

A Nonlinear Fractional Quantum Coupled System: Existence and Mittag-Leffler Ulam-Hyers Stability

Mohamed Bezziou¹, Mohamed Houas² and
Mohammad Esmael Samei^{3,*}

Abstract

Investigating the existence and stability of solutions in nonlinear and discrete cases of fractional differential equations is always useful for complex applied problems. In the current work, we discuss the existence of a solution to a coupled nonlinear system of fractional quantum differential equations involving quantum integrals and derivatives. We prove the existence and uniqueness of solutions by using some classical fixed point theorems in nonlinear analysis, such as Schaefer's fixed point. Also, we define the Mittag-Leffler-Ulam stability and study the stability of solutions for the proposed system. Of course, some examples are demonstrated to confirm our main results for the system. Moreover, a conclusion is given in the last section.

Keywords: Riemann-Liouville fractional q -derivative; Existence and uniqueness solutions; Boundary conditions; Fixed point theorem.

2020 Mathematics Subject Classification: 39A13, 34B15, 30C45.

How to cite this article

M. Bezziou, M. Houas and M. E. Samei, A nonlinear fractional quantum coupled system: Existence and Mittag-Leffler Ulam-Hyers stability, *Math. Interdisc. Res.* **11** (4) (2026) 371-405.

1. Introduction

Differential equations ($\mathbb{D}\mathbb{E}s$) involving fractional quantum derivative (q -derivative) of different orders have diverse applications and have evolved into miscellaneous

*Corresponding author (E-mail: mesamei@basu.ac.ir, mesamei@gmail.com)

Academic Editor: Abbas Saadatmandi

Received 27 January 2026, Accepted 29 June 2026

DOI: 10.22052/mir.2026.258296.1567



subjects [1–4]. Moreover, the $\mathbb{D}\mathbb{E}$ s include fractional quantum calculus ($\mathbf{q}\text{-}\mathbb{F}\mathbb{D}\mathbb{E}$) have been discussed by many scientific scholars [5–7]. Recently, several researchers have examined the uniqueness, existence, Ulam stability, and Mittag-Leffler-Ulam (MLU) stability of solutions for some $\mathbb{D}\mathbb{E}$ s involving fractional $\mathbf{q}$ -derivative [8–12]. We aim to examine the existence and MLU stability of solutions for a fractional quantum coupled system of fractional order $\alpha \in (5, 6]$. In recent years, many scientific researchers have studied the classical types of the boundary value problem ($\mathbb{BVP}$), for more details, see [13–17]. Kaufmann in [18], has established uniqueness and existence results for the fourth-order $\mathbb{D}\mathbb{E}$

$$\mathfrak{s}^{(4)}(z) = \mathfrak{v}\left(z, \mathfrak{s}(z), \mathfrak{s}^{[2]}(z), \dots, \mathfrak{s}^{[m]}(z)\right),$$

with $\mathfrak{s}(a) = \mathfrak{s}(-a) = \mathfrak{s}^{(2)}(a) = \mathfrak{s}^{(2)}(-a) = 0$. Also, Grossinho et al. have discussed the existence results of infinitely many solutions for the following fourth-order nonlinear $\mathbb{BVP}$,

$$\mathfrak{s}^{(4)}(z) + A\mathfrak{s}''(z) + B\mathfrak{s}(z) = \lambda \mathfrak{v}(z, \mathfrak{s}(z)), \quad z \in \Lambda,$$

under conditions $\mathfrak{s}(0) = \mathfrak{s}(1) = 0$, $\mathfrak{s}''(0) = \mathfrak{s}''(1) = 0$ [19]. We also note that the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by a $\mathbb{BVP}$ of sixth-order [13],

$$\mathfrak{s}^{(6)} + 2\mathfrak{s}^{(4)} + \mathfrak{s}'' = \mathfrak{v}(z, \mathfrak{s}),$$

for $z \in \Omega := (0, 1)$, with $\mathfrak{s} = \mathfrak{s}'' = \mathfrak{s}^{(4)} = 0$ on $\partial\Omega$. Khan et al in [20] explored the necessary conditions required for the solutions of an integro-differential system of $n\text{-}\mathbb{F}\mathbb{D}\mathbb{E}$ s in the modified Atangana-Baleanu-Caputo case of derivative, with initial conditions, which is a linearly perturbed system, based on the some classical fixed point (FP) theorems. For more related works, see [21–27]. Moumen et al. studied the existence and uniqueness of solutions for a coupled system of Langevin-type $\mathbb{F}\mathbb{D}\mathbb{E}$ s featuring generalized Ψ -Caputo derivatives in arbitrary Banach spaces and extended the framework to infinite-dimensional settings, offering a more versatile analytical approach [28].

In this work, we discuss the existence and uniqueness solutions of the coupled system of $\mathbf{q}\text{-}\mathbb{F}\mathbb{D}\mathbb{E}$ s involving the Riemann-Liouville (RL) type fractional $\mathbf{q}$ -derivative ${}^{\text{RL}}\mathcal{D}_{\mathbf{q}}^{\alpha_i}$ of order $5 < \alpha_i \leq 6$, is formulated by,

$$\begin{cases} {}^{\text{RL}}\mathcal{D}_{\mathbf{q}}^{\alpha_1} \mathfrak{s}_1(z) = \lambda_1 \mathfrak{v}_1(z, {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_1(z), {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_2(z)) \\ \quad + \mu_1 \mathfrak{h}_1(z, {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_1(z), {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_1 \mathfrak{g}_1(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)), \\ {}^{\text{RL}}\mathcal{D}_{\mathbf{q}}^{\alpha_2} \mathfrak{s}_2(z) = \lambda_2 \mathfrak{v}_2(z, {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_1(z), {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_2(z)) \\ \quad + \mu_2 \mathfrak{h}_2(z, {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_1(z), {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_2 \mathfrak{g}_2(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)), \end{cases} \quad (1)$$

for $z \in \bar{\Omega} := [0, 1]$, and $\lambda_i, \mu_i, \delta_i \in \mathbb{R}$, under conditions

$$\begin{cases} ({}_q\mathcal{D}^m \mathfrak{s}_1)(0) = ({}_q\mathcal{D}^m \mathfrak{s}_2)(0) = 0, & m = \overline{0, 4}, \\ ({}_q\mathcal{D}^4 \mathfrak{s}_i)(1) = \sigma_i [\alpha_i]_q \int_0^1 \mathfrak{s}_i(\varpi) d_q \varpi, & \sigma_i \in \mathbb{R}, \end{cases} \quad (2)$$

where $\mathfrak{v}_i, \mathfrak{h}_i, \mathfrak{g}_i \in C(\bar{\Omega} \times \mathbb{R}^2)$ and ${}_q\mathcal{D}^m \mathfrak{s}_i(z)$ is the general q -derivatives of the function $\mathfrak{s}_i \in C(\bar{\Omega})$ with respect to z , $i = 1, 2$. The current work in the field of fractional calculus can be used to obtain the solution of the related system in [34–37].

Our performance is in this direction. Section 2 contains an auxiliary result which we need in the sequel to analyze the mentioned q -FDE. The main results for the q -BVP (1) are discussed in Section 3. We will introduce the existence and uniqueness results using famous FP theorems. In the final parts of the paper, in Sections 4 and 5, we confirm the achievements of the article by presenting examples and a conclusion, respectively.

2. Preliminaries on quantum calculus

We introduce some fractional quantum calculus by the following notations definitions, which can be found in [29–32].

Let $q \in \Omega$ and define $[x]_q = \frac{1-q^x}{1-q}$, with a real element x . The q -analogue of the power $(a-b)^n$ with $n \in \mathbb{N}$ for $z, \dot{z} \in \mathbb{R}$ is,

$$(z - \dot{z})^{(0)}_q = 1, \quad (z - \dot{z})^{(n)}_q = \prod_{k=0}^{n-1} (z - \dot{z}q^k), \quad n \in \{0\} \cup \mathbb{N}.$$

In general,

$$(z - \dot{z})^{(\alpha)}_q = z^\alpha \prod_{n=0}^{\infty} \frac{z - \dot{z}q^n}{z - \dot{z}q^{\alpha+n}}, \quad \alpha \in \mathbb{R}.$$

The q -gamma function is formulated by:

$$\Gamma_q(z) := \frac{(1-q)_q^{(z-1)}}{(1-q)^{z-1}}, \quad z \in \mathbb{R} \setminus \{0, -1, -2, \dots\},$$

such that $\Gamma_q(\alpha + 1) := [\alpha]_q \Gamma_q(\alpha)$ is fulfilled. The q -derivative of a function $\mathfrak{s}$ is defined by:

$${}_q\mathcal{D}\mathfrak{s}(z) = \frac{\mathfrak{s}(z) - \mathfrak{s}(qz)}{(1-q)z},$$

and ${}_q\mathcal{D}\mathfrak{s}(0) = \lim_{z \rightarrow 0} {}_q\mathcal{D}\mathfrak{s}(z)$. The following formulas will be used in the sequel:

$$\begin{aligned} {}_q\mathcal{D}z^{(\alpha)} &= [\alpha]_q z^{(\alpha-1)}, \\ {}_q\mathcal{D}^\sigma q^{(\alpha)} &= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-\sigma+1)} z^{(\alpha-\sigma)}, \quad \sigma > 0, \\ \left({}_q\mathcal{D} \int_0^z \mathfrak{v}(z, \varpi) d_q \varpi \right)(z) &= \int_0^z {}_q\mathcal{D}\mathfrak{v}(z, \varpi) d_q \varpi + \mathfrak{v}(qz, z). \end{aligned} \quad (3)$$

Definition 2.1. Let $q \in \Omega$. Then, the RL fractional q -integral of order $\alpha > 0$ and q -derivative of order $n - 1 < \alpha \leq n$ for $\mathfrak{s} \in C(\bar{\Omega})$ are described for $z \in \bar{\Omega}$ by,

$${}^{\text{RL}}_q \mathcal{J}^\alpha \mathfrak{s}(z) = \begin{cases} \int_0^z \frac{(z-q\omega)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mathfrak{s}(\omega) d_q \omega, & \alpha > 0, \\ \mathfrak{s}(z), & \alpha = 0, \end{cases}$$

and

$${}^{\text{RL}}_q \mathcal{D}^\alpha \mathfrak{s}(z) = \left({}_q \mathcal{D}_z^{[\alpha]} {}^{\text{RL}}_q \mathcal{J}^{[\alpha]-\alpha} \mathfrak{s} \right)(z) \quad z \in \bar{\Omega}.$$

We need the essential lemmas mentioned below [6, 29, 33].

Lemma 2.2. Let $\alpha > 0$. Then, the following equality holds:

$$\left({}^{\text{RL}}_q \mathcal{J}^\alpha {}^{\text{RL}}_q \mathcal{D}^\alpha \mathfrak{s} \right)(z) = \mathfrak{s}(z) - \sum_{k=0}^{[\alpha]-1} \frac{z^{(\alpha-k)}}{\Gamma_q(\alpha+1)} {}^{\text{RL}}_q \mathcal{D}^\alpha \mathfrak{s}(0).$$

In the case $\alpha \in \Omega$, we have $\left({}^{\text{RL}}_q \mathcal{J}^\alpha {}^{\text{RL}}_q \mathcal{D}^\alpha \mathfrak{s} \right)(z) = \mathfrak{s}(z) - \mathfrak{s}(0)$.

Lemma 2.3. For $\beta > 0$, using q -integration by parts, we have

$${}^{\text{RL}}_q \mathcal{J}^\beta 1(z) = \frac{z^{(\beta)}}{\Gamma_q(\beta+1)}, \quad \text{or} \quad \int_0^z (z-q\omega)^{\beta-1} d_q \omega = \frac{z^{(\beta)}}{[\beta]_q}.$$

Lemma 2.4. Let $\alpha, \beta \geq 0$ and the function $\mathfrak{s}$ defined in $\bar{\Omega}$, the following formulas hold:

$$\left({}^{\text{RL}}_q \mathcal{J}^\alpha {}^{\text{RL}}_q \mathcal{J}^\beta \mathfrak{s} \right)(z) = {}^{\text{RL}}_q \mathcal{J}^{\alpha+\beta} \mathfrak{s}(z), \quad \left({}^{\text{RL}}_q \mathcal{D}^\alpha {}^{\text{RL}}_q \mathcal{J}^\alpha \mathfrak{s} \right)(z) = \mathfrak{s}(z), \quad z \in \bar{\Omega}.$$

3. Main results

First, we prove the key [Lemma 3.1](#).

Lemma 3.1. For any $\varphi_i \in C(\bar{\Omega})$ ($i = 1, 2$), the unique solution of the linear q -FDE:

$$\begin{cases} {}^{\text{RL}}_q \mathcal{D}^{\alpha_i} \mathfrak{s}_i(z) = \varphi_i(z), & z \in \bar{\Omega}, 5 < \alpha_i \leq 6, \\ {}_q \mathcal{D}^m \mathfrak{s}_i(0) = 0, & m = \overline{0, 4}, \\ {}_q \mathcal{D}^4 \mathfrak{s}_i(1) = \sigma_i [\alpha_i]_q \int_0^1 \mathfrak{s}_i(\omega) d_q \omega, & i = 1, 2, \end{cases} \quad (4)$$

is formulated by

$$\mathfrak{s}_i(z) = \int_0^z \frac{(z-q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \varphi_i(q) d_q \omega$$

$$\begin{aligned}
& + \frac{\sigma_i [\alpha_i]_q z^{(\alpha_i-1)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1 - q\omega)^{(\alpha_i)} \varphi_i(\omega) d_q \omega \\
& - \frac{z^{(\alpha_i-1)} (\mathfrak{I}_i + \sigma_i [\alpha_i]_q)}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1 - q\omega)^{(\alpha_i-5)} \varphi_i(\omega) d_q \omega,
\end{aligned} \quad (5)$$

such that $\mathfrak{I}_i := (\Pi_{j=0}^4 [\alpha_i - j]_q) - \sigma_i [\alpha_i]_q \neq 0$, $i = 1, 2$.

