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# A Simple Classification of Finite Groups of Order $p^2q^2$

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**Abstract**

Suppose $G$ is a group of order $p^2q^2$ where $p > q$ are prime numbers and suppose $P$ and $Q$ are Sylow $p$-subgroup and Sylow $q$-subgroup of $G$, respectively. In this paper, we show that up to isomorphism, there are four groups of order $p^2q^2$ when $Q$ and $P$ are cyclic, three groups when $Q$ is a cyclic and $P$ is an elementary abelian group, $p^2 + 3p/2 + 7$ groups when $Q$ is an elementary abelian group and $P$ is a cyclic group and finally, $p + 5$ groups when both $Q$ and $P$ are elementary abelian groups.

Keywords: Semi-direct product, $p$-group, Sylow subgroup.

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## 1. Introduction

All finite groups can be broken down into a series of finite simple groups which have been called the building blocks of finite groups. The history of finite simple groups originated in the 1830's with Evariste Galois and the solution of fifth degree polynomial equations. In the twentieth century, the recognition of the importance of finite simple groups inspired a huge effort to find all finite simple groups, see [9] for more details. But the classification of finite groups is still an open problem. In this paper, by using the notation of [1], we determine all groups of order $p^2q^2$ by a simple method.

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## 2. Preliminaries and Notations

In this section, we first recall some concepts, notations and results in group theory, which are used in the next section. We use standard basic group theory terminology and notations, see [2, 4, 5, 7, 10, 11] as well as [6, 8]. The set of all Sylow $p$-subgroups of $G$ are denoted by $Syl_p(G)$.

A semi-direct product is a generalization of a direct product. Let $N$ be a normal subgroup of $G$, each element $g \in G$ defines an automorphism of $N$, $n \to gng^{-1}$, and this defines a homomorphism

$$\theta: \ G \to Aut(N), \ g \to i_g|N.$$

If there exists a subgroup $Q$ of $G$ such that $G \to G/N$ maps $Q$ isomorphically onto $G/N$, then we can reconstruct $G$ from $N, Q$ and the restriction of $\theta$ to $Q$. Indeed, an element $g \in G$ can be written uniquely in the form

$$g = nq, \ n \in N, \ q \in Q.$$

$q$ must be the unique element of $Q$ mapping to $gN \in G/N$ and $n$ must be $gq^{-1}$. Thus, we have a one-to-one correspondence of sets

$$G \leftrightarrow N \times Q.$$

If $g = nq$ and $g' = n'q'$, then

$$gg' = (nq)(n'q') = n(qn'q^{-1})qq' = n\theta(q)(n').qq'.$$

Equivalently, $G$ is a semi-direct product of subgroups $N$ and $Q$ if

$$N \trianglelefteq G, \ NQ = G, \ N \cap Q = \{1\}.$$

Note that $Q$ need not be a normal subgroup of $G$. When $G$ is the semi-direct product of subgroups $N$ and $Q$, we write $G = N \rtimes Q$ or $N \rtimes_\varphi Q$.

**Theorem 2.1.** *[8] Let $P$ be a $p$-group, $Q$ be a $q$-group and $\psi, \varphi : Q \longrightarrow Aut(P)$ be two homomorphisms. Then $Q \ltimes_\varphi P \cong Q \ltimes_\psi P$ if and only if $\psi \circ \gamma$ and $\varphi$ are conjugate in $Aut(P)$ for some $\gamma \in Aut(Q)$.*

**Lemma 2.2.** *[8] Up to isomorphism, there is a 1-1 correspondence between groups $G = Q \ltimes_\varphi P$ and the number of orbits of action $Aut(P) \times Aut(Q)$ on the set $Hom(Q, Aut(P))$ where for all $\alpha \in Aut(P), \beta \in Aut(Q)$ and $y \in Q$, we have*

$$\varphi^{(\alpha,\beta)}(y) = \alpha \circ [\varphi \circ \beta^{-1}(y)] \circ \alpha^{-1}.$$

Suppose $Q = \langle y \rangle$ is a cyclic $q$-group. For every non-trivial homomorphism $\varphi : Q \to Aut(P)$, $Q_y = \langle \varphi(y) \rangle$ is a subgroup of $Aut(P)$ of order $|Q|$. On the other hand, all automorphisms of $Q$ map $y$ to $y^j$ for some $j$ and hence $Q_{y^j} = (Q_y)^j = Q_y$. Thus, we can deduce the following result.

