

A Useful Mathematical Lemma: Shifted Coordinate Transformations for Analytical Modeling of Plasma Multi-Ring Antennas

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Abstract

Transformations of scalar and vector quantities from one Cartesian coordinate system to a shifted Cartesian frame constitute a powerful tool in linear algebra with wide applicability in solving physical problems. In this work, we employ such coordinate transformations to construct a unified analytical formulation for multi-ring plasma antennas. By expressing the vector and scalar potentials in systematically shifted Cartesian coordinates, the electromagnetic fields and radiation impedance of single-ring and multi-ring structures are derived in closed form. The method clarifies how radiation resistance and field distributions depend on ring number, radii, and vertical displacements, while showing that symmetric arrangements lead to notable mathematical simplifications. Because the framework is built upon generalizable coordinate transformations, it naturally extends to a broad class of antenna geometries composed of repetitive or modular elements. This highlights the strength of the mathematical approach: many complex antenna configurations become analytically tractable once reformulated through shifted Cartesian transformation techniques.

Keywords: Plasma antenna, Multi-ring structure, Far-field radiation, Radiation impedance, Vector potential formulation, Cartesian coordinate transformation, Electromagnetic wave propagation, Telescopic antenna design

2020 Mathematics Subject Classification: 78A50, 78A40.

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Academic Editor: Behrouz Mirza
Received 27 November 2025, Accepted 16 February 2026
DOI: 10.22052/mir.2026.258025.1557

How to cite this article

T. Taheri and B. Jazi, A useful mathematical lemma: shifted coordinate transformations for analytical modeling of plasma multi-ring antennas, *Math. Interdisc. Res.* x (x) (202x) xx-yy.

1. Introduction

A key factor in creating radiation in an antenna is the generation of fluctuations in the surface and volumetric current densities that occur within the antenna. As we know, extensive research has recently been conducted on the characteristics of thin two-dimensional plasma antennas in the shape of circles or straight rods [1–5]. One of the fundamental parameters in antenna studies is the radiation resistance and the radiation pattern, both of which are highly dependent on the antenna's shape, operating frequency, and the materials used in the antenna's structure [6–9]. Since the phase and amplitude of the waves received from an antenna are highly dependent on the geometric shape and the observer's position, it is evident that a repetitive and periodic structure of a particular type of antenna can create an interference process (constructive or destructive interference) at the observer's location [10–13]. In reference [14], the authors examined how this approach is closely related to multilayered planar antennas, as both exploit advanced polarization control and phase manipulation to enhance performance in communication and signal processing applications. This very aspect can have a significant impact on the radiation impedance of periodic multi-segment antenna systems. In other words, the arrangement and number of antennas can create conditions that optimize or minimize the radiation impedance. Extensive experimental research has been conducted on two-dimensional Archimedean antennas and telescopic antennas, demonstrating that these antennas exhibit relatively high radiative power in specific regions of space [14–17]. In this research, as we will see, by employing transformations in two parallel Cartesian coordinate systems, a generalized relation to the Cartesian components of physical quantities such as the vector potential will be provided for the far-field regions resulting from multi-segment structures used as antennas. It should be emphasized that one of the main design ideas for plasma antennas is the development of a thin, three-dimensional plasma antenna in the form of an Archimedean spiral. The primary motivation for investigating and constructing a multi-layer Archimedean plasma antenna is to achieve a desirable energy distribution profile along the antenna's axis of symmetry, arising from the constructive superposition of the rings in each layer. Although the current in each layer is assumed to be in phase, due to the adjustable spacing between the layers, the fields from each layer travel different wave paths before reaching a point along the axis. This can lead to the appearance of peak intensity points along the axis. However, this article does not delve into such tuning details. This paper consists of four sections, with the "Introduction" being presented as the first section. The second section provides a mathematical formalism for the radiation from a thin three-dimensional plasma antenna shaped in cylindrical coordinates as $r' = f(\theta')$

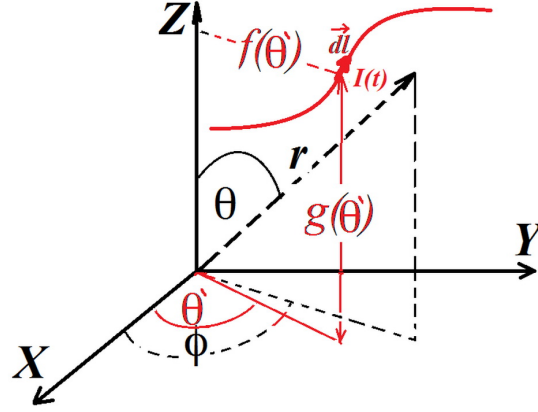


Figure 1: The positioning of a three-dimensional antenna according to the equations $r' = f(\theta')$ and $z' = g(\theta')$ and the selection of an element on it for calculating the vector potential component.

and $z' = g(\theta')$. This is followed by the derivation of the electric E and magnetic H fields generated by a circular antenna. Then, using the properties of vector quantity transformations in Cartesian coordinates, a generalization is made. This generalization applies to the pattern of electric and magnetic fields produced by similar multi-segment structures. The section also examines strategies for calculating radiation resistance. Finally, it presents a relation for the radiation pattern of a multi-ring plasma thin antenna. The third section presents the obtained results. It also includes diagrams showing changes in radiation impedance and variations in the radiation pattern for different configurations. Finally, the fourth section provides a summary and conclusion of the research conducted in this paper.

2. Formalism for three-dimensional plasma antennas and their transformations generalization to multi-segment structures transformations

As Figure 1 illustrates, it is assumed that an alternating current $I(t) = i_m e^{i\omega t}$ with an angular frequency ω is passing through a three-dimensional length element in cylindrical coordinates, which is part of a thin antenna described by the equations $r' = f(\theta')$ and $z' = g(\theta')$ [18, 19]. The observer is typically located at a point like $\vec{r}$ in three-dimensional coordinates. The Cartesian components of the vector potential (specifically just a component, not the entire vector potential) resulting

from a length element $\vec{dl'}$ can simply be derived from the following equation:

$$\vec{dA}(\vec{r}, t) = \frac{\mu_0 i_m e^{i\omega t}}{4\pi} \frac{e^{\frac{-2\pi i}{\lambda_0} [r^2 + f(\theta')^2 - 2r \sin(\theta) f(\theta') \cos(\varphi - \theta') - g(\theta')^2 - 2rg(\theta') \cos(\theta)]^{1/2}}}{[r^2 + f(\theta')^2 - 2r \sin(\theta) f(\theta') \cos(\varphi - \theta') - g(\theta')^2 - 2rg(\theta') \cos(\theta)]^{1/2}} \\ \times \left\{ \begin{aligned} &\left(\cos \theta' \frac{df(\theta')}{d\theta'} - f(\theta') \sin \theta' \right) \hat{x} \\ &+ \left(\sin \theta' \frac{df(\theta')}{d\theta'} - f(\theta') \cos \theta' \right) \hat{y} \\ &+ \frac{dg(\theta')}{d\theta'} \hat{z} \end{aligned} \right\} d\theta'.$$

In the above equation, θ' is a variable, and depending on the range of the curve $r' = f(\theta')$ and $z' = g(\theta')$, integration will be performed over it [19–21].