Proof. From Definition 2.1 and Lemma 2.4, the general solution of the q -FDE (4) of fractional order $5 < \alpha_i \leq 6$, $i = 1, 2$, is given by

$$\begin{aligned}
\mathfrak{s}_i(z) = & \int_0^z \frac{(z - q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \varphi_i(q) d_q \omega + C_1 z^{(\alpha_i-1)} + C_2 z^{(\alpha_i-2)} \\
& + C_3 z^{(\alpha_i-3)} + C_4 z^{(\alpha_i-4)} + C_5 z^{(\alpha_i-5)} + C_6 z^{(\alpha_i-6)},
\end{aligned} \quad (6)$$

where $C_k \in \mathbb{R}$, $k = \overline{1, 6}$. Since $\mathfrak{s}_i(0) = 0$, we obtain $C_6 = 0$. Taking the derivative of both sides of Equation (6), with the help of Equation (3), we obtain:

$$\begin{aligned}
{}_q \mathcal{D} \mathfrak{s}_i(z) = & \frac{[\alpha_i-1]_q}{\Gamma_q(\alpha_i)} \int_0^z (z - q\omega)^{(\alpha_i-2)} \varphi_i(\omega) d_q \omega + C_1 [\alpha_i - 1]_q z^{(\alpha_i-2)} \\
& + C_2 [\alpha_i - 2]_q z^{(\alpha_i-3)} + C_3 [\alpha_i - 3]_q z^{(\alpha_i-4)} + C_4 [\alpha_i - 4]_q z^{(\alpha_i-5)} \\
& + C_5 [\alpha_i - 5]_q z^{(\alpha_i-6)}, \quad i = 1, 2,
\end{aligned}$$

Using the boundary condition of ${}_q \mathcal{D} \mathfrak{s}_i(0) = 0$, $i = 1, 2$, we get $C_5 = 0$. Further,

$$\begin{aligned}
{}_q \mathcal{D}^2 \mathfrak{s}_i(z) = & \frac{[\alpha_i-1]_q [\alpha_i-2]_q}{\Gamma_q(\alpha_i)} \int_0^z (z - q\omega)^{(\alpha_i-3)} \varphi_i(\omega) d_q \omega \\
& + C_1 [\alpha_i - 1]_q [\alpha_i - 2]_q z^{(\alpha_i-3)} + C_2 [\alpha_i - 2]_q [\alpha_i - 3]_q z^{(\alpha_i-4)} \\
& + C_3 [\alpha_i - 3]_q [\alpha_i - 4]_q z^{(\alpha_i-5)} + C_4 [\alpha_i - 4]_q [\alpha_i - 5]_q z^{(\alpha_i-6)},
\end{aligned}$$

for $i = 1, 2$. Form ${}_q \mathcal{D}^2 \mathfrak{s}_i(0) = 0$, $i = 1, 2$, implies that $C_4 = 0$. It follows as before,

$${}_q \mathcal{D}^3 \mathfrak{s}_i(0) = {}_q \mathcal{D}^4 \mathfrak{s}_i(0) = 0,$$

which implies $C_2 = C_3 = 0$. By substituting these values into the Equation (6), we find

$$\mathfrak{s}_i(z) = \int_0^z \frac{(z - q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \varphi_i(\omega) d_q \omega + C_1 z^{(\alpha_i-1)}, \quad i = 1, 2, \quad (7)$$

and

$${}_q \mathcal{D}^4 \mathfrak{s}_i(1) = \sigma_i [\alpha_i]_q \int_0^1 \mathfrak{s}(z) dz,$$

if and only if

$$\frac{[\alpha_i-1]_q [\alpha_i-2]_q [\alpha_i-3]_q [\alpha_i-4]_q}{\Gamma_q(\alpha_i)} \int_0^1 (1 - qs)^{(\alpha_i-5)} \varphi_i(\omega) d_q \omega$$

$$\begin{aligned}
& + C_1 [\alpha_i - 1]_{\mathbf{q}} [\alpha_i - 2]_{\mathbf{q}} [\alpha_i - 3]_{\mathbf{q}} [\alpha_i - 4]_{\mathbf{q}q} \\
& = \sigma_i [\alpha_i]_{\mathbf{q}} \int_0^1 \mathfrak{s}_i(z) d_{\mathbf{q}} z, \quad i = 1, 2.
\end{aligned}$$

Finally,

$$C_1 = \frac{\sigma_i [\alpha_i]_{\mathbf{q}}}{\mathfrak{I}_i + \sigma_i [\alpha_i]_{\mathbf{q}}} \int_0^1 \mathfrak{s}_i(\mathfrak{s}) d_{\mathbf{q}} \mathfrak{w} - \int_0^1 \frac{(1 - \mathbf{q}\mathfrak{w})^{(\alpha_i - 5)}}{\Gamma_{\mathbf{q}}(\alpha_i)} \varphi_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w}.$$

By replacing C_k , $k = \overline{1, 6}$ with expressions in Equation (6), we get

$$\begin{aligned}
\mathfrak{s}_i(z) &= \int_0^z \frac{(z - \mathbf{q}\mathfrak{w})^{(\alpha_i - 1)}}{\Gamma_{\mathbf{q}}(\alpha_i)} \varphi_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w} + \frac{\sigma_i [\alpha_i]_{\mathbf{q}} z^{(\alpha_i - 1)}}{\mathfrak{I}_i + \sigma_i [\alpha_i]_{\mathbf{q}}} \int_0^1 \mathfrak{s}_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w} \\
&\quad - z^{(\alpha_i - 1)} \int_0^1 \frac{(1 - \mathbf{q}\mathfrak{s})^{(\alpha_i - 5)}}{\Gamma_{\mathbf{q}}(\alpha_i)} \varphi_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w}, \quad i = 1, 2.
\end{aligned} \tag{8}$$

Integrating (8) with respect to z from 0 to 1, we get

$$\begin{aligned}
\int_0^1 \mathfrak{s}_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w} &= \frac{\mathfrak{I}_i + \sigma_i [\alpha_i]_{\mathbf{q}}}{\mathfrak{I}_i \Gamma_{\mathbf{q}}(\alpha_i)} \left[\int_0^1 (1 - \mathbf{q}\mathfrak{w})^{(\alpha_i)} \varphi_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w} \right. \\
&\quad \left. - \int_0^1 (1 - \mathbf{q}\mathfrak{w})^{(\alpha_i - 5)} \varphi_i(\mathfrak{w}) d_{\mathbf{q}} \mathfrak{w} \right].
\end{aligned} \tag{9}$$

Now, by inserting (9) into (8), we deduce (5). $\square$

We define the set

$$\Upsilon = \left\{ \mathfrak{s}_i : \mathfrak{s}_i, {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_i, {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_i \in C(\Omega) \right\}, \quad i = 1, 2.$$

Based on nonlinear analysis tools, we study $\mathbf{q}\text{-}\mathbb{BV}\mathbb{P}$ (4), in the Banach product space $(\Upsilon^2, \|\cdot\|_{\Upsilon^2})$ endowed with the norms, $i = 1, 2$,

$$\| {}_{\mathbf{q}}\mathcal{D}^n \mathfrak{s}_i \|_{\infty} = \sup \left\{ | {}_{\mathbf{q}}\mathcal{D}^n \mathfrak{s}_i(z) | : z \in \bar{\Omega} \right\}, \quad n \in \{0, 2, 4\},$$

and $\|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Upsilon^2} = \|\mathfrak{s}_1\|_{\Upsilon} + \|\mathfrak{s}_2\|_{\Upsilon}$, where

$$\|\mathfrak{s}_i\|_{\Omega} = \|\mathfrak{s}_i\|_{\infty} + \| {}_{\mathbf{q}}\mathcal{D}^2 \mathfrak{s}_i \|_{\infty} + \| {}_{\mathbf{q}}\mathcal{D}^4 \mathfrak{s}_i \|_{\infty}.$$

Thanks to Lemma 3.1, we can consider an operator

$$\begin{cases} \mathcal{T} : \Upsilon^2 \rightarrow \Upsilon^2, \\ \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2)(z) = \begin{pmatrix} \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2)(z) \\ \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2)(z) \end{pmatrix}, \end{cases} \tag{10}$$

where, for any $0 \leq z \leq 1$ and $i = 1, 2$, we have

$$\begin{aligned} \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) = & \int_0^z \frac{(z-q\varpi)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \left[\lambda_i \mathfrak{v}_i(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\ & + \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \left. \right] d_q \varpi \\ & + \frac{\sigma_i [\alpha_i]_q z^{(\alpha_i-1)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1-q\varpi)^{(\alpha_i)} \left[\lambda_i \mathfrak{v}_i(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\ & + \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \left. \right] d_q \varpi \\ & - \frac{(\mathfrak{I}_i + \sigma_i [\alpha_i]_q) z^{(\alpha_i-1)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1-q\varpi)^{(\alpha_i-5)} \left[\lambda_i \mathfrak{v}_i(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\ & + \mu_i \mathfrak{h}_i(z, {}^{\text{RL}}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}^{\text{RL}}_q \mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \left. \right] d_q \varpi. \quad (11) \end{aligned}$$

Lemma 3.2. Assume that $\mathfrak{v}_i, \mathfrak{g}_i \in C(\bar{\Omega} \times \mathbb{R}^2)$. Then $(\mathfrak{s}_1, \mathfrak{s}_2) \in \Upsilon^2$ is a solution of the coupled system (1) if and only if $(\mathfrak{s}_1, \mathfrak{s}_2)$ is a solution of the equation

$$(\mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2)(z), \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2)(z)) = (\mathfrak{s}_1(z), \mathfrak{s}_2(z)).$$

Proof. The proof is immediate from Lemma 3.1. $\square$

Theorem 3.3. Assume that the following assumption is valid:

(H₁) Suppose the existence of nonnegative reals numbers ${}_{\mathfrak{v}}\zeta_i, {}_{\mathfrak{h}}\zeta_i, {}_{\mathfrak{g}}\zeta_i$, such that for every $z \in \bar{\Omega}$ and $\mathfrak{s}_j, \mathfrak{s}_j \in \mathbb{R}$, $i = 1, 2$, we have

$$\begin{aligned} |\mathfrak{v}_i(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathfrak{v}_i(z, \mathfrak{s}_1, \mathfrak{s}_2)| & \leq {}_{\mathfrak{v}}\zeta_i \sum_{j=1}^2 |\mathfrak{s}_j - \mathfrak{s}_j|, \\ |\mathfrak{h}_i(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathfrak{h}_i(z, \mathfrak{s}_1, \mathfrak{s}_2)| & \leq {}_{\mathfrak{h}}\zeta_i \sum_{j=1}^2 |\mathfrak{s}_j - \mathfrak{s}_j|, \\ |\mathfrak{g}_i(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathfrak{g}_i(z, \mathfrak{s}_1, \mathfrak{s}_2)| & \leq {}_{\mathfrak{g}}\zeta_i \sum_{j=1}^2 |\mathfrak{s}_j - \mathfrak{s}_j|. \end{aligned}$$

Then, q -BVP (1) has a unique solution over $\bar{\Omega}$ provided that

$$K_1 \Delta_1 + K_2 \Delta_2 < 1, \quad (12)$$

where for $i = 1, 2$, $\Delta_i = |\lambda_i| {}_{\mathfrak{v}}\zeta_i + |\mu_i| {}_{\mathfrak{h}}\zeta_i + |\delta_i| {}_{\mathfrak{g}}\zeta_i$, and $K_i = \tilde{M}_i + \tilde{L}_i + \tilde{N}_i$, with

$$\tilde{M}_i = \frac{1}{\Gamma_q(\alpha_i+1)} + \frac{|\sigma_i| [\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} + \frac{\prod_{j=0}^3 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)},$$

$$\begin{aligned}\tilde{L}_i &= \frac{[\alpha_i-1]_q}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i| \prod_{j=0}^2 [\alpha-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} + \frac{\prod_{j=0}^3 [\alpha_i-j]_q \prod_{j=1}^2 [\alpha_i-j]}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)}, \\ \tilde{N}_i &= \frac{\prod_{j=1}^3 [\alpha_i-j]_q}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i| \prod_{j=0}^4 [\alpha-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} + \frac{\prod_{j=0}^3 [\alpha_i-j]_q \prod_{j=1}^4 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)}.\end{aligned}\quad (13)$$

Proof. For finite real numbers $\mathfrak{v}_i^\circ, \mathfrak{h}_i^\circ, \mathfrak{g}_i^\circ > 0, i = 1, 2$, we fix

$$\mathfrak{v}_i^\circ = \sup |\mathfrak{v}_i(z, 0, 0)|, \quad \mathfrak{h}_i^\circ = \sup |\mathfrak{h}_i(z, 0, 0)|, \quad \mathfrak{g}_i^\circ = \sup |\mathfrak{g}_i(z, 0, 0)|,$$

for $0 \leq z \leq 1$ and define $\mathcal{B}_r = \{(\mathfrak{s}_1, \mathfrak{s}_2) \in \Upsilon^2 : \|(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \leq r\}$, in which

$$r \geq \left[1 - \sum_{i=1}^2 K_i \Delta_i\right]^{-1} \sum_{i=1}^2 K_i [\mathfrak{v}_i^\circ + \mathfrak{h}_i^\circ + \mathfrak{g}_i^\circ].$$

Then, we show that $\mathcal{T}_i \mathcal{B}_r \subset \mathcal{B}_r$, where the operator $\mathcal{T}_i$ is defined in (11). For any $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{B}_r, z \in \bar{\Omega}$, the assumption (H₂) implies

$$\begin{aligned}|\mathfrak{v}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z))| &\leq |\mathfrak{v}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) - \mathfrak{v}_i(z, 0, 0)| + |\mathfrak{f}_i(z, 0, 0)| \\ &\leq \mathfrak{v}_i \zeta_i (\|\mathfrak{s}_1\| + \|\mathfrak{s}_2\|) + \mathfrak{v}_i^\circ \leq \mathfrak{v}_i \zeta_i r + \mathfrak{v}_i^\circ, \quad i = 1, 2.\end{aligned}$$

Similarly, we can find that

$$\begin{aligned}|\mathfrak{h}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z))| &\leq \mathfrak{h}_i \zeta_i r + \mathfrak{h}_i^\circ, \\ |\mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z))| &\leq \mathfrak{g}_i \zeta_i r + \mathfrak{g}_i^\circ,\end{aligned}$$

for $i = 1, 2$. In consequence, we obtain for $i = 1, 2$,

$$\begin{aligned}\|\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)\|_\infty &= \sup_{z \in \bar{\Omega}} |\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z)| \\ &\leq \int_0^1 \frac{(1-q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \left| \lambda_i \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. + \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right| d_q \omega \\ &\quad + \frac{|\sigma_i| [\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \int_0^1 (1-q\omega)^{(\alpha_i)} \left| \lambda_i \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. + \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(z)) \right| d_q \omega \\ &\quad + \frac{\mathfrak{I}_i + \sigma_i [\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \int_0^1 (1-q\omega)^{(\alpha_i-5)} \left| \lambda_i \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. + \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right| d_q \omega \\ &\leq \frac{1}{\Gamma_q(\alpha_i+1)} \left[(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right] \\ &\quad + \frac{|\sigma_i| [\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} \left[(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right]\end{aligned}$$

$$\begin{aligned}
& + \frac{\prod_{j=0}^3 [\alpha_i - j]_q}{[\mathfrak{I}_i] \Gamma_q(\alpha_i)} \left[(|\lambda_i|_v \zeta_i + |\delta_i|_h \zeta_i + |\mu_i|_h \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right] \\
& \leq \tilde{M}_i \left[\Delta_i r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right].
\end{aligned}$$