**Lemma 2.3.** *Let $Q$ be a cyclic group, then up to isomorphism, all groups of form $G = Q \ltimes P$ are corresponding to conjugacy classes of subgroups of $Aut(P)$ of order dividing $\mid Q \mid$.*

The general linear group of degree $n$ is the set of $n \times n$ non-singular matrices, together with ordinary multiplication of matrices as its binary operation. Because the product of two non-singular matrices is again non-singular, and the inverse of a non-singular matrix is non-singular, then it forms a group. Generally, the general linear group of degree $n$ over any field $F$ is the set of $n \times n$ non-singular matrices with entries from $F$ denoted by $GL_n(F)$ or $GL(n, F)$.

A field $F$ that contains a finite number of elements is called Galois field. A finite field of order $q$ exists if and only if the $q$ is a prime power $p^k$ where $p$ is a prime number and $k$ is a positive integer. All fields of a given order are isomorphic. By this notation, if $F$ is a finite field with $q = p^n$ elements, then the general linear group of degree $n$ over the filed $F$ is denoted by $GL(n, q)$.

**Theorem 2.4.** *[8] The conjugacy classes of $GL(2, p)$ are as reported in Table 1. In this table $\rho, \sigma$ are primitive elements of $GF(p)$ and $GF(p^2)$, respectively.*

Table 1: Conjugacy classes of group $GL(2, p)$.

| Conjugacy classes | Number of such classes | No. Elements |
|---|---|---|
| $\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^a \end{pmatrix}$ | $(p-1)$ | 1 |
| $\begin{pmatrix} \rho^a & 1 \\ 0 & \rho^a \end{pmatrix}$ | $(p-1)$ | $(p-1)(p+1)$ |
| $\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^b \end{pmatrix}$ | $\frac{1}{2}(p-1)(p-2)$ | $p(p+1)$ |
| $\begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^{ap} \end{pmatrix}$ | $\frac{1}{2}p(p-1)$ | $p(p-1)$ |

# 3. Main Results

In this section, the consequences of the Sylow Theorems are cases where the size of $G$ forces it to have a non-trivial normal subgroup.

**Theorem 3.1.** *Let $G$ be a group of order $p^2q^2$. If $pq = 6$, then $G$ is isomorphic with one of the following groups:*

1. $C_{36}$,
2. $C_{18} \times C_2$,
3. $C_6 \times C_6$,

4. $C_{12} \times C_3$,

5. $D_6 \times C_6 \cong S_3 \times C_6 \cong D_{12} \times C_3$,

6. $D_6 \times D_6 \cong S_3 \times S_3$,

7. $D_{18} \times C_2$,

8. $A_4 \times C_3$,

9. $D_{36}$,

10. $H \times C_3$,

11. $K \times C_2$,

12. $\langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = a^{-1}\rangle$,

13. $\langle a, b, c \mid a^2 = b^2 = c^9 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = ab\rangle$,

14. $\langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$,

*where*

$$H = \langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^{-1}\rangle$$

*and*

$$K = \langle a, b, c \mid a^2 = b^2 = c^2 = (abc)^2 = (ab)^3 = (ac)^3 = 1\rangle.$$

*If* $pq \neq 6$*, then* $G$ *has one of the following structures:*

$$C_{q^2} \ltimes C_{p^2},\ C_{q^2} \ltimes (C_p \times C_p),\ (C_q \times C_q) \ltimes C_{p^2},\ (C_q \times C_q) \ltimes (C_p \times C_p).$$