Additionally, λ_0 represents the wavelength corresponding to the operating frequency ω of the antenna in free space. In the above equation, r , θ and φ correspond to the observer's position and are constant values, while $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in Cartesian coordinates. Since computing and integrating the above relation is challenging even for simple curves such as a simple circle, approximations can be made for regions far from the antenna, which are also suitable for calculating the radiation impedance. Here, for far-field regions and long wavelengths where, $\lambda_0 \gg r \gg \{\max \text{ of } r' = f(\theta') \text{ and } z' = g(\theta')\}$ we can simplify the above equation for integration more conveniently.

$$dA_x = \frac{\mu_0 i_m e^{i\omega t}}{4\pi} [\Psi_1^x - i\Psi_2^x] d\theta', \quad (1)$$

$$dA_y = \frac{\mu_0 i_m e^{i\omega t}}{4\pi} [\Psi_1^y - i\Psi_2^y] d\theta', \quad (2)$$

$$dA_z = \frac{\mu_0 i_m e^{i\omega t}}{4\pi} [\Psi_1^z - i\Psi_2^z] d\theta', \quad (3)$$

where we have introduced

$$\begin{aligned} \Psi_1^x &= \chi_1 \Theta^x(\theta'), & \Psi_2^x &= \frac{2\pi}{\lambda_0} r^2 \chi_1 \chi_2 \Theta^x(\theta'), \\ \Psi_1^y &= \chi_1 \Theta^y(\theta'), & \Psi_2^y &= \frac{2\pi}{\lambda_0} r^2 \chi_1 \chi_2 \Theta^y(\theta'), \\ \Psi_1^z &= \chi_1 \Theta^z(\theta'), & \Psi_2^z &= \frac{2\pi}{\lambda_0} r^2 \chi_1 \chi_2 \Theta^z(\theta'), \\ \chi_1 &= \frac{1}{r} + \frac{f(\theta')}{r^2} \sin(\theta) \cos(\varphi - \theta') + \frac{g(\theta')}{r^2} \cos(\theta), \\ \chi_2 &= \frac{1}{r} - \frac{f(\theta')}{r^2} \sin(\theta) \cos(\varphi - \theta') - \frac{g(\theta')}{r^2} \cos(\theta), \end{aligned}$$

$$\begin{aligned}\Theta^x &= \cos(\theta') \frac{df(\theta')}{d\theta'} - f(\theta') \sin(\theta'), \\ \Theta^y &= \sin(\theta') \frac{df(\theta')}{d\theta'} + f(\theta') \cos(\theta'), \\ \Theta^z &= \frac{dg(\theta')}{d\theta'}.\end{aligned}$$

As shown by the set of Equations (1), (2), and (3), as well as the accompanying relations presented below them, the existence of a phase mismatch between the time-dependent function of the current source and the time-dependent function of the vector potential components implicitly originates from the complex nature of terms such as $[\Psi_1^x - i\Psi_2^x]$, $[\Psi_1^y - i\Psi_2^y]$, and $[\Psi_1^z - i\Psi_2^z]$. This is because these complex expressions contain phase information. Each length element of the thin antenna is considered as a small current source. Consequently, just as current-carrying length elements in traditional metallic antennas contribute to generating the electromagnetic vector potential A , we use the same description for the length elements in the thin plasma antenna. However, the strategy in this article is different. It is based on integral formulations and shifted coordinate transformations. It focuses only on far-field behavior. Additionally, solving the differential equations inside the plasma tube (as discussed in the next section) has been deliberately avoided due to the tube's small thickness. As a result, such numerical methods are neither applicable nor helpful in this context. It must also be emphasized that in these numerical methods, knowledge of near-field data is essential. Since the main objective of this article is to estimate the radiation impedance and the far-field radiation pattern and because radiation impedance is inherently linked to the far-field approximation using direct integration methods offers a simpler and more reliable approach. This is especially true if the asymptotic behavior of the vector potential functions is properly considered in the integrals.

By performing calculations for a circular antenna with radius a_0 described by the equation $r' = a_0$ and $z' = g(\theta') = 0$ each of the Cartesian components of the vector potential under the condition $\lambda_0 \gg r \gg \{\max \text{ of } r' = f(\theta') \text{ and } z' = g(\theta')\}$, will be obtained as follows denoted here as the single ring representation

$$\begin{aligned}A_x^{single \text{ ring}}(x, y, z, a_0, t) &= -\frac{\mu_0 i_m a_0^2}{4r^2} \sin(\varphi) \sin(\theta) \left\{ \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \right. \\ &\quad \left. - \frac{r\omega}{c} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right\}.\end{aligned}\quad (4)$$

$$\begin{aligned}A_y^{single \text{ ring}}(x, y, z, a_0, t) &= \frac{\mu_0 i_m a_0^2}{4r^2} \cos(\varphi) \sin(\theta) \left\{ \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \right. \\ &\quad \left. - \frac{r\omega}{c} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right\}.\end{aligned}\quad (5)$$

Figure 2 shows a schematic approximation upon which the calculations are based. As illustrated in Figure 2, the maximum occupied dimensions of the antenna are

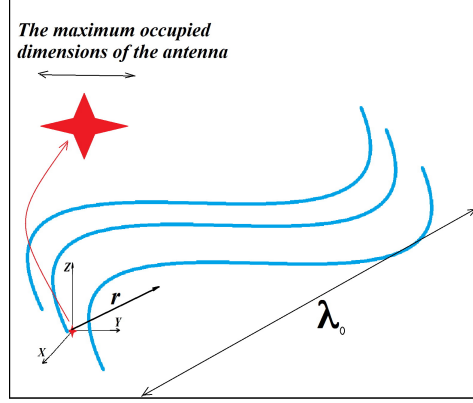


Figure 2: An overview for comparing the antenna dimensions, the observer's distance, and the operating wavelength of the antenna.

significantly smaller than the distance of the observer from the center of the antenna (r). Meanwhile, the system radiates a wavelength that is both greater than the distance to the observer and substantially larger than the occupied dimensions of the antenna. Under the assumption

$$\lambda_0 \gg r \gg \max\{f(\theta'), g(\theta')\},$$

the phase term in the exponential and the denominator of the Green's function can be expanded in powers of $\frac{f(\theta')}{r}$ and $\frac{g(\theta')}{r}$. By retaining only the leading-order terms proportional to $1/r$ and neglecting higher-order contributions, the integral form of the vector potential reduces to a tractable expression. For further details on how to obtain the Cartesian components of the vector potential, and subsequently the electromagnetic scalar potential function, readers are referred to [20, 21]. In these references, the procedure for calculating the electric and magnetic fields within the approximation presented above is also provided. This approximation directly leads to the simplified closed-form results for a circular ring antenna, as given in Equations (4) and (5) [20, 21]. In Equations (4) and (5), each of the coordinates r, θ and φ for the observer in spherical polar coordinates relates to the Cartesian coordinates of the same point, x, y , and z in the following way:

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad (6)$$

$$\sin(\theta) = \left[\frac{x^2 + y^2}{x^2 + y^2 + z^2} \right]^{1/2}, \quad (7)$$

$$\sin(\varphi) = \frac{y}{(x^2 + y^2)^{1/2}}, \quad (8)$$

$$\cos(\varphi) = \frac{x}{(x^2 + y^2)^{1/2}}. \quad (9)$$

One of the advantages of obtaining the components of physical quantities in the Cartesian coordinate system is the ease of applying the superposition principle in this coordinate system. As illustrated in Figure 3-A, if we intend to determine the Cartesian components of the vector potential resulting from two parallel rings in the XYZ coordinates, this can be achieved as described below. In Figure 3-A, a ring in the first level with radius a_1 is located at a distance d_1 from the XOY plane, which we consider as the ground plane. By appropriately choosing a shifted coordinate system x_1, y_1, z_1 and specifying the observer's position relative to these new coordinates, denoted as $\vec{r}_1$, we will be able to estimate the components of the vector potential resulting from this new ring located in the ground plane with respect to the x_1, y_1, z_1 coordinates. As a result, we have:

$$A'_{x_1}(x_1, y_1, z_1, a_1, t) = -\frac{\mu_0 i_m a_1^2}{4r_{(1)}^2} \sin(\varphi_{(1)}) \sin(\theta_{(1)}) \left\{ \cos \left[\omega \left(t - \frac{r_{(1)}}{c} \right) \right] - \frac{r_{(1)}\omega}{c} \sin \left[\omega \left(t - \frac{r_{(1)}}{c} \right) \right] \right\}, \quad (10)$$

$$A'_{y_1}(x_1, y_1, z_1, a_1, t) = \frac{\mu_0 i_m a_1^2}{4r_{(1)}^2} \cos(\varphi_{(1)}) \sin(\theta_{(1)}) \left\{ \cos \left[\omega \left(t - \frac{r_{(1)}}{c} \right) \right] - \frac{r_{(1)}\omega}{c} \sin \left[\omega \left(t - \frac{r_{(1)}}{c} \right) \right] \right\}. \quad (11)$$

In Equations (10) and (11), the observer's position relative to the primed spherical coordinates is specified, and its relation to the Cartesian coordinates x_1, y_1 and z_1 for the same point in the same shifted coordinate system is defined by the following set of equations:

$$r_{(1)} = (x_1^2 + y_1^2 + z_1^2)^{1/2}, \quad \sin(\theta_{(1)}) = \left[\frac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + z_1^2} \right]^{1/2}, \quad (12)$$

$$\sin(\varphi_{(1)}) = \frac{y_1}{(x_1^2 + y_1^2)^{1/2}}, \quad \cos(\varphi_{(1)}) = \frac{x_1}{(x_1^2 + y_1^2)^{1/2}}. \quad (13)$$

Certainly, by employing a series of simple coordinate transformations from the primed coordinate system to the unprimed one, and considering that in Cartesian coordinate systems that are merely shifted in parallel, the unit vectors of the coordinate axes do not change, we can express the potential relationship resulting from two rings in terms of the observer's spatial coordinates relative to the unprimed coordinates. As a result, we would have:

$$\vec{d}_1 + \vec{r}_{(1)} = \vec{r} \implies x_1 = x, \quad y_1 = y, \quad z_1 = z - d_1,$$

$$A_x^{double \ rings}(r, t) = A_x^{single \ ring}(x, y, z, a_0, t)$$

$$\begin{aligned}
& + A_x^{single \ ring}(x, y, z - d_1, a_1, t), \\
A_y^{duble \ rings}(r, t) &= A_y^{single \ ring}(x, y, z, a_0, t) \\
& + A_y^{single \ ring}(x, y, z - d_1, a_1, t)
\end{aligned} \tag{14}$$

In other words, the above equations provide a suitable tool for calculating the components of the vector potential resulting from a multi-rings system (Figure 3-A) arranged in parallel with specified distances d_1 from the ground ring. Below, we observe the presentation of these equations for such a system including m rings.

$$\begin{aligned}
A_x^{multy \ rings}(r, t) &= A_x^{single \ ring}(r, a_0, t) \\
& + \sum_{n=1}^m A_x^{single \ ring}(r_{(n)}, a_n, \theta_n, \varphi_n, t), \\
A_y^{multy \ rings}(r, t) &= A_y^{single \ ring}(r, a_0, t) \\
& + \sum_{n=1}^m A_y^{single \ ring}(r_{(n)}, a_n, \theta_n, \varphi_n, t),
\end{aligned} \tag{15}$$

$$\begin{aligned}
r_{(n)} &= (x^2 + y^2 + (z - d_n)^2)^{1/2}, \quad \sin(\theta_n) = \left[\frac{x^2 + y^2}{x^2 + y^2 + (z - d_n)^2} \right]^{1/2}, \\
\sin(\varphi_n) &= \frac{y_1}{(x_1^2 + y_1^2)^{1/2}} = \frac{y}{(x^2 + y^2)^{1/2}}.
\end{aligned}$$

A set of Equations given in (15) is presented for a total of m parallel rings, as depicted in Figure 3-B, where the distance of each rings from the ground is d_n and their corresponding radii are a_n , relative to the plane on which each ring is situated. As we know, the backbone and validity of the superposition principle for physical quantities for an observer is the linearity of the differential equation obeyed by that quantity [20, 21]. This is true for both vector potential $\vec{A}$ and electromagnetic scalar potential Φ . According to the Lorentz's gauge for electromagnetic wave radiation, it is easy to obtain the electromagnetic scalar potential Φ from the divergence and integration of the vector potential, as follows:

$$\begin{aligned}
\Phi(x, y, z, t)^{multy \ rings} &= -c^2 \int_t \left[\frac{\partial}{\partial x} A_x^{multy \ rings}(x, y, z, t') \right. \\
& \left. + \frac{\partial}{\partial y} A_y^{multy \ rings}(x, y, z, t') \right] dt'.
\end{aligned} \tag{16}$$

Ultimately, each of the equations for the electric field E and the magnetic field H can be derived from the following well-known relationships, and we have:

$$\vec{E}(x, y, z, t)^{multy \ rings} = -\vec{\nabla} \Phi(x, y, z, t)^{multy \ rings}$$

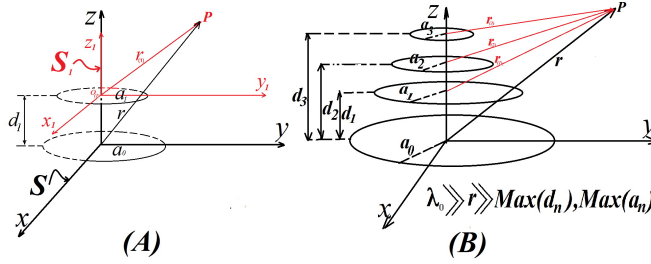


Figure 3: (A): Configuration and arrangement of two parallel circular antennas in S_1 -marked and S -marked coordinate systems. (B): The positioning of multiple circular antennas on the ground floor, first floor, second floor, and third floor, and the arrangement and order of their geometric characteristics.

$$- \frac{\partial}{\partial t} \vec{A}^{multy\ rings}(x, y, z, t), \quad (17)$$

$$\vec{H}(x, y, z, t)^{multy\ rings} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}^{multy\ rings}(x, y, z, t). \quad (18)$$