Also, we can observe that

$$\begin{aligned}
{}_q \mathcal{D}^2 \mathcal{I}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) &= \frac{\prod_{j=1}^2 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} \int_0^z (z - q\omega)^{(\alpha_i - 5)} \left[\lambda_i \mathfrak{v}_i(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\
& \quad \left. + \mu_i \mathfrak{h}_i(z, {}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}_q \mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \right] d_q \omega \\
& + \frac{\sigma_i \prod_{j=0}^2 [\alpha_i - j]_q z^{(\alpha_i - 3)}}{[\mathfrak{I}_i] \Gamma_q(\alpha_i)} \\
& \times \int_0^1 (1 - q\omega)^{(\alpha_i)} \left[\lambda_i \mathfrak{v}_i(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\
& \quad \left. + \mu_i \mathfrak{h}_i(z, {}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}_q \mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \right] d_q \omega \\
& - \frac{\prod_{j=0}^4 [\alpha_i - j]_q \prod_{j=1}^2 [\alpha_i - j]_q z^{(\alpha_i - 3)}}{[\mathfrak{I}_i] \Gamma_q(\alpha_i)} \\
& \times \int_0^1 (1 - q\omega)^{(\alpha_i - 5)} \left[\lambda_i \mathfrak{v}_i(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) \right. \\
& \quad \left. + \mu_i \mathfrak{h}_i(z, {}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}_q \mathcal{D}^2 \mathfrak{s}_2(z)) + \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \right] d_q \omega,
\end{aligned}$$

which implies that

$$\begin{aligned}
\| {}_q \mathcal{D}^2 \mathcal{I}_i(x_1, x_2) \|_\infty &= \sup \left\{ | {}_q \mathcal{D}^2 \mathcal{I}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) | : z \in \bar{\Omega} \right\} \\
&\leq \frac{[\alpha_i - 1]_q}{\Gamma_q(\alpha_i)} \left[(|\lambda_i|_v \zeta_i + |\delta_i|_g \zeta_i + |\mu_i|_h \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right] \\
& + \frac{\sigma_i \prod_{j=0}^2 [\alpha_i - j]_q}{[\mathfrak{I}_i] \Gamma_q(\alpha_i) [\alpha_i + 1]_q} \\
& \times \left[(|\lambda_i|_v \zeta_i + |\delta_i|_g \zeta_i + |\mu_i|_h \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right] \\
& + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^2 [\alpha_i - j]_q}{[\mathfrak{I}_i] \Gamma_q(\alpha_i)} \\
& \times \left[(|\lambda_i|_v \zeta_i + |\delta_i|_g \zeta_i + |\mu_i|_h \zeta_i) r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ \right] \\
&\leq \tilde{L}_i [\Delta_i r + \mathfrak{v}_i^\circ + \mathfrak{g}_i^\circ + \mathfrak{h}_i^\circ], \quad i = 1, 2.
\end{aligned}$$

and

$${}_q \mathcal{D}^4 \mathcal{I}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) = \frac{\mathfrak{I}_i + \sigma_i [\alpha_i]_q}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 5)} \left[\lambda_i \mathfrak{v}_i(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) \right.$$

$$\begin{aligned}
& + \mu_i \mathfrak{h}_i \left(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z) \right) + \delta_i \mathfrak{g}_i \left(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z) \right) \Big] d_q \omega \\
& + \frac{\sigma_i (\mathfrak{I}_i + \sigma_i [\alpha_i]_q) z^{(\alpha_i - 5)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1 - q\omega)^{(\alpha_i)} \\
& \times \left[\lambda_i \mathfrak{v}_i \left(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z) \right) \right. \\
& + \mu_i \mathfrak{h}_i \left(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z) \right) + \delta_i \mathfrak{g}_i \left(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z) \right) \Big] d_q \omega \\
& - \frac{\prod_{j=0}^4 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q z^{(\alpha_i - 5)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1 - q\omega)^{(\alpha_i - 5)} \\
& \times \left[\lambda_i \mathfrak{v}_i \left(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z) \right) \right. \\
& + \mu_i \mathfrak{h}_i \left(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z) \right) + \delta_i \mathfrak{g}_i \left(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z) \right) \Big] d_q \omega.
\end{aligned}$$

It yields that,

$$\begin{aligned}
& \left\| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\infty} = \sup_{z \in \Omega} \left| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\
& \leq \frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} \left[\left(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i \right) r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ} \right] \\
& \leq \frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} \left[\left(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i \right) r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ} \right] \\
& \quad + \frac{\sigma_i (\mathfrak{I}_i + \sigma_i [\alpha_i]_q)}{\mathfrak{I}_i \Gamma_q(\alpha_i) [\alpha_i + 1]_q} \left[\left(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i \right) r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ} \right] \\
& \quad + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \\
& \quad \times \left[\left(|\lambda_i| \mathfrak{v}_i \zeta_i + |\delta_i| \mathfrak{g}_i \zeta_i + |\mu_i| \mathfrak{h}_i \zeta_i \right) r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ} \right] \\
& \leq \tilde{N}_i [\Delta_i r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ}], \quad i = 1, 2.
\end{aligned}$$

In consequence, we get

$$\begin{aligned}
\| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\Upsilon} &= \| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} \\
&\leq K_i [\Delta_i r + \mathfrak{v}_i^{\circ} + \mathfrak{g}_i^{\circ} + \mathfrak{h}_i^{\circ}].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\| \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\Upsilon} &= \| \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^2 \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^4 \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} \\
&\leq K_1 [\Delta_1 r + \mathfrak{v}_1^{\circ} + \mathfrak{g}_1^{\circ} + \mathfrak{h}_1^{\circ}],
\end{aligned}$$

$$\begin{aligned}
\| \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\Upsilon} &= \| \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^2 \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} + \| {}_q\mathcal{D}^4 \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} \\
&\leq K_2 [\Delta_2 r + \mathfrak{v}_2^{\circ} + \mathfrak{g}_2^{\circ} + \mathfrak{h}_2^{\circ}],
\end{aligned}$$

which implies that

$$\begin{aligned} \|\mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Omega \times \Omega} &= \|\mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Omega} + \|\mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Omega} \\ &\leq K_1 [\Delta_1 r + \mathfrak{v}_1^\circ + \mathfrak{g}_1^\circ + \mathfrak{h}_1^\circ] + K_2 [\Delta_2 r + \mathfrak{v}_2^\circ + \mathfrak{g}_2^\circ + \mathfrak{h}_2^\circ] \leq r. \end{aligned}$$

Hence, we prove that, $\mathcal{T}\mathcal{B}_r \subset \mathcal{B}_r$, as $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{B}_r$. Next, we show that $\mathcal{T}$ is a contraction. Using the condition (H₂), for any $(\mathfrak{s}_1, \mathfrak{s}_2), (\mathfrak{s}_1, \mathfrak{s}_2) \in \Upsilon^2$, we can write

$$\begin{aligned} |\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z)| &\leq \int_0^1 \frac{(1-q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \left| \lambda_i \left[\mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad - \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) + \mu_i \left[\mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. \left. - \mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right] + \delta_i \left[\mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad \left. \left. - \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right] \right| d_q \omega + \frac{|\sigma_i|[\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \\ &\quad \times \int_0^1 (1-q\omega)^{(\alpha_i)} \left| \lambda_i \left[\mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad - \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) + \mu_i \left[\mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. \left. - \mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right] + \delta_i \left[\mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad \left. \left. - \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right] \right| d_q \omega + \frac{|\mathfrak{I}_i + \sigma_i|[\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \\ &\quad \times \int_0^1 (1-q\omega)^{(\alpha_i-5)} \left| \lambda_i \left[\mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad - \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) + \mu_i \left[\mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right. \\ &\quad \left. \left. - \mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) \right] + \delta_i \left[\mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right. \right. \\ &\quad \left. \left. - \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right] \right| d_q \omega \\ &\leq \frac{1}{\Gamma_q(\alpha_i+1)} \left[|\lambda_i| \zeta_i \left(\|{}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1\|_{\infty} + \|{}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2\|_{\infty} \right) \right. \\ &\quad + |\mu_i| \zeta_i \left(\|{}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1\|_{\infty} + \|{}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2\|_{\infty} \right) \\ &\quad \left. + |\delta_i| \zeta_i \left(\|\mathfrak{s}_1 - \mathfrak{s}_1\|_{\infty} + \|\mathfrak{s}_2 - \mathfrak{s}_2\|_{\infty} \right) \right] \\ &\quad + \frac{|\sigma_i|[\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)[\alpha_i+1]_q} \\ &\quad \times \left[|\lambda_i| \zeta_i \left(\|{}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1\|_{\infty} + \|{}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2\|_{\infty} \right) \right. \\ &\quad + |\mu_i| \zeta_i \left(\|{}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1\|_{\infty} + \|{}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2\|_{\infty} \right) \\ &\quad \left. + |\delta_i| \zeta_i \left(\|\mathfrak{s}_1 - \mathfrak{s}_1\|_{\infty} + \|\mathfrak{s}_2 - \mathfrak{s}_2\|_{\infty} \right) \right] + \frac{\prod_{j=0}^3 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \end{aligned}$$

$$\begin{aligned}
& \times \left[|\lambda_i|_{\mathfrak{V}} \zeta_i \left(\| {}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2 \|_{\infty} \right) \right. \\
& + |\mu_i|_{\mathfrak{H}} \zeta_i \left(\| {}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2 \|_{\infty} \right) \\
& \left. + |\delta_i|_{\mathfrak{G}} \zeta_i \left(\| \mathfrak{s}_1 - \mathfrak{s}_1 \|_{\infty} + \| \mathfrak{s}_2 - \mathfrak{s}_2 \|_{\infty} \right) \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} \\
& \leq \Delta_i \left[\frac{1}{\Gamma_q(\alpha_i+1)} + \frac{|\sigma_i|_{\mathfrak{A}} [\alpha_i]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\
& \quad \times \left[\| {}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2 \|_{\infty} \right] \\
& + \Delta_i \left[\frac{1}{\Gamma_q(\alpha_i+1)} + \frac{|\sigma_i|_{\mathfrak{A}} [\alpha_i]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\
& \quad \times \left[\| {}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2 \|_{\infty} \right] \\
& + \Delta_i \left[\frac{1}{\Gamma_q(\alpha_i+1)} + \frac{|\sigma_i|_{\mathfrak{A}} [\alpha_i]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\
& \quad \times \left[\| \mathfrak{s}_1 - \mathfrak{s}_1 \|_{\infty} + \| \mathfrak{s}_2 - \mathfrak{s}_2 \|_{\infty} \right] \\
& \leq \Delta_i \tilde{M}_i \left[\| (\mathfrak{s}_1 - \mathfrak{s}_1) \|_{\Upsilon} + \| (\mathfrak{s}_2 - \mathfrak{s}_2) \|_{\Upsilon} \right], \quad i = 1, 2.
\end{aligned}$$

Similarly, it can be shown that for $i = 1, 2$,

$$\begin{aligned}
& \| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_{\infty} \\
& = \sup_{z \in \Omega} | {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) - {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) | \\
& \leq \Delta_i \left[\frac{[\alpha_i-1]_{\mathfrak{q}}}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|_{\mathfrak{A}} \prod_{j=0}^2 [\alpha-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}} \prod_{j=1}^2 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\
& \quad \times \left[\| {}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2 \|_{\infty} \right] \\
& + \Delta_i \left[\frac{[\alpha_i-1]_{\mathfrak{q}}}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|_{\mathfrak{A}} \prod_{j=0}^2 [\alpha-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}} \prod_{j=1}^2 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\
& \quad \times \left[\| {}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1 \|_{\infty} + \| {}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2 \|_{\infty} \right] \\
& + \Delta_i \left[\frac{[\alpha_i-1]_{\mathfrak{q}}}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|_{\mathfrak{A}} \prod_{j=0}^2 [\alpha-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_{\mathfrak{q}}} + \frac{\prod_{j=0}^3 [\alpha_i-j]_{\mathfrak{q}} \prod_{j=1}^2 [\alpha_i-j]_{\mathfrak{q}}}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right]
\end{aligned}$$

$$\begin{aligned} & \times \left[\|\mathfrak{s}_1 - \mathfrak{s}_1\|_\infty + \|\mathfrak{s}_2 - \mathfrak{s}_2\|_\infty \right] \\ & \leq \Delta_i \tilde{L}_i \left[\|(\mathfrak{s}_1 - \mathfrak{s}_1)\|_\Omega + \|(\mathfrak{s}_2 - \mathfrak{s}_2)\|_\Upsilon \right], \end{aligned}$$

and

$$\begin{aligned} & \left\| {}_q\mathcal{D}^4 {}_q\mathcal{D}_i(\mathfrak{s}_1, \mathfrak{s}_2) - {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\infty \\ & = \sup_{z \in \Omega} \left| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\mathfrak{s}_1, \mathfrak{s}_2) - {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\ & \leq \Delta_i \left[\frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|(\mathfrak{I}_i + \sigma_i[\alpha_i]_q)}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)[\alpha_i + 1]_q} + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\ & \quad \times \left[\|{}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1\|_\infty + \|{}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2\|_\infty \right] \\ & \quad + \Delta_i \left[\frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|(\mathfrak{I}_i + \sigma_i[\alpha_i]_q)}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)[\alpha_i + 1]_q} + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\ & \quad \times \left[\|{}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1\|_\infty + \|{}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2\|_\infty \right] \\ & \quad + \Delta_i \left[\frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} + \frac{|\sigma_i|(\mathfrak{I}_i + \sigma_i[\alpha_i]_q)}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)[\alpha_i + 1]_q} + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \right] \\ & \quad \times \left[\|\mathfrak{s}_1 - \mathfrak{s}_1\|_\infty + \|\mathfrak{s}_2 - \mathfrak{s}_2\|_\infty \right] \\ & \leq \Delta_i \tilde{N}_i \left[\|(\mathfrak{s}_1 - \mathfrak{s}_1)\|_\Upsilon + \|(\mathfrak{s}_2 - \mathfrak{s}_2)\|_\Upsilon \right]. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \left\| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\Upsilon = \left\| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\infty \\ & \quad + \left\| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\infty \\ & \quad + \left\| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) - {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\infty \\ & \leq \Delta_i K_i \left[\|(\mathfrak{s}_1 - \mathfrak{s}_1)\|_\Upsilon + \|(\mathfrak{s}_2 - \mathfrak{s}_2)\|_\Upsilon \right], \quad i = 1, 2. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \left\| \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\Upsilon \leq \Delta_1 K_1 \left(\|(\mathfrak{s}_1 - \mathfrak{s}_1)\|_\Upsilon + \|(\mathfrak{s}_2 - \mathfrak{s}_2)\|_\Upsilon \right), \\ & \left\| \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_\Upsilon \leq \Delta_2 K_2 \left(\|(\mathfrak{s}_1 - \mathfrak{s}_1)\|_\Upsilon + \|(\mathfrak{s}_2 - \mathfrak{s}_2)\|_\Upsilon \right), \end{aligned} \quad (14)$$

from (14), we have

$$\left\| \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Upsilon^2} \leq \sum_{i=1}^2 K_i \Delta_i \left\| (\mathfrak{s}_1, \mathfrak{s}_2), (\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Upsilon^2}.$$

We deduce that $\mathcal{T}$ is contractive which implies $\mathcal{T}$ has a unique FP which is the solution of given system (1). $\square$

Now, we prove the existence solution to system (1) by applying following Lemma of Schaefer's FP.

Lemma 3.4. *Let E be a Banach space. Then, a completely continuous operator $\Lambda : E \rightarrow E$ has a FP, whenever the set $\Sigma = \{z \in E : z = \rho \Lambda(z), 0 < \rho < 1\}$, is bounded.*

Theorem 3.5. *Assume that the following hypotheses is satisfied.*

(H₂) *There exist positive constants ${}_{\mathfrak{v}}\xi_i, {}_{\mathfrak{h}}\xi_i, {}_{\mathfrak{g}}\xi_i, i = 1, 2$, such that, for every $z \in \bar{\Omega}$ and $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{R}$, and $i = 1, 2$,*

$$|{}_{\mathfrak{v}}i(z, \mathfrak{s}_1, \mathfrak{s}_2)| \leq {}_{\mathfrak{v}}\xi_i, \quad |{}_{\mathfrak{h}}i(z, \mathfrak{s}_1, \mathfrak{s}_2)| \leq {}_{\mathfrak{h}}\xi_i, \quad |{}_{\mathfrak{g}}i(z, \mathfrak{s}_1, \mathfrak{s}_2)| \leq {}_{\mathfrak{g}}\xi_i.$$

Then, (1) has at least one solution defined over $\bar{\Omega}$.

