
GA	$\frac{432\sqrt{3}}{7}$	$k^2 GA(G) = \frac{432\sqrt{3}}{7}$

$$H(G) = 6 \times \frac{2}{4} = 3,$$

$$M_2(G) = 6(2 \times 2) = 24,$$

$$GA(G) = 6 \frac{2\sqrt{4}}{4} = 6.$$

If we let $k = 3$, then $|V(R_3(G))| = 18$, $|E(R_3(G))| = 54$, and

$$SO(R_3(G)) = 54\sqrt{72} = 324\sqrt{2},$$

$$R(R_3(G)) = 54(6 \times 6)^{-\frac{1}{2}} = 9,$$

$$RR(R_3(G)) = 54 \times 6 = 324,$$

$$DSO(R_3(G)) = 54 \times \frac{\sqrt{72}}{12} = 27\sqrt{2},$$

$$H(R_3(G)) = 54 \times \frac{2}{12} = 9,$$

$$M_2(R_3(G)) = 54(6 \times 6) = 1944,$$

$$GA(R_3(G)) = 54 \times \frac{12}{12} = 54.$$

[Table 2](#) compares the topological indices of $R_3(G)$ obtained by direct calculation with those obtained using the formula.

Table 2: Comparison of computations.

Index	Direct calculation	Formula
SO	$324\sqrt{2}$	$k^3 SO(G) = 324\sqrt{2}$
R	9	$kR(G) = 9$
RR	324	$k^3 RR(G) = 324$
DSO	$27\sqrt{2}$	$k^2 DSO(G) = 27\sqrt{2}$
H	9	$kH(G) = 9$
M_2	1944	$k^4 M_2(G) = 1944$
GA	54	$k^2 GA(G) = 54$

As shown in [Table 2](#), these values confirm our results.

Example 3.3. Let $G=S_4$. Then, $|V(G)| = 5$, $|E(G)| = 4$, and

$$SO(G) = 4\sqrt{4^2 + 1^2} = 4\sqrt{17},$$

$$R(G) = 4 \times (4 \times 1)^{-\frac{1}{2}} = 2,$$

$$\begin{aligned}
RR(G) &= 4\sqrt{4 \times 1} = 8, & DSO(G) &= 4 \times \frac{\sqrt{17}}{5}, \\
H(G) &= 4 \times \frac{2}{4+1} = \frac{8}{5}, & M_2(G) &= 4 \times (4 \times 1) = 16, \\
GA(G) &= 4 \times \frac{2\sqrt{4 \times 1}}{4+1} = \frac{16}{5}.
\end{aligned}$$

If we let $k = 3$, then $|V(R_3(G))| = 15$, $|E(R_3(G))| = 36$, and

$$\begin{aligned}
SO(R_3(G)) &= 36\sqrt{12^2 + 3^2} = 108\sqrt{17}, & R(R_3(G)) &= 36(12 \times 3)^{\frac{-1}{2}} = 6, \\
RR(R_3(G)) &= 36 \times 6 = 216, & DSO(R_3(G)) &= 36 \times \frac{\sqrt{17}}{5}, \\
H(R_3(G)) &= 36 \times \frac{2}{15} = \frac{24}{5}, & M_2(R_3(G)) &= 36 \times (12 \times 3) = 1296, \\
GA(R_3(G)) &= 36 \times \frac{12}{15} = \frac{144}{5}.
\end{aligned}$$

Table 3 compares the topological indices of $R_3(G)$ obtained by direct calculation with those obtained using the formula.

Table 3: Comparison of computations.

Index	Direct calculation	Formula
SO	$108\sqrt{17}$	$k^3 SO(G) = 108\sqrt{17}$
R	6	$kR(G) = 6$
RR	216	$k^3 RR(G) = 216$
DSO	$\frac{36\sqrt{17}}{5}$	$k^2 DSO(G) = \frac{36\sqrt{17}}{5}$
H	$\frac{24}{5}$	$kH(G) = \frac{24}{5}$
M₂	1296	$k^4 M_2(G) = 1296$
GA	$\frac{144}{5}$	$k^2 GA(G) = \frac{144}{5}$

As shown in **Table 3**, these values confirm our results.

4. Conclusions

In this paper, the structural properties of the R_k -Construction and its effect on degree-based topological indices were investigated. By exploiting the homogeneity property of the defining functions, a general relation was derived in terms of a power of the parameter k for a class of indices, which significantly simplifies computations. For the remaining indices that do not satisfy this property, direct calculations were performed using the vertex degrees in the original graph. It was also shown that this transformation preserves the diameter and connectivity of the original graph and maps triangle-free graphs to triangle-free graphs. The

validity of the obtained relations was confirmed through numerous numerical examples. The results of this study, in addition to correcting and generalizing some previous findings, provide an effective tool for generating a family of graphs with predictable topological indices, which can be useful in the design of new molecules with desired chemical properties.

Conflicts of Interest. The author declare that he has no conflicts of interest regarding the publication of this article.

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