Here, it is time to take the first step in simulating a multi-ring system. Initially, in the approximations provided in Equations (4) and (5) above, we obtain the electric and magnetic fields of a single ring, which can serve as a generator function for multi-ring systems. As a result, we have:

$$\begin{aligned} \vec{E}^{single\ ring}(r, a_0, t) &= \frac{-i_m a_0^2 \omega^2}{4\epsilon_0 c^3 r} \sin(\theta) \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \times \\ &\quad [-\sin(\varphi)\hat{x} + \cos(\varphi)\hat{y}], \end{aligned} \quad (19)$$

$$\begin{aligned} \vec{H}^{single\ ring}(r, a_0, t) &= \frac{i_m a_0^2 \omega^2}{4c^2 r} \sin(\theta) \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \times \\ &\quad [\cos(\theta) \cos(\varphi)\hat{x} + \cos(\theta) \sin(\varphi)\hat{y} - \sin(\theta)\hat{z}]. \end{aligned} \quad (20)$$

For Equations (19) and (20), the same approximation has been applied. This allows for a simplified evaluation of the expressions while retaining the essential physical behavior. A more detailed derivation, including the step-by-step application of this approximation, is provided in the references [20, 21].

Without a doubt, since the governing differential equations for the electric and magnetic fields are linear, according to the mentioned discussions, we can infer the electric and magnetic fields for a system consisting of m layers of circular ring antennas as follows. However, it should be noted that the following equations are valid only under approximation $\lambda_0 \gg r \gg \{\text{max of } a_n \text{ and max of } d_n\}$:

$$\vec{E}^{multy\ rings}(r, t) = \vec{E}^{single\ ring}(r, a_0, t)$$

$$+ \sum_{n=1}^m \vec{E}^{single\ ring}(r_{(n)}, a_n, \theta_n, \varphi_n, t) \quad (21)$$

$$\begin{aligned} \vec{H}^{multy\ rings}(r, t) &= \vec{H}^{single\ ring}(r, a_0, t) \\ &+ \sum_{n=1}^m \vec{H}^{single\ ring}(r_{(n)}, a_n, \theta_n, \varphi_n, t). \end{aligned} \quad (22)$$

As mentioned in the previous paragraphs, the only condition for applying the superposition principle is the linearity of the governing differential equations of the quantities. Therefore, in order to obtain the Poynting vector S , the only permissible relationship for its application can be calculated as follows:

$$\vec{S}^{multy\ rings}(r, t) = \vec{E}^{multy\ rings}(r, t) \times \vec{H}^{multy\ rings}(r, t). \quad (23)$$

By calculating the flux of the Poynting vector over a mathematical closed sphere with radius R and applying the far-field approximation E and $H \sim (1/r)$ for each of the magnetic and electric fields, and averaging it over one period, the average radiated power by a repetitive and regular multi-element antenna can be estimated as follows:

$$\begin{aligned} \langle P^{multy\ rings} \rangle_t &= \frac{1}{T} \int_0^T \left[\oint \vec{E}^{multy\ rings} \times \vec{H}^{multy\ rings} \cdot \vec{da} \right] dt = \\ &\left\langle \oint \vec{E}^{multy\ rings} \times \vec{H}^{multy\ rings} \cdot \vec{da} \right\rangle_t = \\ &\frac{1}{2} \oint \left[\vec{E}^{multy\ rings} \times \vec{H}^{multy\ rings*} \right] \cdot \vec{da}. \end{aligned} \quad (24)$$

The vector $\vec{da} = r_{far}^2 \sin \theta d\theta d\varphi \hat{r}_0$ in Equation (24) represents the surface element vector perpendicular to the far sphere. Additionally, the symbol $*$ denotes the complex conjugate of the magnetic field function, which was previously introduced in Equation (20). As previously mentioned, since a closed mathematical sphere is considered in the process of calculating the radiation impedance under the far-field approximation, for each of the parameters such as $r_n = (x^2 + y^2 + (z - d_n)^2)^{1/2}$, $\sin(\theta_n) = \left[\frac{x^2 + y^2}{x^2 + y^2 + (z - d_n)^2} \right]^{1/2}$, $\cos(\theta_n) = \left[\frac{(z - d_n)^2}{x^2 + y^2 + (z - d_n)^2} \right]^{1/2}$, and d_n that implicitly appear in Equations (21) and (22) and consequently in Equation (24), we have the following approximation:

$$\sin \theta_n = \frac{d_n}{r} \sin(\theta) \cos(\theta), \quad \cos(\theta_n) = \frac{-d_n}{r} \sin^2(\theta) + \cos(\theta), \quad (25)$$

$$\frac{1}{r_n} \approx \frac{d_n \cos(\theta)}{r^2} + \frac{1}{r}, \quad r_n \approx -d_n \cos(\theta) + r, \quad (26)$$

$$\exp \left[\frac{i\omega}{c} d_n \cos(\theta) \right] = \exp \left[\frac{2\pi i}{\lambda_0} d_n \cos(\theta) \right] \approx \left[1 + \frac{2\pi i}{\lambda_0} d_n \cos(\theta) \right]. \quad (27)$$

In the set of Equations (25), (26) and (27), a_n and d_n represent the radius of the n – th ring located in the n – th layer and the height of the n – th ring from the xy plane, respectively and r_n represents the magnitude of the position vector of the observer relative to the center of the n – th ring in the n – th layer. It should be noted that the a_n values are always positive, whereas the d_n values can be negative or positive, which the negative values indicating rings located below the xy plane. Additionally, it should be noted that in Equation (27), λ_0 represents the wavelength in a vacuum, which corresponds to the operating frequency of the antenna. The approximation $\lambda_0 \gg r \gg \{max\ of\ a_n, d_n\}$ has been made for this scenario. After applying the proposed approximations and retaining the terms of the electric and magnetic fields up to the order of $(\frac{1}{r})$, we equate the average radiated power by the antenna with the average power consumed by a linear resistor R^{rad} as follows:

$$\frac{1}{2} \oint \left[\overrightarrow{E^{multy\ rings}} \times \overrightarrow{H^{multy\ rings*}} \right] \cdot \overrightarrow{da} = \frac{1}{2} R^{rad} i_m^2. \quad (28)$$

The radiation impedance of this multi-element antenna can be estimated as follows:

$$R^{rad} = \frac{\pi^5}{c\epsilon_0\lambda_0^4} \left[\frac{8}{3}a_0^4 + \frac{16}{3}a_0^2 \sum_{n=1}^m a_n^2 + \frac{8}{3} \sum_{l=1}^m \sum_{n=1}^m a_l^2 a_n^2 + \frac{32\pi^2}{15\lambda_0^2} \sum_{l=1}^m \sum_{n=1}^m a_l^2 a_n^2 d_n d_l \right]. \quad (29)$$

In other words, as observed in Equation (29), the radiation resistance for a segmented ring antenna, with varying radii and different heights for each segment, is a function of the geometric dimensions as well as the wavelength λ_0 corresponding to the antenna's operating frequency ω .