Proof. The continuity of ${}_{\mathfrak{v}}i, {}_{\mathfrak{h}}i$ and ${}_{\mathfrak{g}}i, i = 1, 2$, we find that $\mathcal{T}$ is continuous. Now, we show that $\mathcal{T}$ is completely continuous. We show that $\mathcal{T}$ maps bounded set $F \subset \Omega \times \Omega$ into bounded set of F . In this regard, for all $(\mathfrak{s}_1, \mathfrak{s}_2) \in F$, we have

$$\begin{aligned} \|\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\infty} &= \sup_{z \in \bar{\Omega}} |\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z)| \\ &\leq \frac{1}{\Gamma_q(\alpha_i+1)} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\quad + \frac{|\sigma_i| [\alpha_i]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\quad + \frac{\prod_{j=0}^3 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\leq \tilde{M}_i \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \quad i = 1, 2. \end{aligned} \quad (15)$$

Likewise, for $i = 1, 2$, we get

$$\begin{aligned} \|{}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\infty} &= \sup_{z \in \bar{\Omega}} |{}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z)| \\ &\leq \frac{[\alpha_i-1]_q}{\Gamma_q(\alpha_i)} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\quad + \frac{\sigma_i \prod_{j=0}^2 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\quad + \frac{\prod_{j=0}^3 [\alpha_i-j]_q \prod_{j=1}^2 [\alpha_i-j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right] \\ &\leq \tilde{L}_i \left[|\lambda_i| {}_{\mathfrak{v}}\xi_i + |\mu_i| {}_{\mathfrak{h}}\xi_i + |\delta_i| {}_{\mathfrak{g}}\xi_i \right], \end{aligned} \quad (16)$$

and

$$\begin{aligned}
 \left\| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\infty} &= \sup_{z \in \bar{\Omega}} \left| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\
 &\leq \frac{\prod_{j=1}^3 [\alpha_i - j]_q}{\Gamma_q(\alpha_i)} \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right] \\
 &\quad + \frac{\sigma_i \prod_{j=0}^4 [\alpha_i - j]_q}{\mathfrak{I}_i \Gamma_q(\alpha_i) [\alpha_i + 1]_q} \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right] \\
 &\quad + \frac{\prod_{j=0}^3 [\alpha_i - j]_q \prod_{j=1}^4 [\alpha_i - j]_q}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right] \\
 &\leq \tilde{N}_i \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right]. \tag{17}
 \end{aligned}$$

Now, by helping Equations (15), (16) and (17), for $i = 1, 2$, we have

$$\begin{aligned}
 \left\| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Omega} &= \left\| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\infty} + \left\| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\infty} + \left\| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\infty} \\
 &\leq \left(\tilde{M}_i + \tilde{L}_i + \tilde{N}_i \right) \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right] \\
 &\leq K_i \left[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \left\| \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Upsilon} &\leq K_1 \left[|\lambda_1|_v \xi_1 + |\mu_1|_h \xi_1 + |\delta_1|_g \xi_1 \right], \\
 \left\| \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Upsilon} &\leq K_2 \left[|\lambda_2|_v \xi_2 + |\mu_2|_h \xi_2 + |\delta_2|_g \xi_2 \right]. \tag{18}
 \end{aligned}$$

Thus, thanks to the inequalities in Equation (18), we get

$$\begin{aligned}
 \left\| \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2) \right\|_{\Upsilon^2} &\leq K_1 \left[|\lambda_1|_v \xi_1 + |\mu_1|_h \xi_1 + |\delta_1|_g \xi_1 \right] \\
 &\quad + K_2 \left[|\lambda_2|_v \xi_2 + |\mu_2|_h \xi_2 + |\delta_2|_g \xi_2 \right].
 \end{aligned}$$

Indeed, $\mathcal{T}$ is uniformly bounded. Next, we show that $\mathcal{T}$ is equicontinuous. Let $(\mathfrak{s}_1, \mathfrak{s}_2) \in F$ and $0 \leq z < \mathfrak{z} \leq 1$, it follows that:

$$\begin{aligned}
 &\left| \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\mathfrak{z}) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\
 &\leq \frac{1}{\Gamma_q(\alpha)} \left| \int_0^z \left((\mathfrak{z} - q\omega)^{(\alpha_i-1)} - (z_1 - q\omega)^{(\alpha_i-1)} \right) \right. \\
 &\quad \times \left[\lambda_i \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \\
 &\quad \left. + \mu_i \mathfrak{h}_i(\omega, {}_q\mathcal{D}^2 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^2 \mathfrak{s}_2(\omega)) + \delta_i \mathfrak{g}_i(\omega, \mathfrak{s}_1(\omega), \mathfrak{s}_2(\omega)) \right] d_q \omega \left. \right| \\
 &\quad + \frac{1}{\Gamma_q(\alpha)} \left| \int_z^{\mathfrak{z}} (\mathfrak{z} - q\omega)^{(\alpha_i-1)} \left[\lambda_i \mathfrak{v}_i(\omega, {}_q\mathcal{D}^4 \mathfrak{s}_1(\omega), {}_q\mathcal{D}^4 \mathfrak{s}_2(\omega)) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \mu_i \mathfrak{h}_i(\mathfrak{w}, {}_q\mathcal{D}^2 \mathfrak{s}_1(\mathfrak{w}), {}_q\mathcal{D}^2 \mathfrak{s}_2(\mathfrak{w})) + \delta_i \mathfrak{g}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(z)) \Big] d_q \mathfrak{w} \Big| \\
& + \frac{|\sigma_i| [\alpha_i]_q \left| \dot{z}^{(\alpha_i-1)} - z_1^{(\alpha_i-1)} \right|}{\left| \prod_{j=0}^4 [\alpha_i - j]_q - \sigma_i [\alpha_i]_q \right| \Gamma_q(\alpha_i)} \\
& \times \int_0^1 (1 - q\mathfrak{w})^{(\alpha_i)} \left| \lambda_i \mathfrak{v}_i(\mathfrak{w}, {}_q\mathcal{D}^4 \mathfrak{s}_1(\mathfrak{w}), {}_q\mathcal{D}^4 \mathfrak{s}_2(\mathfrak{w})) \right. \\
& + \mu_i \mathfrak{h}_i(\mathfrak{w}, {}_q\mathcal{D}^2 \mathfrak{s}_1(\mathfrak{w}), {}_q\mathcal{D}^2 \mathfrak{s}_2(\mathfrak{w})) + \delta_i \mathfrak{g}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w})) \Big| d_q \mathfrak{w} \\
& + \frac{(\mathfrak{I}_i + \sigma_i [\alpha_i]_q) \left| \dot{z}^{(\alpha_i-1)} - z^{(\alpha_i-1)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)} \\
& \times \int_0^1 (1 - q\mathfrak{w})^{(\alpha-5)} \left| \lambda_i \mathfrak{v}_i(\mathfrak{w}, {}_q\mathcal{D}^4 \mathfrak{s}_1(\mathfrak{w}), {}_q\mathcal{D}^4 \mathfrak{s}_2(\mathfrak{w})) \right. \\
& + \mu_i \mathfrak{h}_i(\mathfrak{w}, {}_q\mathcal{D}^2 \mathfrak{s}_1(\mathfrak{w}), {}_q\mathcal{D}^2 \mathfrak{s}_2(\mathfrak{w})) + \delta_i \mathfrak{g}_i(\mathfrak{w}, \mathfrak{s}_1(\mathfrak{w}), \mathfrak{s}_2(\mathfrak{w})) \Big| d_q \mathfrak{w} \\
& \leq \frac{|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i}{\Gamma_q(\alpha_i+1)} \left[2(\dot{z} - z)^{(\alpha)} + \left| \dot{z}^{(\alpha)} - z^{(\alpha)} \right| \right] \\
& + \frac{|\sigma_i| [\alpha_i]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right] \left| \dot{z}^{(\alpha_i-1)} - z^{(\alpha_i-1)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} \\
& + \frac{\prod_{j=0}^3 [\alpha-j]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right] \left| \dot{z}^{(\alpha-1)} - z^{(\alpha-1)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)}, \quad i = 1, 2,
\end{aligned}$$

which implies

$$|\mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\dot{z}) - \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z)| \rightarrow 0, \quad (19)$$

as $\dot{z} \rightarrow z$. Similarly, one finds for $i = 1, 2$,

$$\begin{aligned}
& \left| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\dot{z}) - {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\
& \leq \frac{[\alpha_i-1]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right]}{\Gamma_q(\alpha_i)} \\
& \times \left[2(\dot{z} - z)^{(\alpha-2)} + \left| \dot{z}^{(\alpha-2)} - z^{(\alpha-2)} \right| \right] \\
& + \frac{|\sigma_i| \prod_{j=0}^2 [\alpha_i-j]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right] \left| \dot{z}^{(\alpha_i-3)} - z^{(\alpha_i-3)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i) [\alpha_i+1]_q} \\
& + \frac{\prod_{j=0}^3 [\alpha_i-j]_q \prod_{j=1}^2 [\alpha_i-j]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right] \left| \dot{z}^{(\alpha-3)} - z^{(\alpha-3)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)},
\end{aligned}$$

and

$$\begin{aligned}
& \left| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\dot{z}) - {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \\
& \leq \frac{\prod_{j=1}^3 [\alpha_i-j]_q \left[|\lambda_i| {}_v\xi_i + |\mu_i| {}_h\xi_i + |\delta_i| {}_g\xi_i \right]}{\Gamma_q(\alpha_i)} \\
& \times \left[2(\dot{z} - z)^{(\alpha-4)} + \left| \dot{z}^{(\alpha-4)} - z^{(\alpha-4)} \right| \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\sigma_i|(\mathfrak{I}_i + \sigma_i[\alpha_i]_q)[|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i] \left| z^{(\alpha_i-5)} - z^{(\alpha_i-5)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)[\alpha_i+1]_q} \\
& + \frac{\prod_{j=0}^3 [\alpha_i-j]_q \prod_{j=1}^4 [\alpha_i-j]_q [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i] \left| z^{(\alpha-5)} - z^{(\alpha-5)} \right|}{|\mathfrak{I}_i| \Gamma_q(\alpha_i)},
\end{aligned}$$

which imply that

$$\begin{aligned}
& \left| {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\dot{z}) - {}_q\mathcal{D}^2 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \rightarrow 0, \\
& \left| {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(\dot{z}) - {}_q\mathcal{D}^4 \mathcal{T}_i(\mathfrak{s}_1, \mathfrak{s}_2)(z) \right| \rightarrow 0,
\end{aligned} \tag{20}$$

as $\dot{z} \rightarrow z$. It follows that the operators $\mathcal{T}_i$, $i = 1, 2$ are equicontinuous. At present, the Arzelà-Ascoli theorem with these results confirm the completely continuity of the operator $\mathcal{T}$. Finally, we consider the set

$$\Theta = \left\{ (\mathfrak{s}_1, \mathfrak{s}_2) \in \Upsilon^2 : (\mathfrak{s}_1, \mathfrak{s}_2) = \varrho \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2), 0 < \varrho < 1 \right\},$$

and we prove that it is bounded. For $(\mathfrak{s}_1, \mathfrak{s}_2) \in \Theta$ and $z \in \bar{\Omega}$, we have $(\mathfrak{s}_1, \mathfrak{s}_2)(z) = \varrho \mathcal{T}(\mathfrak{s}_1, \mathfrak{s}_2)(\mathfrak{s}_1, \mathfrak{s}_2)$. It follows that

$$\mathfrak{s}_1(z) = \varrho \mathcal{T}_1(\mathfrak{s}_1, \mathfrak{s}_2)(z), \quad \mathfrak{s}_2(z) = \varrho \mathcal{T}_2(\mathfrak{s}_1, \mathfrak{s}_2)(z).$$

Using the condition (H_2) , we obtain

$$\|\mathfrak{s}_i\|_\infty = \sup_{z \in \bar{\Omega}} |\mathfrak{s}_i(z)| \leq \varrho \tilde{M}_i [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i], \quad i = 1, 2. \tag{21}$$

Also, we have

$$\begin{aligned}
\|{}_q\mathcal{D}^2 \mathfrak{s}_i\|_\infty &= \sup_{z \in \bar{\Omega}} |{}_q\mathcal{D}^2 \mathfrak{s}_i(z)| \leq \varrho \tilde{L}_i [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i], \\
\|{}_q\mathcal{D}^4 \mathfrak{s}_i\|_\infty &= \sup_{z \in \bar{\Omega}} |{}_q\mathcal{D}^4 \mathfrak{s}_i(z)| \leq \varrho \tilde{N}_i [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i].
\end{aligned} \tag{22}$$

It follows from (21) and (22), that

$$\begin{aligned}
\|\mathfrak{s}_i\|_\Upsilon &= \|\mathfrak{s}_i\|_\infty + \|{}_q\mathcal{D}^2 \mathfrak{s}_i\|_\infty + \|{}_q\mathcal{D}^4 \mathfrak{s}_i\|_\infty \\
&\leq \varrho \left(\tilde{M}_i + \tilde{L}_i + \tilde{N}_i \right) [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i] \\
&\leq \left(\tilde{M}_i + \tilde{L}_i + \tilde{N}_i \right) [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i] \\
&\leq K_i [|\lambda_i|_v \xi_i + |\mu_i|_h \xi_i + |\delta_i|_g \xi_i].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathfrak{s}_1\|_\Upsilon &\leq K_1 [|\lambda_1|_v \xi_1 + |\mu_1|_h \xi_1 + |\delta_1|_g \xi_1], \\
\|\mathfrak{s}_2\|_\Upsilon &\leq K_2 [|\lambda_2|_v \xi_2 + |\mu_2|_h \xi_2 + |\delta_2|_g \xi_2].
\end{aligned}$$

Indeed,

$$\begin{aligned} \|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Upsilon^2} &= \|\mathfrak{s}_1\|_{\Upsilon} + \|\mathfrak{s}_2\|_{\Upsilon} \\ &\leq K_1 K_1 [|\lambda_1|_{\mathfrak{v}} \xi_1 + |\mu_i|_{\mathfrak{h}} \xi_1 + |\delta_1|_{\mathfrak{g}} \xi_1] \\ &\quad + K_2 [|\lambda_2|_{\mathfrak{v}} \xi_2 + |\mu_2|_{\mathfrak{h}} \xi_2 + |\delta_2|_{\mathfrak{g}} \xi_2]. \end{aligned}$$

Consequently, $\|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{\Upsilon^2} < \infty$. Indeed, Θ is bounded and so, [Lemma 3.4](#) implies that $\mathcal{T}$ has at least one FP, which is a solution of q-FDE (1). $\square$

4. The stability of Mittag-Leffler-Ulam-Hyers

We provide results regarding Ulam-Hyers-Mittag-Leffler (UHML) stability for our problem. We consider the following inequalities,

$$\begin{aligned} \left| {}_q\mathcal{D}^{\alpha_i} \mathfrak{s}_i(z) - \lambda_i \mathfrak{v}_i(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z)) - \mu_i \mathfrak{h}_i(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) \right. \\ \left. - \delta_i \mathfrak{g}_i(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) \right| \leq \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}), \quad \epsilon > 0, \end{aligned} \quad (23)$$

with Mittag-Leffler function $\mathcal{E}_{\alpha_i}(\cdot)$ is formulated by:

$$\mathcal{E}_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+1)},$$

under conditions,

$$\begin{cases} ({}_q\mathcal{D}^m \mathfrak{s}_1)(0) = ({}_q\mathcal{D}^m \mathfrak{s}_2)(0) = 0, & m = \overline{0, 4}, \\ ({}_q\mathcal{D}^4 \mathfrak{s}_i)(1) = \sigma_i [\alpha_i]_q \int_0^1 \mathfrak{s}_i(\varpi) d_q \varpi, & \sigma_i \in \mathbb{R}, \end{cases} \quad (24)$$

where $\lambda_i, \mu_i, \delta_i \in \mathbb{R}$, $5 < \alpha_i \leq 6$, $i = 1, 2$, $z \in \bar{\Omega}$.