*Proof.* If $pq = 6$, then the proof is clear. Suppose $pq \neq 6$, since $p > q$ according to Sylow theorem, it is clear that Sylow $p$-subgroup of $G$ is normal. We assume that $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. This implies that $Q \cap P = \langle 1\rangle$, $P \lhd G$, $G = PQ$. Hence, $G$ is isomorphic with semi-direct product $Q \ltimes_\varphi P$, where $\varphi : Q \longrightarrow Aut(P)$ is a homomorphism. Since, every group of order $p^2$ is abelian, then

$$Syl_p(G) = \{C_{p^2}, C_p \times C_p\} \text{ and } Syl_q(G) = \{C_{q^2}, C_q \times C_q\}.$$

By using Lemma 2.2 and Theorem 2.1, the group $G$ is isomorphic with one of the following presentation:

$$C_{q^2} \ltimes_\varphi C_{p^2}, (C_q \times C_q) \ltimes_\varphi C_{p^2}, C_{q^2} \ltimes_\varphi (C_p \times C_p), (C_q \times C_q) \ltimes_\varphi (C_p \times C_p).$$

In continuing, we determine the presentation of all these groups. To do this, we can consider the following cases:

1. $G \cong C_{q^2} \ltimes_{\varphi} C_{p^2}$, then $Aut(C_{p^2}) \cong C_{p(p-1)}$ and consider the homomorphism $\varphi : C_{q^2} \hookrightarrow C_{p(p-1)}$. If $Im\varphi \cong \langle 1 \rangle$, then $G \cong C_{q^2} \times C_{p^2}$. If $Im\varphi \cong C_q$ and $q \mid (p-1)$, since $Aut(P)$ is cyclic, according to Lemma 2.3 there exists only one group with the following presentation:

$$\begin{aligned} G &\cong \langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r^q \equiv 1(mod\ p^2) \rangle \\ &= \langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r = r_0^{\frac{(p-1)p}{q}} \rangle, \end{aligned}$$

where $r_0$ is $p^2$-th root of unity. If $Im\varphi \cong C_{q^2}$ and $q^2 \mid (p-1)$, since $Aut(P)$ is cyclic, by Lemma 2.3, there exists only one group with the following presentation:

$$\begin{aligned} G &\cong \langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r^{q^2} \equiv 1(mod\ p^2) \rangle \\ &= \langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r = r_0^{\frac{(p-1)p}{q^2}} \rangle, \end{aligned}$$

where $r_0$ is $p^2$-th root of unity.

2. $G \cong (C_q \times C_q) \ltimes_{\varphi} C_{p^2}$ and assume that the image of $\varphi$ is not trivial where $\varphi : C_q \times C_q \hookrightarrow Aut(C_{p^2})$. If $Im\varphi \cong \langle 1 \rangle$, then $G \cong C_q \times C_q \times C_{p^2}$. Let $Im\varphi \cong C_q$ and $q \mid (p-1)$. Since $Aut(P)$ is cyclic, by using Lemma 1.3, there exists only one group with the following presentation:

$$G \cong \langle a, b, c \mid a^q = b^q = c^{p^2} = 1, [a, b] = [b, c] = 1, a^{-1}ca = c^r \rangle,$$

where $r_0$ is $p^2$-th root of unity and $r^q \equiv 1(mod\ p^2)$.

3. $G \cong C_{q^2} \ltimes_{\varphi} (C_p \times C_p)$ and so $Aut(C_p \times C_p) \cong GL(2, p)$. If $Im\varphi \cong \langle 1 \rangle$, then $G \cong C_{q^2} \times (C_p \times C_p)$ and so $G$ is abelian. If $Im\varphi \cong C_q$, then for given homomorphism $\varphi : C_{q^2} \to GL(2, p)$, $Im\varphi \cong \langle 1 \rangle$ or $C_q$ or $C_{q^2}$. Suppose the subgroups of order $q$ are in the first class in Table 1. Hence, $q|p-1$ and $\alpha$ is a primitive root of unity of $F_p$ and so

$$Im\varphi \cong C_q \cong \langle \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \mid \beta = \alpha^{\frac{p-1}{q}} \rangle.$$