Equation (29) provides a generalized expression for calculating the radiation resistance of a regular array of ring antennas arranged in parallel layers with specific spacing and systematically varying radii. It is easily seen that if we wish to verify Equation (29) for the special case where only a single ring exists in the system, we simply eliminate all the radii corresponding to rings 1, 2, 3, and so on. As a result, the well-known relation to the radiation resistance of a single ring can be derived from it as follows [20, 21]

$$R_{single\ ring}^{rad} = 3.07 \times 10^5 \left(\frac{a_0}{\lambda_0} \right)^4, \quad for \quad \lambda_0 \gg a_0.$$

Undoubtedly, if we wish to obtain a wavelength-independent radiation resistance for a single-ring antenna structure to serve as a reference and act as a constant in

subsequent calculations, the ring radius must be considered as a small fraction of the wavelength. Here, we have taken $a_0 = (1/50)\lambda_0$. In this case, the radiation resistance of a single-ring will be $R_{single\ ring}^{rad} = 0.0491\ \Omega$. In the following section, this value will increase for multi-ring systems. An important and noteworthy point to consider when applying Equation (29) is that if it is used for an antenna system arranged symmetrically with respect to the XOY plane, the rings placed below the XOY plane must be assigned negative height values $d_n < 0$. Equation (29) consists of four separate terms, with the fourth term representing the coupled contribution of the antenna ring radii and their placement heights. In the first, second, and third terms of Equation (29), the ring radii ultimately contribute only to the fourth power. Since, in the following section, these radii are considered as multiples of the wavelength, the only wavelength-dependent term is the fourth term of Equation (29). This term vanishes for configurations in which the number of rings is symmetric, and the upper and lower rings are exactly identical.

Up to this point in the article, there has been no discussion on how to drive the antenna current $I(t) = i_m e^{i\omega t}$, and only in specific approximations, for the case where the antenna's longitudinal elements are very thin, the calculations related to the radiation impedance were performed and presented, which are independent of the amplitude of the current i_m passing through the antenna. Here, to continue presenting the theoretical formulation for what will be presented in the next section, we assume that the multi-ring, multi-layer antenna is a very thin plasma antenna tube biased by a high-frequency alternating RF voltage source. Undoubtedly, if we consider the amplitude of the biased RF voltage as V_m , the amplitude of the current i_m passing through the plasma tube can be easily estimated from the following relationship:

$$i_m = \frac{\sigma(\omega, \omega_{pe}, \gamma) A V_m}{2\pi a_0 + \sum_{n=1}^m 2\pi a_n}. \quad (30)$$

In the above equation, $\sigma(\omega, \omega_{pe})$ is the plasma conductivity and A is the cross-sectional area of the glass tube containing the plasma. The entire expression in the denominator of Equation (30), $2\pi a_0 + \sum_{n=1}^m 2\pi a_n$ is considered equivalent to the total length of the plasma tube. Since it is assumed that the gas inside the very thin glass tube is converted into plasma through an AC discharge process and is constantly being charged, undoubtedly the plasma generated under these conditions will fall into the category of weakly ionized plasmas (or partially ionized plasmas). In this situation, the plasma conductivity relationship can be applied as follows [22–24],

$$\sigma(\omega, \omega_{pe}, \gamma) = \frac{\varepsilon_0 \omega_{pe}^2 \gamma}{\omega^2 + \gamma^2} \begin{cases} \text{For } \gamma \gg \omega, & \sigma(\omega, \omega_{pe}, \gamma) = \frac{\varepsilon_0 \omega_{pe}^2}{\gamma}, \\ \text{For } \gamma \ll \omega, & \sigma(\omega, \omega_{pe}, \gamma) = \frac{\gamma \varepsilon_0 \omega_{pe}^2}{\omega^2}. \end{cases} \quad (31)$$

In Equation (31) and its limiting cases, which are presented subsequently, ω_{pe} is the plasma frequency of the electrons given by the following relationship, which is

related to the electron density, n_{0e} , electron mass, m_e , and electron charge e :

$$\omega_{pe} = \sqrt{\frac{n_{0e}e^2}{\varepsilon_0 m_e}}. \quad (32)$$

Also, in Equation (31), ω and γ represent the frequency of the applied signal and the collision frequency of the electron carriers, respectively. Therefore, in summary, it should be noted that by applying a biased RF voltage to a plasma antenna with a specific geometry, the amplitude of the current i_m generated in the plasma tube can be estimated. It should be noted that all relations are derived under the assumption that the plasma tube is very thin, in which case the self-inductance L (in Henry unit) of the plasma tube will be negligible, and only the ohmic resistance of the plasma tube will play a role in determining the current amplitude. Finally, with the amplitude of the current i_m generated in the very thin quasi-Archimedean plasma antenna, and by applying the presented theory for multi-ring systems, the following relationship can be used for the average power radiated into the solid angle $d\Omega$ by this antenna:

$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{1}{d\Omega} \left\langle \overrightarrow{E^{multy\ rings}} \times \overrightarrow{H^{multy\ rings}} \cdot \overrightarrow{da} \right\rangle_t. \quad (33)$$

After performing calculations similar to those presented for radiation impedance, we have for the average power radiated into the solid angle $d\Omega$:

$$\begin{aligned} \left\langle \frac{dP}{d\Omega} \right\rangle_t &= \Gamma_1 \sin^2(\theta) + \Gamma_2 \sin^2(\theta) \cos^2(\theta), \\ \Gamma_1 &= \frac{i_m^2}{32c\varepsilon_0} \left(\frac{2\pi}{\lambda_0} \right)^4 \left[a_0^4 + 2a_0^2 \sum_{n=1}^m a_n^2 + \sum_{l=1}^m \sum_{n=1}^m a_l^2 a_n^2 \right], \\ \Gamma_2 &= \frac{i_m^2}{32c\varepsilon_0} \left(\frac{2\pi}{\lambda_0} \right)^6 \sum_{l=1}^m \sum_{n=1}^m a_l^2 a_n^2 d_n d_l. \end{aligned} \quad (34)$$

All parameters in Equation (34) have been previously introduced, and θ is the angle of observation at a relatively far distance. In the following section, we will present diagrams of the radiation impedance variations (Equation (29)) as well as the simulation of the radiation pattern of a plasma antenna using Equation (34).

3. Radiation resistance data and diagrams: derived from the presented formulation

In this section, we will begin by simulating and presenting the diagrams of radiation resistance variations in multi-layer ring antennas in various configurations. To verify the accuracy of the provided graphs for radiation resistance variations, each

of the Figures 4 to 6. Here, it should be noted that in all the figures presented and discussed hereafter, the term *nur* is an abbreviation for number of up rings (the number of rings located above the xoy plane), and *ndr* is an abbreviation for number of down rings (the number of rings located below the xoy plane). It should also be noted that, as introduced in the theoretical section in Figure 3, the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively.

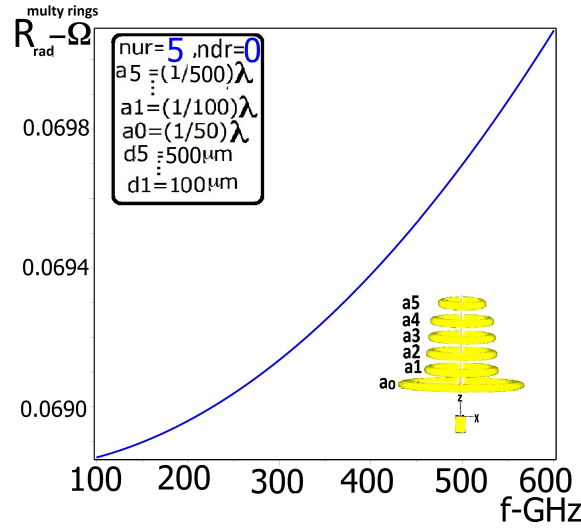


Figure 4: The radiation resistance variation diagram as a function of frequency. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively. Also, λ is considered as the wavelength corresponding to the wave frequency in vacuum.