Definition 4.1. The system (1) is UHML stable with respect to $\mathcal{E}_{\gamma}(z^{\gamma})$, if there exists an element $C_{\gamma} > 0$ such that for each $\epsilon > 0$ and for every solution $\mathfrak{s} \in \Upsilon$ of inequality (23), there exists a solution $\hat{\mathfrak{s}} \in \Upsilon$ of the system (1) with,

$$|\hat{\mathfrak{s}}(z) - \mathfrak{s}(z)| \leq C_{\gamma} \epsilon \mathcal{E}_{\gamma}(z^{\gamma}).$$

Theorem 4.2. Assume that (H_1) in [Theorem 3.3](#) is valid and also the following hypothesis holds:

(H_3) There exists $\hat{\mathfrak{s}} \in \Upsilon$. Such that for each $z \in \bar{\Omega}$, we have $|\hat{\mathfrak{s}}| \leq \epsilon \mathcal{E}_{\alpha}$.

Then, the system (1) is generalized UHML stable.

Proof. Let $\hat{\mathbf{s}}_i, i = 1, 2$ be solution of (23), we suppose that

$$\begin{aligned} \hat{\mathbf{s}}_i(z) = & \int_0^z \frac{(z-q\omega)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} \left[\lambda_i \mathbb{V}_i(z, {}_q\mathcal{D}^4 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^4 \hat{\mathbf{s}}_2(z)) \right. \\ & + \mu_i \mathbb{H}_i(z, {}_q\mathcal{D}^2 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^2 \hat{\mathbf{s}}_2(z)) + \delta_i \mathbb{G}_i(z, \hat{\mathbf{s}}_1(z), \hat{\mathbf{s}}_2(z)) \left. \right] {}_q\mathcal{D} \\ & + \frac{\sigma_i [\alpha_i]_q z^{(\alpha_i-1)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1-q\omega)^{(\alpha_i)} \left[\lambda_i \mathbb{V}_i(z, {}_q\mathcal{D}^4 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^4 \hat{\mathbf{s}}_2(z)) \right. \\ & + \mu_i \mathbb{H}_i(z, {}_q\mathcal{D}^2 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^2 \hat{\mathbf{s}}_2(z)) + \delta_i \mathbb{G}_i(z, \hat{\mathbf{s}}_1(z), \hat{\mathbf{s}}_2(z)) \left. \right] d_q \omega \\ & - \frac{(\mathfrak{I}_i + \sigma_i [\alpha_i]_q) z^{(\alpha_i-1)}}{\mathfrak{I}_i \Gamma_q(\alpha_i)} \int_0^1 (1-q\omega)^{(\alpha_i-5)} \left[\lambda_i \mathbb{V}_i(z, {}_q\mathcal{D}^4 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^4 \hat{\mathbf{s}}_2(z)) \right. \\ & + \mu_i \mathbb{H}_i(z, {}_q\mathcal{D}^2 \hat{\mathbf{s}}_1(z), {}_q\mathcal{D}^2 \hat{\mathbf{s}}_2(z)) + \delta_i \mathbb{G}_i(z, \hat{\mathbf{s}}_1(z), \hat{\mathbf{s}}_2(z)) \left. \right] d_q \omega = I_i, \end{aligned}$$

for $i = 1, 2$. Thanks to (H_3) , we get

$$|\hat{\mathbf{s}}_i(t) - I_i| \leq \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}), \quad i = 1, 2. \quad (25)$$

By Theorem 3.3, the system (1) has a unique solution $\mathbf{s}_i, i = 1, 2$. Then, using (25) and (H_3) , we get

$$\begin{aligned} \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\infty} & \leq \tilde{M}_i \left[(|\lambda_i| \mathbb{V}_i \zeta_i + |\mu_i| \mathbb{H}_i \zeta_i + |\delta_i| \mathbb{G}_i \zeta_i) \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right] \\ & \leq \tilde{M}_i \left[\Delta_i \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right], \end{aligned}$$

and also,

$$\begin{aligned} \|D_q^2 y_i - D_q^2 x_i\|_{\infty} & \leq \tilde{L}_i \left[(|\lambda_i| \mathbb{V}_i \zeta_i + |\mu_i| \mathbb{H}_i \zeta_i + |\delta_i| \mathbb{G}_i \zeta_i) \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right] \\ & \leq \tilde{L}_i \left[\Delta_i \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right], \\ \|{}_q\mathcal{D}^4 \hat{\mathbf{s}}_i - {}_q\mathcal{D}^4 \mathbf{s}_i\|_{\infty} & \leq \tilde{N}_i \left[(|\lambda_i| \mathbb{V}_i \zeta_i + |\mu_i| \mathbb{H}_i \zeta_i + |\delta_i| \mathbb{G}_i \zeta_i) \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right] \\ & \leq \tilde{N}_i \left[\Delta_i \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}) \right], \end{aligned}$$

for $i = 1, 2$. Therefore,

$$\begin{aligned} \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} & = \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\infty} + \|{}_q\mathcal{D}^2 \hat{\mathbf{s}}_i - {}_q\mathcal{D}^2 \mathbf{s}_i\|_{\infty} + \|{}_q\mathcal{D}^4 \hat{\mathbf{s}}_i - {}_q\mathcal{D}^4 \mathbf{s}_i\|_{\infty} \\ & \leq K_i \Delta_i \|\hat{\mathbf{s}}_i - \mathbf{s}_i\|_{\Upsilon} + K_i \epsilon \mathcal{E}_{\alpha_i}(z^{\alpha_i}), \quad i = 1, 2, \end{aligned}$$

implies that

$$\begin{aligned} \|\hat{\mathbf{s}}_1 - \mathbf{s}_1\|_{\Upsilon} & \leq K_1 \Delta_1 \|\hat{\mathbf{s}}_1 - \mathbf{s}_1\|_{\Upsilon} + K_1 \epsilon \mathcal{E}_{\alpha_1}(z^{\alpha_1}), \\ \|\hat{\mathbf{s}}_2 - \mathbf{s}_2\|_{\Upsilon} & \leq K_2 \Delta_2 \|\hat{\mathbf{s}}_2 - \mathbf{s}_2\|_{\Upsilon} + K_2 \epsilon \mathcal{E}_{\alpha_2}(z^{\alpha_2}). \end{aligned}$$

Thus,

$$\|(\hat{\mathfrak{s}}_1 - \mathfrak{s}_1, \hat{\mathfrak{s}}_2 - \mathfrak{s}_2)\|_{\Upsilon^2} = \sum_{i=1}^2 \|\hat{\mathfrak{s}}_i - \mathfrak{s}_i\|_{\Upsilon} \leq \frac{K_1 + K_2}{1 - (K_1 \Delta_1 + K_2 \Delta_2)} \epsilon \mathcal{C}_{\gamma}(z^{\gamma}),$$

where $C_{\gamma} = \frac{K_1 + K_2}{1 - (K_1 \Delta_1 + K_2 \Delta_2)}$, and $\mathcal{C}_{\gamma}(z^{\gamma}) = \max\{\mathcal{C}_{\alpha_1}(z^{\alpha_1}), \mathcal{C}_{\alpha_2}(z^{\alpha_2})\}$. Thus, the system (1) is UHML stable. $\square$

5. Examples

We examine some illustrative examples. First, we control the results when q changes in Ω .

Example 5.1. Consider the following system of q -FDEs for $z \in \bar{\Omega}$:

$$\begin{cases} {}^{\text{RL}}_q \mathcal{D}^{28/5} \mathfrak{s}_1(z) = \frac{\cos(z) |{}_q \mathcal{D}^4 \mathfrak{s}_2(z)|}{15e^{z+\sqrt{2}}(1+|{}_q \mathcal{D}^4 \mathfrak{s}_2(z)|)} \\ \quad + \frac{z^3(|{}_q \mathcal{D}^2 \mathfrak{s}_1(z)| + |{}_q \mathcal{D}^2 \mathfrak{s}_2(z)|)}{10\pi(z^2+1)} + \frac{e^{z^2}(\sin|\mathfrak{s}_1(z)| + |\mathfrak{s}_2(z)|)}{19}, \\ {}^{\text{RL}}_q \mathcal{D}^{11/2} \mathfrak{s}_2(z) = \frac{z \sin({}_q \mathcal{D}^4 \mathfrak{s}_1(z))}{113(3-z)} + \frac{\sqrt{z}(|{}_q \mathcal{D}^2 \mathfrak{s}_2(z)|)}{49 \ln(e^z+2)} + \frac{\sqrt{e^z+1}(|\mathfrak{s}_2(z)|)}{80(1+|\mathfrak{s}_2(z)|)}, \end{cases} \quad (26)$$

under conditions

$$\begin{aligned} ({}_q \mathcal{D}^m \mathfrak{s}_1)(0) &= ({}_q \mathcal{D}^m \mathfrak{s}_2)(0) = 0, \quad m = \overline{0, 4}, \\ ({}_q \mathcal{D}^4 \mathfrak{s}_1)(1) &= \sigma_1 \left[\frac{28}{5} \right]_q \int_0^1 \mathfrak{s}(\omega) d_q \omega, \\ ({}_q \mathcal{D}^4 \mathfrak{s}_2)(1) &= \sigma_2 \left[\frac{11}{2} \right]_q \int_0^1 \mathfrak{s}_2(\omega) d_q \omega, \end{aligned}$$

with $\sigma_1 = 10^{-2}$, $\sigma_2 = 10^{-3}$ and three values of $q = \frac{1}{2}, \frac{7}{11}, \frac{12}{13}$. For this example, we have

$$\begin{aligned} \mathfrak{v}_1(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) &= \frac{\cos(z) |{}_q \mathcal{D}^4 \mathfrak{s}_2(z)|}{e^{z+\sqrt{2}}(1+|{}_q \mathcal{D}^4 \mathfrak{s}_2(z)|)}, \\ \mathfrak{h}_1(z, {}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}_q \mathcal{D}^2 \mathfrak{s}_2(z)) &= \frac{z^3(|{}_q \mathcal{D}^2 \mathfrak{s}_1(z)| + |{}_q \mathcal{D}^2 \mathfrak{s}_2(z)|)}{\pi(z^2+1)}, \\ \mathfrak{g}_1(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) &= \frac{e^{z^2}(\sin|\mathfrak{s}_1(z)| + |\mathfrak{s}_2(z)|)}{6 \times 10^8}, \\ \mathfrak{v}_2(z, {}_q \mathcal{D}^4 \mathfrak{s}_1(z), {}_q \mathcal{D}^4 \mathfrak{s}_2(z)) &= \frac{z \sin({}_q \mathcal{D}^4 \mathfrak{s}_1(z))}{3-z}, \\ \mathfrak{h}_2(z, {}_q \mathcal{D}^2 \mathfrak{s}_1(z), {}_q \mathcal{D}^2 \mathfrak{s}_2(z)) &= \frac{\sqrt{z}(|{}_q \mathcal{D}^2 \mathfrak{s}_2(z)|)}{\ln(e^z+2)}, \\ \mathfrak{g}_2(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) &= \frac{\sqrt{e^z+1}(|\mathfrak{s}_2(z)|)}{(1+|\mathfrak{s}_2(z)|)}, \end{aligned}$$

and $\lambda_1 = \frac{1}{15}$, $\lambda_2 = \frac{1}{113}$, $\mu_1 = 10^{-1}$, $\mu_2 = \frac{1}{49}$, $\delta_1 = \frac{1}{19}$, $\delta_2 = \frac{3}{80}$. Thus,

$$\begin{aligned} & \left| \mathbb{V}_1(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2) - \mathbb{V}_1(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2) \right| \\ &= \left| \frac{\cos(z) |{}_q\mathcal{D}^4 \mathfrak{s}_2|}{e^z + \sqrt{2} (1 + |{}_q\mathcal{D}^4 \mathfrak{s}_2|)} - \frac{\cos(z) |{}_q\mathcal{D}^4 \mathfrak{s}_2|}{e^z + \sqrt{2} (1 + |{}_q\mathcal{D}^4 \mathfrak{s}_2|)} \right| \\ &\leq \frac{1}{e^{\sqrt{2}}} |{}_q\mathcal{D}^4 \mathfrak{s}_2 - {}_q\mathcal{D}^4 \mathfrak{s}_2|, \\ & \left| \mathbb{H}_1(z, {}_q\mathcal{D}^2 \mathfrak{s}_1, {}_q\mathcal{D}^2 \mathfrak{s}_2) - \mathbb{H}_1(z, {}_q\mathcal{D}^2 \mathfrak{s}_1, {}_q\mathcal{D}^2 \mathfrak{s}_2) \right| \\ &= \left| \frac{z^3 (|{}_q\mathcal{D}^2 \mathfrak{s}_1| + |{}_q\mathcal{D}^2 \mathfrak{s}_2|)}{\pi(z^2 + 1)} - \frac{z^3 (|{}_q\mathcal{D}^2 \mathfrak{s}_1| + |{}_q\mathcal{D}^2 \mathfrak{s}_2|)}{\pi(z^2 + 1)} \right| \\ &\leq \frac{1}{\pi} \left[|{}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1| + |{}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2| \right], \\ & \left| \mathbb{G}_1(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathbb{G}_1(z, \mathfrak{s}_1, \mathfrak{s}_2) \right| = \left| \frac{e^{z^2} (\sin|\mathfrak{s}_1| + |\mathfrak{s}_2|)}{6 \times 10^8} - \frac{e^{z^2} (\sin|\mathfrak{s}_1| + |\mathfrak{s}_2|)}{6 \times 10^8} \right| \\ &\leq e \left[|\mathfrak{s}_1 - \mathfrak{s}_1| + |\mathfrak{s}_2 - \mathfrak{s}_2| \right], \end{aligned}$$

and

$$\begin{aligned} & \left| \mathbb{V}_2(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2) - \mathbb{V}_2(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2) \right| \\ &= \left| \frac{z \sin({}_q\mathcal{D}^4 \mathfrak{s}_1)}{3 - z} - \frac{z \sin({}_q\mathcal{D}^4 \mathfrak{s}_1)}{3 - z} \right| \leq \frac{1}{2} |{}_q\mathcal{D}^4 \mathfrak{s}_1 - {}_q\mathcal{D}^4 \mathfrak{s}_1|, \\ & \left| \mathbb{H}_2(z, {}_q\mathcal{D}^2 \mathfrak{s}_1, {}_q\mathcal{D}^2 \mathfrak{s}_2) - \mathbb{H}_2(z, {}_q\mathcal{D}^2 \mathfrak{s}_1, {}_q\mathcal{D}^2 \mathfrak{s}_2) \right| = \left| \frac{\sqrt{z} (|{}_q\mathcal{D}^2 \mathfrak{s}_2|)}{\ln(e^z + 2)} - \frac{\sqrt{z} (|{}_q\mathcal{D}^2 \mathfrak{s}_2|)}{\ln(e^z + 2)} \right| \\ &\leq \frac{1}{\ln(3)} \left[|{}_q\mathcal{D}^2 \mathfrak{s}_1 - {}_q\mathcal{D}^2 \mathfrak{s}_1| + |{}_q\mathcal{D}^2 \mathfrak{s}_2 - {}_q\mathcal{D}^2 \mathfrak{s}_2| \right], \\ & \left| \mathbb{G}_2(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathbb{G}_2(z, \mathfrak{s}_1, \mathfrak{s}_2) \right| = \left| \frac{\sqrt{e^z + 1} (|\mathfrak{s}_2|)}{(1 + |\mathfrak{s}_2|)} - \frac{\sqrt{e^z + 1} (|\mathfrak{s}_2|)}{(1 + |\mathfrak{s}_2|)} \right| \\ &\leq \sqrt{e + 1} \left[|\mathfrak{s}_1 - \mathfrak{s}_1| + |\mathfrak{s}_2 - \mathfrak{s}_2| \right]. \end{aligned}$$

These mean that, the condition (H_1) holds with ${}_v\zeta_1 = \frac{1}{e^{\sqrt{2}}}$, ${}_v\zeta_2 = \frac{1}{2}$, ${}_h\zeta_1 = \frac{1}{\pi}$, ${}_h\zeta_2 = \frac{1}{\ln(3)}$, ${}_g\zeta_1 = e$, ${}_g\zeta_2 = \sqrt{e + 1}$, and

$$\Delta_1 \approx \begin{cases} 0.1624, & q = 1/2, \\ 0.1624, & q = 7/11, \\ 0.1624, & q = 12/13, \end{cases} \quad \Delta_2 \approx \begin{cases} 0.0953, & q = 1/2, \\ 0.0953, & q = 7/11, \\ 0.0953, & q = 12/13. \end{cases}$$

By this data, from relation in (13), we find that

$$\tilde{M}_1 \approx \begin{cases} 0.1666, & q = 1/2, \\ 0.0749, & q = 7/11, \\ 0.0061, & q = 12/13, \end{cases} \quad \tilde{M}_2 \approx \begin{cases} 0.2450, & q = 1/2, \\ 0.1258, & q = 7/11, \\ 0.0137, & q = 12/13, \end{cases}$$

$$\begin{aligned} \tilde{L}_1 &\approx \begin{cases} 0.5973, & q = 1/2, \\ 0.4115, & q = 7/11, \\ 0.0851, & q = 12/13, \end{cases} & \tilde{L}_2 &\approx \begin{cases} 1.3702, & q = 1/2, \\ 1.2823, & q = 7/11, \\ 0.5204, & q = 12/13, \end{cases} \\ \tilde{N}_1 &\approx \begin{cases} 1.5452, & q = 1/2, \\ 1.3499, & q = 7/11, \\ 0.3983, & q = 12/13, \end{cases} & \tilde{N}_2 &\approx \begin{cases} 3.1041, & q = 1/2, \\ 3.2151, & q = 7/11, \\ 1.3221, & q = 12/13, \end{cases} \end{aligned}$$

and

$$K_1 \approx \begin{cases} 2.3092, & q = 1/2, \\ 1.8364, & q = 7/11, \\ 0.5271, & q = 12/13, \end{cases} \quad K_2 \approx \begin{cases} 4.7194, & q = 1/2, \\ 4.6233, & q = 7/11, \\ 1.9762, & q = 12/13. \end{cases}$$

One can check the estimated results of $\tilde{M}_i$, $\tilde{L}_i$, $\tilde{N}_i$, k_i , $i = 1, 2$ and $\Delta_1 K_1 + \Delta_2 K_2 < 1$ for the system (26) in Tables 1 to 3. When as q approaches to 1, parameters $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{N}_1$, $\tilde{N}_2$ decrease. The curves of these data are plotted in Figures 1a and 1b. Consequently, the following inequalities hold:

$$\sum_{i=1}^2 K_i \Delta_i \approx \begin{cases} 0.8249, & q = 1/2, \\ 0.7390, & q = 7/11, \\ 0.2739, & q = 12/13. \end{cases} < 1. \quad (27)$$

The curves in Figure 2 define well that the condition $K_1 \Delta_1 + K_2 \Delta_2 < 1$ is met. The reader remarks that the conditions of Theorem 3.3 are satisfied. Hence, the system (26) has a unique solution on $\bar{\Omega}$. If we consider the case

$$C_\alpha = \frac{K_1 + K_2}{1 - (K_1 \Delta_1 + K_2 \Delta_2)} \approx \begin{cases} 40.1592, & q = 1/2, \\ 24.7520, & q = 7/11, \\ 3.4767, & q = 12/13. \end{cases}$$

in the above example, then the solution $(\mathfrak{s}_1(z), \mathfrak{s}_2(z))$ is UHML stable.

In the next example, we explore the mentioned theorems for changing the derivative order α_1 .

Example 5.2. Consider the system of q -FDEs of the form

$$\begin{cases} {}^{\text{RL}}_{2/5} \mathcal{D}^{\alpha_1} \mathfrak{s}_1(z) = \frac{|\sin({}_{2/5} \mathcal{D}^4 \mathfrak{s}_1(z))|}{\sqrt{105} e^{z+1} (1 + |{}_{2/5} \mathcal{D}^4 \mathfrak{s}_1(z)|)} \\ \quad + \frac{|z| (|{}_{2/5} \mathcal{D}^2 \mathfrak{s}_1(z)| + |{}_{2/5} \mathcal{D}^2 \mathfrak{s}_2(z)|)}{2\sqrt{5}\pi(|z|+12)} + \frac{e^{|z|} (|\mathfrak{s}_1(z)| + \tan^{-1} |\mathfrak{s}_2(z)|)}{39(1+|z|)}, \\ {}^{\text{RL}}_{2/5} \mathcal{D}^{23/4} \mathfrak{s}_2(z) = \frac{12z \tan({}_{2/5} \mathcal{D}^4 \mathfrak{s}_2(z))}{251(\sqrt{3}-z)} + \frac{\sqrt{z} (|{}_{2/5} \mathcal{D}^2 \mathfrak{s}_1(z)|)}{49 \ln(e^z + 2)} \\ \quad + \frac{\ln(z+1) |\mathfrak{s}_1(z)| |\mathfrak{s}_2(z)|}{25(1+|\mathfrak{s}_2(z)|)(4+|\mathfrak{s}_1(z)|)}, \end{cases} \quad (28)$$

for $z \in \bar{\Omega}$, and for $m = \overline{0, 4}$, $({}_{2/5} \mathcal{D}^m \mathfrak{s}_1)(0) = ({}_{2/5} \mathcal{D}^m \mathfrak{s}_2)(0) = 0$, via conditions

$$({}_{2/5} \mathcal{D}^4 \mathfrak{s}_1)(1) = \frac{2}{89} [\alpha_1]_{2/5} \int_0^1 \mathfrak{s}(\omega) d_{2/5} \omega,$$

Table 1: Estimated values for $\tilde{M}_i, \tilde{L}_i, i = 1, 2$ for system (26) with three values of q in Example 5.1.

n	$q = 1/2$				$q = 7/11$				$q = 12/13$			
	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$
1	0.130	0.192	0.470	1.079	0.039	0.066	0.219	0.686	0.000	0.000	0.001	0.006
2	0.148	0.218	0.531	1.219	0.050	0.085	0.280	0.875	0.000	0.000	0.002	0.012
3	0.157	0.231	0.564	1.293	0.059	0.099	0.324	1.011	0.000	0.000	0.003	0.019
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	0.166	0.244	0.595	1.365	0.072	0.121	0.396	1.235	0.001	0.002	0.010	0.067
8	0.166	0.245	0.596	1.368	0.073	0.123	0.402	1.252	0.001	0.002	0.013	0.083
9	0.167	0.245	0.597	1.369	0.074	0.124	0.405	1.263	0.001	0.003	0.016	0.099
10	0.167	0.245	0.597	1.370	0.074	0.125	0.408	1.270	0.001	0.003	0.018	0.117
11	0.167	0.245	0.597	1.370	0.074	0.125	0.409	1.275	0.001	0.003	0.021	0.134
12	0.167	0.245	0.597	1.370	0.075	0.125	0.410	1.277	0.002	0.004	0.024	0.152
13	0.167	0.245	0.597	1.370	0.075	0.126	0.411	1.279	0.002	0.004	0.027	0.170
14	0.167	0.245	0.597	1.370	0.075	0.126	0.411	1.280	0.002	0.005	0.030	0.188
15	0.167	0.245	0.597	1.370	0.075	0.126	0.411	1.281	0.002	0.005	0.033	0.206
16	0.167	0.245	0.597	1.370	0.075	0.126	0.411	1.282	0.002	0.006	0.036	0.223
17	0.167	0.245	0.597	1.370	0.075	0.126	0.411	1.282	0.003	0.006	0.039	0.240
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
41	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.005	0.012	0.077	0.470
42	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.013	0.077	0.473
43	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.013	0.078	0.477
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
59	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.013	0.083	0.508
60	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.083	0.509
61	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.083	0.510
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
73	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.084	0.517
74	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.517
75	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.517
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
89	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.519
90	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.520
91	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.520
92	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.520
93	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.520
94	0.167	0.245	0.597	1.370	0.075	0.126	0.412	1.282	0.006	0.014	0.085	0.520

$$\left({}_{2/5}\mathcal{D}^4\mathfrak{s}_2\right)(1) = \frac{15}{163} \left[\frac{23}{4}\right]_{2/5} \int_0^1 \mathfrak{s}_2(\varpi) d_{2/5}\varpi,$$

with three values for $\alpha_1 = \frac{51}{10}, \frac{27}{5}, \frac{57}{10}$. For this example, we have

$$\mathfrak{v}_1(z, {}_q\mathcal{D}^4\mathfrak{s}_1(z), {}_q\mathcal{D}^4\mathfrak{s}_2(z)) = \frac{\left|\sin\left({}_{2/5}\mathcal{D}^4\mathfrak{s}_1(z)\right)\right|}{e^{z+1}\left(1+\left|{}_{2/5}\mathcal{D}^4\mathfrak{s}_1(z)\right|\right)},$$

Table 2: Estimated values for $\tilde{N}_i$, $i = 1, 2$ for system (26) with three values of q in Example 5.1.

n	$q = 1/2$		$q = 7/11$		$q = 12/13$	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_1$	$\tilde{N}_2$
1	1.215	2.444	0.717	1.720	0.005	0.017
2	1.374	2.762	0.917	2.194	0.009	0.032
3	1.458	2.930	1.062	2.536	0.015	0.053
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	1.544	3.101	1.329	3.167	0.081	0.278
10	<u>1.545</u>	3.103	1.337	3.184	0.095	0.326
11	1.545	3.103	1.342	3.195	0.110	0.375
12	1.545	<u>3.104</u>	1.345	3.203	0.125	0.425
13	1.545	3.104	1.347	3.207	0.140	0.475
14	1.545	3.104	1.348	3.210	0.155	0.525
15	1.545	3.104	1.349	3.212	0.170	0.575
16	1.545	3.104	1.349	3.213	0.184	0.623
17	1.545	3.104	1.349	3.214	0.199	0.670
18	1.545	3.104	<u>1.350</u>	3.214	0.213	0.716
19	1.545	3.104	1.350	<u>3.215</u>	0.226	0.760
20	1.545	3.104	1.350	3.215	0.239	0.803
21	1.545	3.104	1.350	3.215	0.251	0.844
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
92	1.545	3.104	1.350	3.215	0.438	1.452
93	1.545	3.104	1.350	3.215	0.438	1.452
94	1.545	3.104	1.350	3.215	<u>0.439</u>	1.452
95	1.545	3.104	1.350	3.215	0.439	1.452
96	1.545	3.104	1.350	3.215	0.439	1.452
97	1.545	3.104	1.350	3.215	0.439	1.452
98	1.545	3.104	1.350	3.215	0.439	<u>1.453</u>
99	1.545	3.104	1.350	3.215	0.439	1.453

$$\begin{aligned} \mathfrak{h}_1(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) &= \frac{|z| \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| + \left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2(z) \right| \right)}{(|z|+12)}, \\ \mathfrak{g}_1(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) &= \frac{e^{|z|} (|\mathfrak{s}_1(z)| + \tan^{-1} |\mathfrak{s}_2(z)|)}{1+|z|}, \\ \mathfrak{v}_2(z, {}_q\mathcal{D}^4 \mathfrak{s}_1(z), {}_q\mathcal{D}^4 \mathfrak{s}_2(z)) &= \frac{z \tan({}_{2/5}\mathcal{D}^4 \mathfrak{s}_2(z))}{(\sqrt{3}-z)}, \\ \mathfrak{h}_2(z, {}_q\mathcal{D}^2 \mathfrak{s}_1(z), {}_q\mathcal{D}^2 \mathfrak{s}_2(z)) &= \frac{\sqrt{z} \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| \right)}{\ln(e^z+2)}, \end{aligned}$$

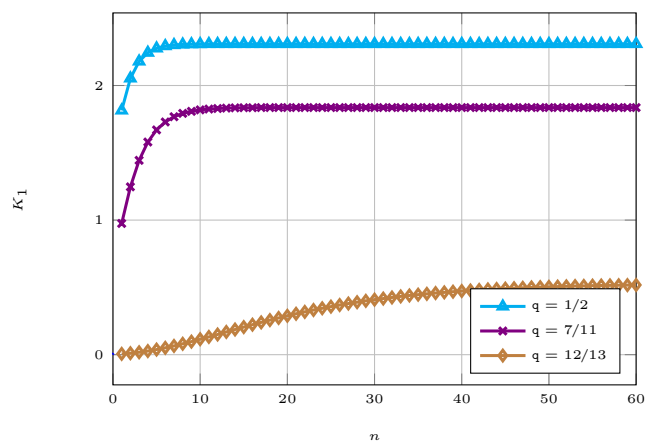
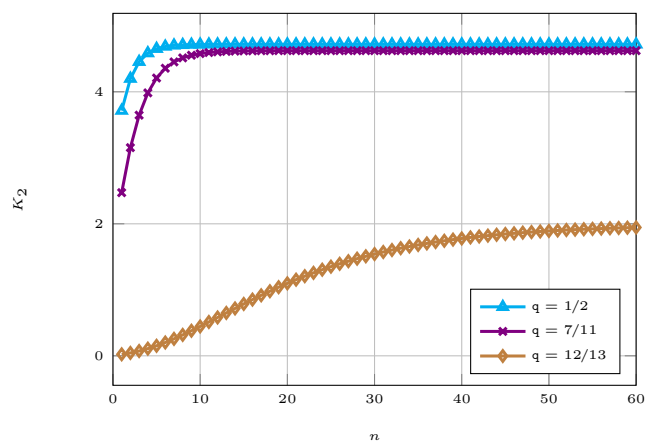

 (a) K_1

 (b) K_2

 Figure 1: The curves of K_i , $i = 1, 2$ for system (26) with three values of q in Example 5.1.