With an increase in the number of rings above the XOY plane, the average values of the radiation resistance show a relative increase. This is clearly observed in Figure 5, which has been theoretically derived from Equation (29). In Figure 5, the number of rings above the XOY plane has been increased to ten. As Figure 5 also shows, with an increase in the antenna's operating frequency, the radiation resistance rises. Compared to Figure 6, since the number of rings has increased, the average radiation resistance has grown. An interesting point mentioned below the presentation of Equation (29) the disappearance of the radiation resistance's relative dependence on frequency for structures that are completely symmetric with respect to the XOY plane is observed in Figure 7. In all configurations shown in Figure 7, the number of rings above and below the XOY plane is not only considered equal, but their spacing from each other is also uniform. This

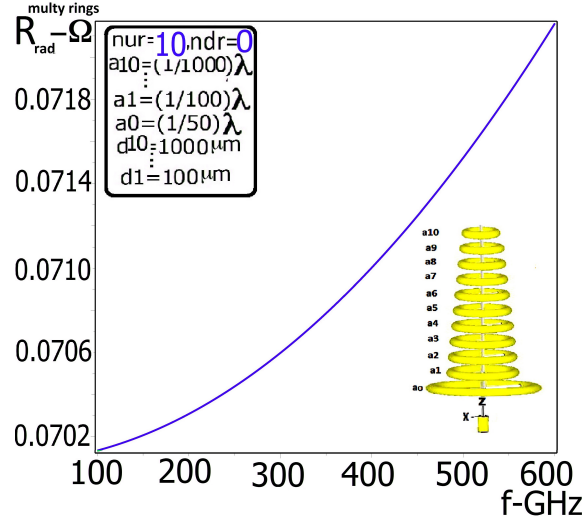


Figure 5: The curve of radiation resistance variations versus frequency when the number of rings is increased compared to Figure 4. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively. Also, λ is considered as the wavelength corresponding to the wave frequency in vacuum.

means the system is completely symmetric in terms of both the number and the arrangement of the rings. It is again emphasized here that the first, second, and third terms of Equation (29), due to the presence of the inverse fourth power of the wavelength behind the parentheses, is in fact independent of the wavelength. The only term that indicates a dependence on the wavelength is the fourth term of Equation (29), which contains the coupling of the radii with the ring spacings. To demonstrate the role of the fourth term in Equation (29) regarding the dependence of the radiation resistance on frequency for asymmetric systems, it is sufficient, for instance, to introduce an asymmetry in the spacing of the lower rings in the data of Figure 5. As a result, the diagram of radiation resistance variation with frequency takes the forms shown in Figure 8. The data corresponding to Figure 8 are consistent with the data presented by the blue curve in Figure 7, with the difference that the spacing between the lower rings is considered to be closer and denser. As can be seen, in this case, the fourth term of Equation (29) plays an effective role, and the dependence of the radiation resistance on frequency reappears.

Next, we examine a new configuration in which the ring radii are alternately equal, and we compare it with the case where all the radii are identical. As shown

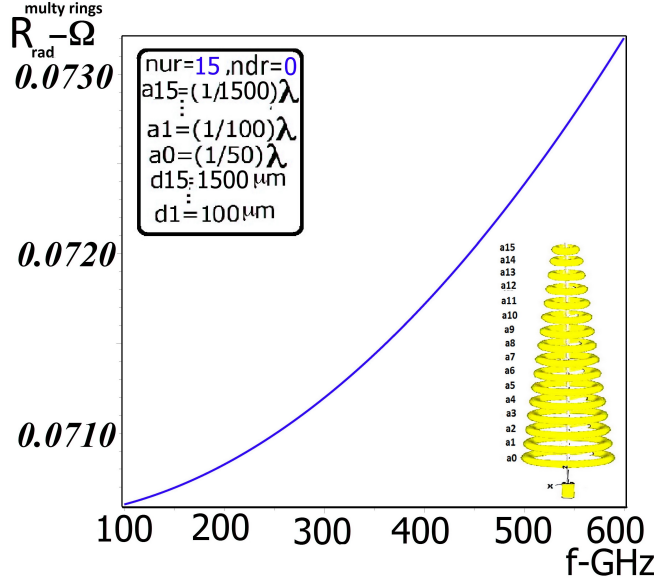


Figure 6: Diagram of the flocculation reduction in radiation resistance variations versus frequency changes that occur with an increase in the number of rings. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively. Also, λ is considered as the wavelength corresponding to the wave frequency in vacuum.

in Figure 9 for a 15-ring structure, the radii of the rings in the odd layers (1, 3, 5, ...) are considered different from those in the even layers (2, 4, 6, ...). Thus, it can be said that the rings alternate between smaller and larger sizes. For a clearer comparison, these diagrams are presented together with those of the case where all ring radii are identical in a single figure. Following the approach presented in this paper, Figure 9 represents the variations of radiation resistance versus frequency obtained from the theoretical Equation (29). A clear and evident conclusion that can be drawn from the diagrams presented in Figure 9 is that with the appearance of alternating radii in each layer at a fixed frequency, the radiation resistance decreases. As seen in Figure 9, when the ring radii alternate, the radiation resistance becomes much more uniform and tends toward an asymptotic behavior.

Here, we delve into examining the radiation pattern of such configurations, specifically of plasma antennas. For this purpose, Figures 10 to 12 are provided. Since an increase in the voltage amplitude V_m naturally leads to an increase in radiated power due to the increased current density, in all the considered configurations for investigating the radiation pattern, the voltage amplitude parameter is set con-

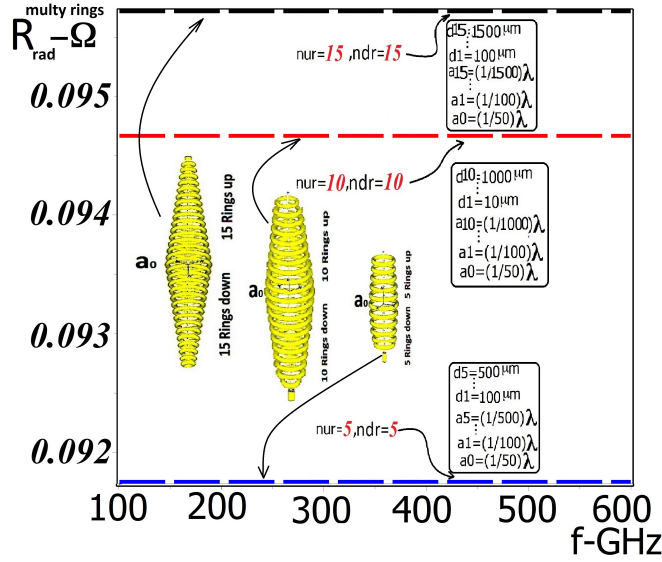


Figure 7: Diagram of the stability of radiation resistance variations versus frequency changes for configurations that are fully symmetrical in terms of the number of rings and their spacing relative to the XOY plane. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively. Also, λ is considered as the wavelength corresponding to the wave frequency in vacuum.

stant at $V_m = 4KV$. Indeed, as customary in plotting radiation pattern diagrams, the distance of each point from the coordinate origin represents the power radiated at that angle. As shown in Figure 10-a, if we consider a plasma antenna composed of very thin ring (assuming the radius of the glass tube containing the plasma h is very small), if the distance between the rings is very small, it can be simulated using the theory of a segmented, multi-layer ring antenna. Such antennas, as shown in Figure 10-a, take the form of a three-dimensional Archimedean spiral, and for this reason, they can be referred to as three-dimensional Archimedean (quasi-Archimedean) spiral plasma antennas [25–27]. The previously presented set of relations is valid for Figure 10-a when the spiral pitch (distance between layers) is very small, and the antenna is also very thin.