$$\mathfrak{G}_2(z, \mathfrak{s}_1(z), \mathfrak{s}_2(z)) = \frac{\ln(z+1)|\mathfrak{s}_1(z)||\mathfrak{s}_2(z)|}{(1+|\mathfrak{s}_2(z)|)(4+|\mathfrak{s}_1(z)|)},$$

and $\lambda_1 = \frac{1}{\sqrt{105}}$, $\lambda_2 = \frac{12}{251}$, $\mu_1 = \frac{1}{2\sqrt{5}\pi}$, $\mu_2 = \frac{1}{49}$, $\delta_1 = \frac{1}{39}$, $\delta_2 = \frac{1}{25}$. Thus,

$$\begin{aligned} & \left| \mathfrak{V}_1 \left(z, {}_{2/5}\mathcal{D}^4 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^4 \mathfrak{s}_2 \right) - \mathfrak{V}_1 \left(z, {}_{2/5}\mathcal{D}^4 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^4 \mathfrak{s}_2 \right) \right| \\ &= \left| \frac{|\sin({}_{2/5}\mathcal{D}^4 \mathfrak{s}_1(z))|}{e^{z+1}(1+|{}_{2/5}\mathcal{D}^4 \mathfrak{s}_1(z)|)} - \frac{|\sin({}_{2/5}\mathcal{D}^4 \mathfrak{s}_1(z))|}{e^{z+1}(1+|{}_{2/5}\mathcal{D}^4 \mathfrak{s}_1(z)|)} \right| \\ &\leq \frac{1}{e} \left| {}_{2/5}\mathcal{D}^4 \mathfrak{s}_1 - {}_{2/5}\mathcal{D}^4 \mathfrak{s}_1 \right|, \end{aligned}$$

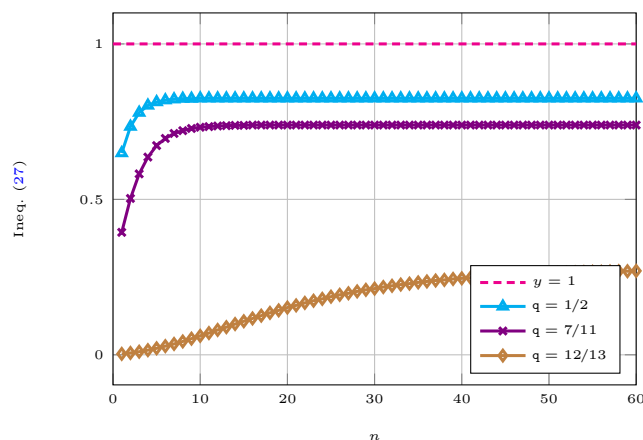


Figure 2: Inequality (27), $\sum_{i=1}^2 K_i \Delta_i < 1$ for system (26) in Example 5.1 with three values of q .

$$\begin{aligned}
 & \left| \mathbb{h}_1 \left(z, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right) - \mathbb{h}_1 \left(z, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right) \right| \\
 &= \left| \frac{|z| \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| + \left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2(z) \right| \right)}{|z|+12} - \frac{|z| \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| + \left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2(z) \right| \right)}{|z|+12} \right| \\
 &\leq \frac{1}{12} \left[\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1 - {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1 \right| + \left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 - {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right| \right],
 \end{aligned}$$

$$\begin{aligned}
 & \left| \mathbb{G}_1(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathbb{G}_1(z, \mathfrak{s}_1, \mathfrak{s}_2) \right| = \left| \frac{e^{|z|} \left(|\mathfrak{s}_1(z)| + \tan^{-1} |\mathfrak{s}_2(z)| \right)}{1+|z|} - \frac{e^{|z|} \left(|\mathfrak{s}_1(z)| + \tan^{-1} |\mathfrak{s}_2(z)| \right)}{1+|z|} \right| \\
 &\leq e \left[|\mathfrak{s}_1 - \mathfrak{s}_1| + |\mathfrak{s}_2 - \mathfrak{s}_2| \right],
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \mathbb{v}_2 \left(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2 \right) - \mathbb{v}_2 \left(z, {}_q\mathcal{D}^4 \mathfrak{s}_1, {}_q\mathcal{D}^4 \mathfrak{s}_2 \right) \right| \\
 &= \left| \frac{z \tan \left({}_{2/5}\mathcal{D}^4 \mathfrak{s}_2(z) \right)}{\sqrt{3}-z} - \frac{z \tan \left({}_{2/5}\mathcal{D}^4 \mathfrak{s}_2(z) \right)}{\sqrt{3}-z} \right| \\
 &\leq \frac{1}{\sqrt{3}-1} \left| {}_{2/5}\mathcal{D}^4 \mathfrak{s}_2 - {}_{2/5}\mathcal{D}^4 \mathfrak{s}_2 \right|, \\
 & \left| \mathbb{h}_2 \left(z, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right) - \mathbb{h}_2 \left(z, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1, {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right) \right| \\
 &= \left| \frac{\sqrt{z} \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| \right)}{\ln(e^z+2)} - \frac{\sqrt{z} \left(\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1(z) \right| \right)}{\ln(e^z+2)} \right| \\
 &\leq \frac{1}{\ln(3)} \left[\left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1 - {}_{2/5}\mathcal{D}^2 \mathfrak{s}_1 \right| + \left| {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 - {}_{2/5}\mathcal{D}^2 \mathfrak{s}_2 \right| \right], \\
 & \left| \mathbb{G}_2(z, \mathfrak{s}_1, \mathfrak{s}_2) - \mathbb{G}_2(z, \mathfrak{s}_1, \mathfrak{s}_2) \right|
 \end{aligned}$$

Table 3: Estimated values for K_i , $i = 1, 2$ and $K_1\Delta_1 + K_2\Delta_2 < 1$ for system (26) with three values of q in Example 5.1.

n	$q = 1/2$			$q = 7/11$			$q = 12/13$		
	K_1	K_2	$\sum_{i=1}^2 K_i \Delta_i$	K_1	K_2	$\sum_{i=1}^2 K_i \Delta_i$	K_1	K_2	$\sum_{i=1}^2 K_i \Delta_i$
1	1.815	3.715	0.649	0.975	2.472	0.394	0.006	0.023	0.003
2	2.053	4.199	0.734	1.247	3.154	0.503	0.011	0.044	0.006
3	2.179	4.455	0.779	1.444	3.646	0.582	0.018	0.073	0.010
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	2.307	4.715	0.824	1.808	4.554	0.728	0.097	0.380	0.052
10	2.308	4.717	<u>0.825</u>	1.819	4.579	0.732	0.115	0.445	0.061
11	<u>2.309</u>	4.718	0.825	1.825	4.595	0.734	0.132	0.513	0.070
12	2.309	<u>4.719</u>	0.825	1.829	4.605	0.736	0.150	0.581	0.080
13	2.309	4.719	0.825	1.832	4.612	0.737	0.169	0.650	0.089
14	2.309	4.719	0.825	1.834	4.616	0.738	0.187	0.718	0.099
15	2.309	4.719	0.825	1.835	4.619	0.738	0.205	0.786	0.108
16	2.309	4.719	0.825	1.835	4.620	<u>0.739</u>	0.223	0.852	0.117
17	2.309	4.719	0.825	<u>1.836</u>	4.622	0.739	0.240	0.917	0.126
18	2.309	4.719	0.825	1.836	4.622	0.739	0.257	0.979	0.135
19	2.309	4.719	0.825	1.836	<u>4.623</u>	0.739	0.273	1.040	0.143
20	2.309	4.719	0.825	1.836	4.623	0.739	0.289	1.098	0.152
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
76	2.309	4.719	0.825	1.836	4.623	0.739	0.527	1.976	0.274
77	2.309	4.719	0.825	1.836	4.623	0.739	0.527	1.977	0.274
78	2.309	4.719	0.825	1.836	4.623	0.739	0.528	1.978	0.274
79	2.309	4.719	0.825	1.836	4.623	0.739	0.528	1.979	0.274
80	2.309	4.719	0.825	1.836	4.623	0.739	0.528	1.980	0.274
81	2.309	4.719	0.825	1.836	4.623	0.739	0.528	1.980	<u>0.275</u>
82	2.309	4.719	0.825	1.836	4.623	0.739	0.528	1.981	0.275
83	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.982	0.275
84	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.982	0.275
85	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.983	0.275
86	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.983	0.275
87	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.983	0.275
88	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.984	0.275
89	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.984	0.275
90	2.309	4.719	0.825	1.836	4.623	0.739	0.529	<u>1.985</u>	0.275
91	2.309	4.719	0.825	1.836	4.623	0.739	0.529	1.985	0.275
92	2.309	4.719	0.825	1.836	4.623	0.739	<u>0.530</u>	1.985	0.275
93	2.309	4.719	0.825	1.836	4.623	0.739	0.530	1.985	0.275

$$\begin{aligned}
&= \left| \frac{\ln(z+1)|\mathfrak{s}_1(z)||\mathfrak{s}_2(z)|}{(1+|\mathfrak{s}_2(z)|)(4+|\mathfrak{s}_1(z)|)} - \frac{\ln(z+1)|\mathfrak{s}_1(z)||\mathfrak{s}_2(z)|}{(1+|\mathfrak{s}_2(z)|)(4+|\mathfrak{s}_1(z)|)} \right| \\
&\leq \ln 2 [|\mathfrak{s}_1 - \mathfrak{s}_1| + |\mathfrak{s}_2 - \mathfrak{s}_2|].
\end{aligned}$$

These mean that, the condition (H_1) holds with ${}_{\mathfrak{v}}\zeta_1 = \frac{1}{e}$, ${}_{\mathfrak{v}}\zeta_2 = \frac{1}{\sqrt{3}-1}$, ${}_{\mathfrak{h}}\zeta_1 = \frac{1}{12}$, ${}_{\mathfrak{h}}\zeta_2 = \frac{1}{\ln(3)}$, ${}_{\mathfrak{g}}\zeta_1 = e$, ${}_{\mathfrak{g}}\zeta_2 = 4\ln(2)$, and $\Delta_1 \approx 0.1161$, $\Delta_2 \approx 0.1116$ for each α_1 .

Hence, thanks to the equations in (13), we obtain

$$\begin{aligned} \tilde{M}_1 &\approx \begin{cases} 0.411, & \alpha_1 = 51/10, \\ 0.329, & \alpha_1 = 27/5, \\ 0.269, & \alpha_1 = 57/10. \end{cases} & \tilde{M}_2 &\approx \begin{cases} 0.403, & \alpha_1 = 51/10, \\ 0.367, & \alpha_1 = 27/5, \\ 0.345, & \alpha_1 = 57/10. \end{cases} \\ \tilde{L}_1 &\approx \begin{cases} 1.065, & \alpha_1 = 51/10, \\ 0.867, & \alpha_1 = 27/5, \\ 0.720, & \alpha_1 = 57/10. \end{cases} & \tilde{L}_2 &\approx \begin{cases} 1.496, & \alpha_1 = 51/10, \\ 1.369, & \alpha_1 = 27/5, \\ 1.292, & \alpha_1 = 57/10. \end{cases} \\ \tilde{N}_1 &\approx \begin{cases} 1.915, & \alpha_1 = 51/10, \\ 1.761, & \alpha_1 = 27/5, \\ 1.591, & \alpha_1 = 57/10. \end{cases} & \tilde{N}_2 &\approx \begin{cases} 2.535, & \alpha_1 = 51/10, \\ 2.686, & \alpha_1 = 27/5, \\ 2.805, & \alpha_1 = 57/10. \end{cases} \end{aligned}$$

and

Table 4: Estimated values for $\tilde{M}_i, \tilde{L}_i, i = 1, 2$ for system (28) with three values of α_1 in Example 5.2.

n	$\alpha_1 = 51/10$				$\alpha_1 = 27/5$				$\alpha_1 = 57/10$			
	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$	$\tilde{M}_1$	$\tilde{M}_2$	$\tilde{L}_1$	$\tilde{L}_2$
1	0.370	0.362	0.961	1.346	0.296	0.330	0.781	1.232	0.242	0.310	0.648	1.163
2	0.394	0.387	1.023	1.435	0.315	0.352	0.832	1.313	0.258	0.331	0.691	1.240
3	0.404	0.396	1.048	1.472	0.323	0.361	0.853	1.347	0.265	0.339	0.708	1.271
4	0.408	0.400	1.058	1.486	0.326	0.365	0.861	1.360	0.268	0.343	0.715	1.284
5	0.410	0.402	1.062	1.492	0.328	0.366	0.865	1.366	0.269	0.344	0.718	1.289
6	0.411	0.403	1.064	1.495	0.328	0.367	0.866	1.368	0.269	0.345	0.719	1.291
7	0.411	0.403	1.065	1.496	0.328	0.367	0.867	1.369	0.269	0.345	0.720	1.292
8	0.411	0.403	1.065	1.496	0.329	0.367	0.867	1.369	0.269	0.345	0.720	1.292
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$K_1 \approx \begin{cases} 3.391, & \alpha_1 = 51/10, \\ 2.957, & \alpha_1 = 27/5, \\ 2.581, & \alpha_1 = 57/10. \end{cases} \quad K_2 \approx \begin{cases} 4.434, & \alpha_1 = 51/10, \\ 4.423, & \alpha_1 = 27/5, \\ 4.443, & \alpha_1 = 57/10. \end{cases}$$

Tables 4 to 6 show the estimated results of $\tilde{M}_i, \tilde{L}_i, \tilde{N}_i, k_i, i = 1, 2$ and $\Delta_1 K_1 + \Delta_2 K_2 < 1$ for the system (26). When as q approaches to 1, parameters $\tilde{M}_1, \tilde{M}_2, \tilde{L}_1, \tilde{L}_2, \tilde{N}_1, \tilde{N}_2$ decrease. The curves of these data are plotted in Figures 3a and 3b. Consequently, the following inequalities hold:

$$\sum_{i=1}^2 K_i \Delta_i \approx \begin{cases} 0.889, & \alpha_1 = 51/10, \\ 0.837, & \alpha_1 = 27/5, \\ 0.796, & \alpha_1 = 57/10, \end{cases} < 1. \quad (29)$$

The curves in Figure 4 define well that the condition $K_1 \Delta_1 + K_2 \Delta_2 < 1$ is met. The reader remarks that the conditions of Theorem 3.3 are satisfied. Hence, system (28) has a unique solution on Ω . If we consider the case

Table 5: Estimated values for $\tilde{N}_i$, $i = 1, 2$ for system (28) with three values of α_1 in Example 5.2.

n	$\alpha_1 = 51/10$		$\alpha_1 = 27/5$		$\alpha_1 = 57/10$	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_1$	$\tilde{N}_2$
1	1.727	2.281	1.586	2.417	1.432	2.524
2	1.839	2.432	1.690	2.577	1.526	2.691
3	1.884	2.494	1.733	2.642	1.565	2.759
4	1.902	2.519	1.750	2.669	1.581	2.787
5	1.910	2.529	1.757	2.679	1.587	2.798
6	1.913	2.533	1.759	2.684	1.590	2.802
7	1.914	2.534	1.760	2.685	1.591	2.804
8	1.914	2.535	1.761	2.686	1.591	2.805
9	1.915	2.535	1.761	2.686	1.591	2.805
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 6: Estimated values for K_i , $i = 1, 2$ and $K_1\Delta_1 + K_2\Delta_2 < 1$ for system (28) with three values of $\mathbf{q}$ in Example 5.2.

n	$\mathbf{q} = 1/2$			$\mathbf{q} = 7/11$			$\mathbf{q} = 12/13$		
	K_1	K_2	$\sum_{i=1}^2 K_i\Delta_i$	K_1	K_2	$\sum_{i=1}^2 K_i\Delta_i$	K_1	K_2	$\sum_{i=1}^2 K_i\Delta_i$
1	3.058	3.989	0.800	2.663	3.979	0.753	2.322	3.997	0.716
2	3.256	4.253	0.853	2.837	4.242	0.803	2.476	4.261	0.763
3	3.336	4.362	0.874	2.909	4.350	0.823	2.538	4.370	0.782
4	3.369	4.405	0.883	2.937	4.394	0.831	2.564	4.414	0.790
5	3.382	4.423	0.886	2.949	4.411	0.835	2.574	4.431	0.793
6	3.387	4.430	0.888	2.954	4.418	0.836	2.578	4.438	0.795
7	3.389	4.433	0.888	2.955	4.421	0.837	2.580	4.441	0.795
8	3.390	4.434	0.889	2.956	4.422	0.837	2.580	4.442	0.795
9	3.390	4.434	0.889	2.956	4.423	0.837	2.581	4.443	0.795
10	3.391	4.434	0.889	2.957	4.423	0.837	2.581	4.443	0.796
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	1:	$\vdots$	$\vdots$

$$C_\alpha = \frac{K_1 + K_2}{1 - (K_1\Delta_1 + K_2\Delta_2)} \approx \begin{cases} 70.275, & \alpha_1 = 51/10, \\ 45.257, & \alpha_1 = 27/5, \\ 34.352, & \alpha_1 = 57/10. \end{cases}$$

in the above example, then the solution $(\mathfrak{s}_1(z), \mathfrak{s}_2(z))$ is UHML stable.