As observed in Figure 10-b, when the number of rings is relatively high and they are located only in the upper part of the XOY plane, the radiation pattern diagrams exhibit a limiting behavior and become very close to each other. In terms of radiation power, no significant difference is observed when the number of rings is high. In Figure 10-b, the plasma density n_0 , which is synonymous with the plasma

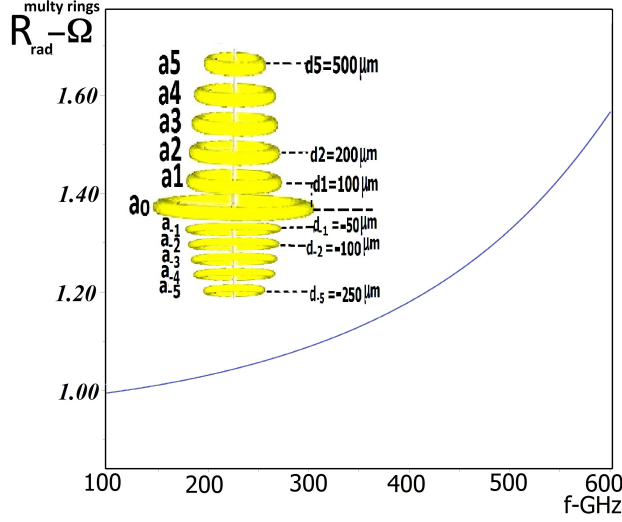


Figure 8: The emergence of the dependence of radiation resistance variations on frequency for the case where the number and radii of the rings above and below the XOY plane are identical, but the ring spacings below are different. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively.

frequency ω_{pe} , and the collision frequency γ are considered constant. Additionally, the radius of the plasma tube cross-section is assumed to be $h = 0.5mm$. If the number of rings is considered to be around *nur* 1, 2, or 3, a significant difference in their radiation power at each angle is observed. However, in both Figure 10-b and Figure 10-c, the optimal radiation power is for an observer located on the XOY plane, i.e., $\theta = \pi/2$. As shown in Figure 10-c, the higher radiation power corresponds to a greater number of rings, which is natural because, in the far-field region, rings with in-phase currents do not create destructive interference due to their dimensions, and the sum of their effects results in higher amplitudes.

Since in this simulation, the bias voltage amplitude is considered a constant value of $V_m = 4KV$, conditions that lead to a decrease in the ohmic resistance of the plasma tube result in an increase in the amplitude of the current intensity i_m and consequently an increase in the radiated power. One of the factors contributing to the increase in the current intensity amplitude is the increase in plasma conductivity σ , which logically has a direct relationship with the increase in the density of free charge carriers, synonymous with an increase in the plasma frequency ω_{pe} .

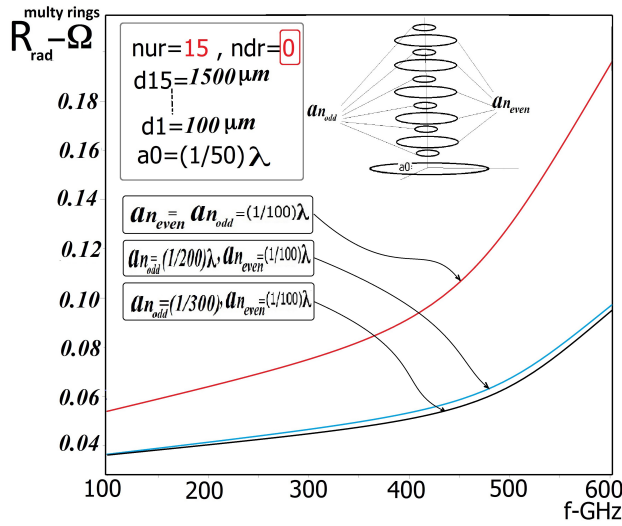


Figure 9: The diagram of radiation resistance variations versus frequency for configurations that include only rings above the XOY plane, with their radii alternately equal and changing periodically in each layer. The presented curve is obtained from the theoretical formula in Equation (29). Here the parameters a_i and d_i are considered for the radius of the i -th ring and its distance from the xoy plane, respectively. Also, λ is considered as the wavelength corresponding to the wave frequency in vacuum.

To demonstrate this effect, Figure 11-a is presented, showing that with an increase in the plasma frequency, the conductivity and consequently the radiated power increase significantly. It should be noted that in the series of Figure 11-a, Figure 11-b, Figure 11-c, and Figure 11-d, all the rings are considered to be located on the upper side of the XOY plane, and the system is not symmetric. One of the most interesting behaviors observed in the radiation power diagrams appears when we examine the effect of collision frequency in the plasma on the antenna's radiation pattern. The origin of this dual behavior relates to the limiting relationships mentioned in Equation (31). As shown in Figure 11-b, since the assumed data is such that the operating frequency of the antenna is much greater than the collision frequency, it is observed that with the increase in collision frequency γ , the conductivity σ increases, and consequently, the current intensity amplitude i_m increases, leading to higher radiated power. This plasma behavior is not always consistent, and as indicated in Equation (31), in another region for the antenna's operating frequency, the radiation pattern behavior changes so that with the increase in collision frequency γ , the conductivity σ decreases, resulting

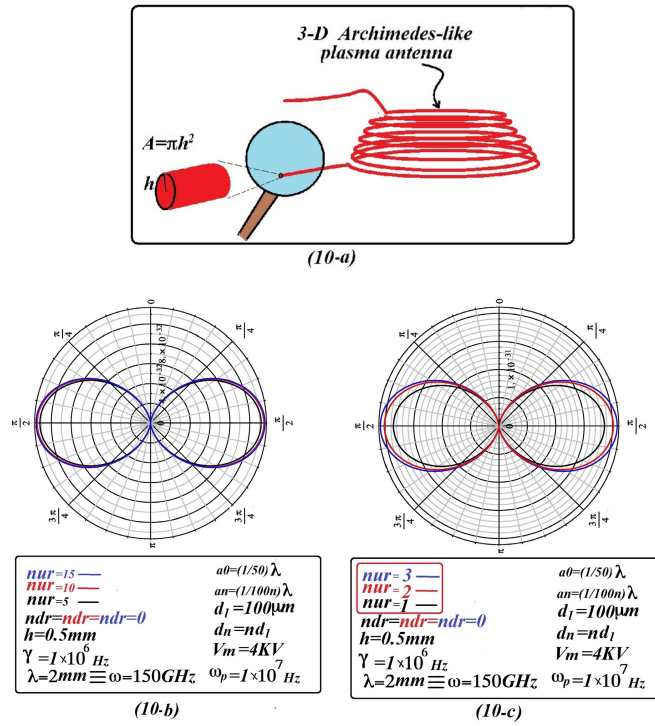


Figure 10: A three-dimensional thin Archimedean plasma antenna configuration with very small pitch steps and a description of its radiation pattern using the theory of successive rings with varying radii. Figure 10-a shows the geometric structure of the three-dimensional thin Archimedean plasma antenna. Figure 10-b illustrates the radiation pattern diagram when the number of rings is large. Figure 10-c shows the radiation pattern of the antenna when the number of rings is relatively small.