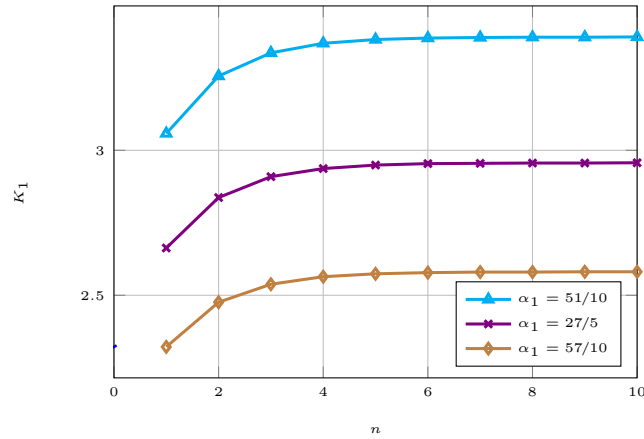
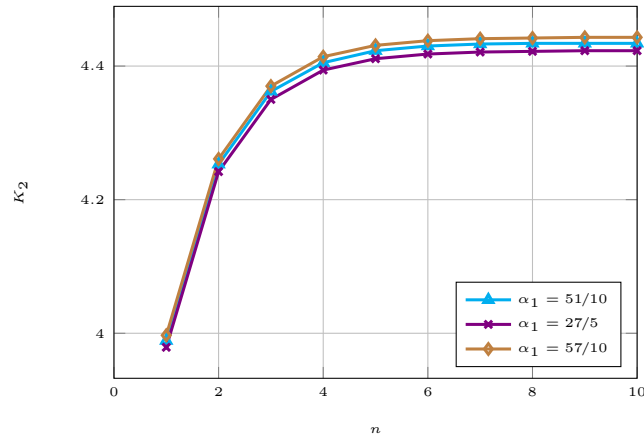
(a) K_1 (b) K_2

Figure 3: The curves of K_i , $i = 1, 2$ for system (28) with three values of α_1 in Example 5.2.

6. Conclusion

This research established new achievements regarding the uniqueness, existence, and stability of solutions for the coupled system of q-FDEs (1) based on famous methods in nonlinear analysis. We also defined and studied the stability of the mentioned coupled system (1). Some remarkable examples were presented, those show well the details of the achievements of the existence and stability of the system solution. Further, the current work in the field of fractional calculus can be used to obtain the solution of the related system in [34–37].

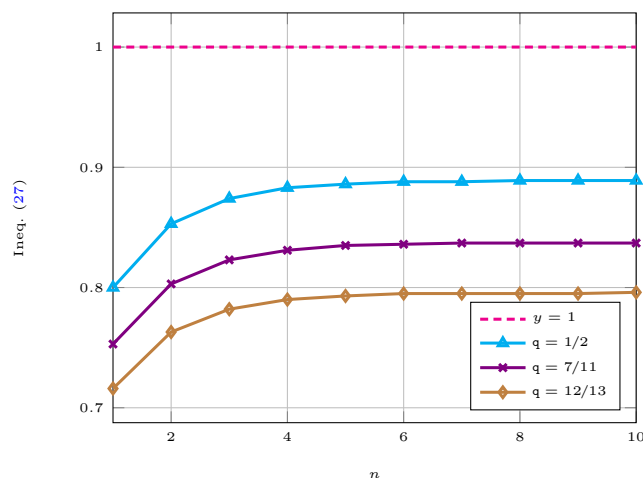


Figure 4: Inequality (27), $\sum_{i=1}^2 K_i \Delta_i < 1$ for system (28) with three values of q in Example 5.2.

Abbreviations. BVP: Boundary value problem; DE: Differential equation; c: Fixed point; FDE: Fractional differential equation; MLU: Mittag-Leffler-Ulam; q-FDE: Quantum fractional differential equation.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

- [1] R. A. C. Ferreira, Positive solutions for a class of boundary value problems with fractional q -differences, *Comput. Math. Appl.* **61** (2011) 367 – 373, <https://doi.org/10.1016/j.camwa.2010.11.012>.
- [2] R. Finkelstein and E. Marcus, Transformation theory of the q -oscillator, *J. Math. Phys.* **36** (1995) 2652 – 2672, <https://doi.org/10.1063/1.531057>.
- [3] P. G. O. Freund and A. V. Zabrodin, The spectral problem for the q -Knizhnik-Zamolodchikov equation and continuous q -Jacobi polynomials, *Commun. Math. Phys.* **173** (1995) 17 – 42, <https://doi.org/10.1007/BF02100180>.
- [4] G. N. Han and J. Zeng, On a q -sequence that generalizes the median Genocchi numbers, *Ann. Sci. Math. Québec* **23** (1999) 63 – 72.

- [5] F. Jarad, T. Abdeljawad and D. Baleanu, Stability of q -fractional non-autonomous systems, *Nonlinear Anal. Real World Appl.* **14** (2013) 780 – 784, <https://doi.org/10.1016/j.nonrwa.2012.08.001>.
- [6] V. Kalvandi and M. E. Samei, New stability results for a sum-type fractional q -integro-differential equation, *J. Adv. Math. Stud.* **12** (2019) 201 – 209.
- [7] S. Rezapour and M. E. Samei, On the existence of solutions for a multi-singular pointwise defined fractional q -integro-differential equation, *Bound. Value Probl.* **2020** (2020) #38, <https://doi.org/10.1186/s13661-020-01342-3>.
- [8] J. Alzabut, M. Houas and M. I. Abbas, Application of fractional quantum calculus on coupled hybrid differential systems within the sequential Caputo fractional q -derivatives, *Demonstr. Math.* **56** (2023) #20220205.
- [9] M. Houas, F. Martínez, M. E. Samei and M. K. A. Kaabar, Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q -differential equations, *J. Inequal. Appl.* **2022** (2022) #93, <https://doi.org/10.1186/s13660-022-02828-7>.
- [10] M. Houas and M. E. Samei, Existence and Mittag-Leffler-Ulam stability results for Duffing type problem involving sequential fractional derivatives, *Int. J. Appl. Comput. Math.* **8** (2022) #185, <https://doi.org/10.1007/s40819-022-01398-y>.
- [11] A. H. Ganie, M. Houas and M. E. Samei, Pantograph system with mixed Riemann-Liouville and Caputo-Hadamard sequential fractional derivatives: existence and ulam-stability, *Math. Interdisc. Res.* **10** (2025) 1 – 33, <https://doi.org/10.22052/mir.2024.254075.1453>.
- [12] M. Houas, M. E. Samei and S. Rezapour, Solvability and stability for a fractional quantum jerk type problem including Riemann-Liouville-Caputo fractional q -derivatives, *Partial Differ. Equ. Appl. Math.* **7** (2023) 100514, <https://doi.org/10.1016/j.padiff.2023.100514>.
- [13] R. P. Agarwal, B. Kovacs and D. O'Regan, Existence of positive solution for a sixth-order differential system with variable parameters, *J. Appl. Math. Comput.* **44** (2014) 437–454, <https://doi.org/https://doi.org/10.1007/s12190-013-0701-1>.
- [14] G. Bonanno and B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equation, *Nonlinear Differ. Equ. Appl.* **18** (2011) 357 – 368, <https://doi.org/10.1007/s00030-011-0099-0>.
- [15] C. P. Danet, Uniqueness in some higher order elliptic boundary value problems in n dimensional domains, *Electron. J. Qual. Theory Differ. Equ.* **2011** (2011) 1 – 12, <https://doi.org/10.14232/ejqtde.2011.1.54>.

-
- [16] C. P. Danet, Existence and uniqueness for a semilinear sixth-order ODE, *Electron. J. Qual. Theory Differ. Equ.* **2022** (2022) 1 – 22, <https://doi.org/10.14232/ejqtde.2022.1.53>.
- [17] J. R. L. Webb, G. Infante and D. Franco, Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **138** (2008) 427 – 446, <https://doi.org/10.1017/S0308210506001041>.
- [18] E. R. Kaufmann, A fourth-order iterative boundary value problem with Lidstone boundary conditions, *Differ. Equ. Appl.* **14** (2022) 305 – 312, <https://doi.org/10.7153/dea-2022-14-21>.
- [19] M. D. R. Grossinho, L. Sanchez and S. A. Tersian, On the solvability of a boundary value problem for a fourth-order ordinary differential equation, *Appl. Math. Lett.* **18** (2005) 439 – 444, <https://doi.org/10.1016/j.aml.2004.03.011>.
- [20] H. Khan, J. Alzabut, D. K. Almutairi, W. K. Alqurashi, S. Pinelas, O. Tunç and M. Athar Azim, A coupled nonlinear system of integro-differential equations using modified ABC operator, *Fractals* **33** (2025) #2540105, <https://doi.org/10.1142/S0218348X2540105X>.
- [21] A. Alsaedi, B. Ahmad and H. Al-Hutami, Nonlinear multi-term impulsive fractional q -difference equations with closed boundary conditions, *Qual. Theory Dyn. Syst.* **23** (2024) #67, <https://doi.org/10.1007/s12346-023-00934-5>.
- [22] A. H. Ganie, M. Houas, F. Mofarreh and S. Rezapour, Existence and UH-Rassias stability for fractional quantum duffing problem with sequential q -fractional derivatives, *New York J. Math.* **30** (2024) 1479 – 1497.
- [23] M. Houas, Existence and stability of solutions for fractional quantum hybrid pantograph problem, *Malaya J. Mat.* **13** (2025) 157 – 168.
- [24] M. Houas and M. E. Samei, Existence and stability of solutions for linear and nonlinear damping of q -fractional Duffing-Rayleigh problem, *Mediterr. J. Math.* **20** (2023) #148.
- [25] M. Houas, M. E. Samei, S. S. Santra and J. Alzabut, On a Duffing-type oscillator differential equation on the transition to chaos with fractional q -derivatives, *J. Inequal. Appl.* **2024** (2024) #12, <https://doi.org/10.1186/s13660-024-03093-6>.
- [26] Z. Liu and S. Sun, Solvability and stability of multi-term fractional delay q -difference equation, *J. Appl. Anal. Comput.* **14** (2024) 1177 – 1197, <https://doi.org/10.11948/20220411>.

- [27] J. Wang, J. Zhang and C. Yu, Hyers-Ulam stability and existence of solutions for high-order fractional q -difference equations on infinite intervals, *J. Appl. Math. Comput.* **69** (2023) 4665 – 4688, <https://doi.org/10.1007/s12190-023-01947-8>.
- [28] A. Moumen, O. Zentar, J. Alzabut H. Saber and T. Alraqad, A coupled system of Langevin-type fractional differential equations involving ψ -Caputo derivatives in arbitrary Banach spaces, *Bound. Value Probl.* **2026** (2026) #103, <https://doi.org/10.1186/s13661-026-02283-z>.
- [29] R. P. Agarwal, Certain fractional q -integrals and q -derivatives, *Mathematical Proc. Cambridge Philos. Soc.* **66** (1969) 365 – 370, <https://doi.org/10.1017/S0305004100045060>.
- [30] M. H. Annaby and Z. S. Mansour, *q -fractional Calculus and Equations, Lecture Notes in Mathematics*, Springer-Verlag Berlin Heidelberg, 2012.
- [31] S. Bermudo, P. Kórus and J. E. Nápoles Valdés, On q -Hermite-Hadamard inequalities for general convex functions, *Acta Math. Hungar.* **162** (2020) 364 – 374, <https://doi.org/10.1007/s10474-020-01025-6>.
- [32] P. Rajković, S. Marinković and M. Stanković, On q -analogues of Caputo derivative and Mittag-Leffler function, *Fract. Calc. Appl. Anal.* **10** (2007) 359 – 373.
- [33] Y. Sheng and T. Zhang, Some results on the q -calculus and fractional q -differential equations, *Mathematics* **10** (2022) #64, <https://doi.org/10.3390/math10010064>.
- [34] H. Bouzid, B. Abdelkader, L. Tabharit and M. E. Samei, Existence of solutions to a fractional differential equation involving the Caputo q -derivative with boundary conditions in Banach spaces, *J. Inequal. Appl.* **2025** (2025) #56, <https://doi.org/10.1186/s13660-025-03302-w>.
- [35] H. R. Sahebi and M. E. Samei, Analysis of the existence of solutions for a general class of equations including the q -Erdélyi-Kober integral, *Beni-Suef Univ. J. Basic Appl. Sci.* **14** (2025) #85, <https://doi.org/10.1186/s43088-025-00672-4>.
- [36] A. Salim, S. T. M. Thabet, A. S. Rafeeq, M. E. Samei, I. Kedim and M. Vivas-Cortez, Exploring the solutions of tempered (κ, ψ) -Hilfer hybrid implicit boundary value problem, *Alex. Eng. J.* **119** (2025) 138 – 148, <https://doi.org/10.1016/j.aej.2025.01.069>.
- [37] A. Berhail, J. Alzabut and M. E. Samei, On multi order nonlinear Langevin type of FDE subject to multi-point boundary conditions, *Qual. Theory Dyn. Syst.* **24** (2025) #58, <https://doi.org/10.1007/s12346-024-01211-9>.

Mohamed Bezziou
Department of Mathematics,
Faculty of Sciences of Matter and Computer Sciences,
Khemis Miliana University, Algeria,
e-mail: m.bezziou@univ-dbk.m.dz

Mohamed Houas
Laboratory, FIMA, UDBKM,
Khemis Miliana University, Algeria
e-mail: m.houas.st@univ-dbk.m.dz

Mohammad Esmail Samei
Department of Mathematics, Faculty of Science,
Bu-Ali Sina University,
Hamedan, I. R. Iran
e-mail: mesamei@basu.ac.ir, mesamei@gmail.com