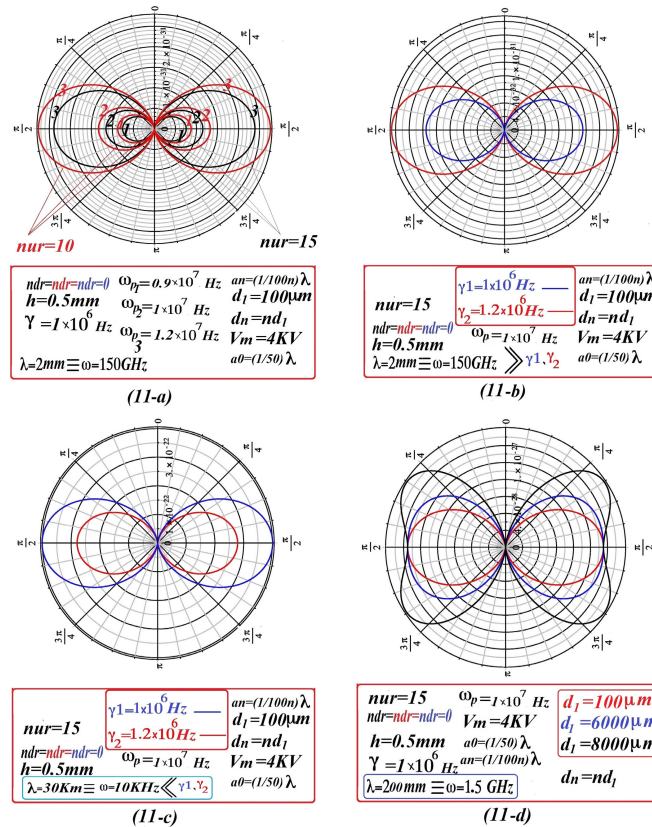


Figure 11: A thin plasma helical Archimedean antenna described using multi-loop antenna theory, and diagrams illustrating their radiation patterns.

in a decrease in the antenna's radiated power, as shown in Figure 11-c. In plotting Figure 11-c, as observed, we had to change the antenna's operating wavelength region to avoid issues with the approximations used in the problem. This is because, in weakly ionized plasmas, as mentioned, the collision frequency ranges are within a specific limit, and our assumptions must align with practical and experimental setups. Another configuration whose radiation pattern we will discuss corresponds to the configuration in Figure 9, where the rings have alternating radii and are not telescopic. As shown in Figure 11-d, the configuration corresponding to Figure 9 is depicted in a state where the spacing between the layers and the rings has also been changed, and its effect on the radiation pattern is illustrated. It should again be noted that in this case, because the geometric dimensions of the problem and the radiated wavelength must conform to a specific approximation,

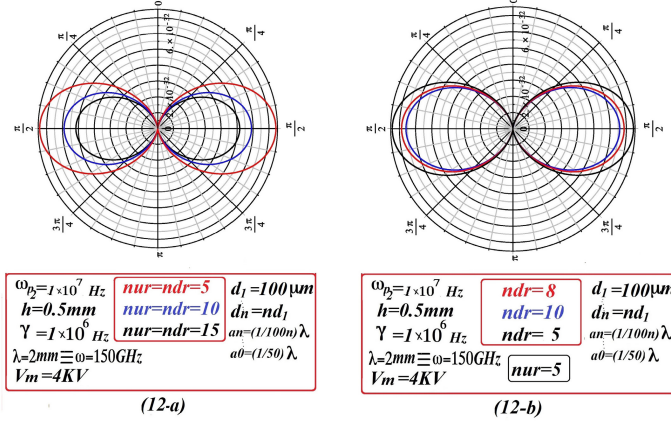


Figure 12: The radiation patterns for cases where the presence of the rings is either symmetric (12-a) or asymmetric relative to the XOY plane(12-b).

we had to increase the operating wavelength of the antenna to $\lambda = 200 \text{ mm}$ to ensure the approximations used were not violated. Figure 11-d shows that with the increase in the distance between the layers (while adhering to the limitations of the approximations), destructive interference patterns can also manifest, and for an observer at $\theta = \pi/2$, a decrease in power can be observed. Another observation in Figure 8-d is that the radiation pattern for an observer with $\theta = \pi/2$ reaches a limiting value for all different layer spacings. Finally, it seems it is time to examine the radiation patterns for cases where the presence of the rings is either symmetric or asymmetric relative to the XOY plane.

For this purpose, Figure 12-a, and Figure 12-b are presented. As seen in Figure 12-a, where the rings are symmetrically distributed on both sides of the XOY plane, the radiated power decreases with an increase in the number of rings under symmetric conditions. The only reason for this can be attributed to the principle of superposition, where the effects of the upper and lower antennas destructively interfere with the radiation pattern. However, this destructive effect disappears when the symmetry in the distribution of the rings is lost. Figure 12-b shows that with an increase in the number of lower rings (number of down rings ndr), the power radiated into space increases again.

4. Conclusion and summary

This work presented a comprehensive theoretical analysis of multi-segment plasma antennas composed of multiple rings positioned at varying heights above and below the XOY plane. The study focused on vector potential, electric and magnetic fields, and radiation resistance calculations, with particular emphasis on vector

and scalar transformations of the electromagnetic quantities between two shifted Cartesian coordinate systems. By employing these mathematical transformations, the complex integrals typically required for direct electromagnetic calculations were avoided, significantly simplifying the analysis. This approach allowed the fields and potentials to be expressed in a more tractable form, making it easier to evaluate the influence of geometric configurations, such as ring radii, layer spacing, and symmetry, as well as plasma parameters including density, collision frequency, and conductivity. Symmetric configurations generally yielded higher and more stable radiation resistance, while asymmetries introduced variability and fluctuations. The results demonstrate that using coordinate-shift transformations as a mathematical tool not only reduces computational complexity but also provides deep insight into the interplay between antenna geometry and plasma properties on key performance metrics, such as radiation resistance and radiation pattern. Overall, this study highlights the power of analytical mathematical methods in modeling multi-segment plasma antennas. The technique of transforming vector and scalar quantities between shifted coordinate systems proves to be a highly effective strategy for simplifying calculations, avoiding cumbersome direct integrals, and providing clear guidance for antenna design and optimization.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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Corrected Proof