This implies that

$$G \cong \langle a, b, c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^{\beta}, a^{-1}ca = c^{\beta}, bc = cb \rangle.$$

Let the subgroups of order $q$ be in the second class of Table 1. Clearly, in this case, one can not find a new presentation for $G$. Let subgroups of order $q$ be in the third class. If $q = 2$, then

$$Im\varphi \cong C_q \cong \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

and so

$$G \cong \langle a,b,c \mid a^{q^2} = b^p = c^p = 1, ab = ba, a^{-1}ca = c^{-1}, bc = cb\rangle.$$

Let $q \mid (p-1)$ and $q \neq 2$, then $\alpha$ is a $p$-th root of unity and $\beta = \alpha^{\frac{p-1}{q}}$. All solutions of equation $x^q \equiv 1(\,mod\,p)$ are

$$x_1 = 1, x_2 = \beta, x_3 = \beta^2, \ldots, x_q = \beta^{q-1}.$$

This means that there are $\frac{q+1}{2}$ non-conjugate cyclic groups of order $q$ as follows:

$$\langle\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\rangle, \langle\begin{pmatrix} \beta & 0 \\ 0 & \beta^i \end{pmatrix}\rangle$$

where $i = 2, 3, \ldots, \frac{q-1}{2}$ and $q-1$. Hence in this case, one can verify that

$$\begin{aligned} G &\cong \langle a,b,c \mid a^{q^2} = b^p = c^p = 1, ab = ba, a^{-1}ca = c^{\beta}, bc = cb\rangle, \\ G &\cong \langle a,b,c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^{\beta}, a^{-1}ca = c^{\beta^i}, bc = cb\rangle. \end{aligned}$$

Finally, let the subgroups of order $q$ be in the fourth row in Table 1. For $q = 2$ there is no group of order two. If $q \neq 2$ and $q|p-1$, then there is no a subgroup of order $q$, but if $q$ divides $p+1$, then we can construct a subgroup of order $q$ as follows:

Suppose $\sigma$ be a primitive root of $GF(p)$ and $\alpha_0 = \sigma^{\frac{p^2-1}{q}}$, then

$$a_0 = \sigma^{\frac{p^2-1}{q}} = \alpha + \beta\sqrt{D} : \alpha, \beta, D \in GF(p), \beta \neq 0, D \ is\ not\ square$$

and so

$$\begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0^p \end{pmatrix} \in [\begin{pmatrix} \alpha & \beta D \\ \beta & \alpha \end{pmatrix}]$$

where for an element $g \in G$, $[g]$ means the conjugacy class of $g$ in $G$. This means that

$$G \cong \langle a,b,c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^{\alpha}c^{\beta D}, a^{-1}ca = b^{\beta}c^{\alpha}, bc = cb\rangle.$$

Suppose $Im\varphi \cong C_{q^2}$, then all non-conjugate cyclic subgroups of order $q^2$ in $GL(2,p)$ are as follows:

(a) The cyclic subgroups of order $q^2$ are in the first row in Table 1. Further, $q^2$ divides $p-1$ and $\alpha$ is a $p$-primitive root of unity in $F_p$, thus

$$Im\varphi \cong C_{q^2} \cong \langle\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}\rangle, \quad \beta = \alpha^{\frac{p-1}{q^2}}.$$

Hence

$$G \cong \langle a,b,c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^{\beta}, a^{-1}ca = c^{\beta}, bc = cb\rangle.$$

(b) The cyclic subgroups of order $q^2$ are in the second row. In this case, there is no a cyclic subgroup of order $q^2$.

(c) The cyclic subgroups of order $q^2$ are in the third row. If $q^2$ divides $p-1$, then the equation $x^{q^2} \equiv 1 \pmod p$ has exactly $q^2$ roots. Among them, $q(q-1)$ subgroups are of order $q^2$ and so there are $\frac{q^2+q}{2}$ non-conjugate subgroups of order $q^2$ as follows:

Let $\alpha$ be a $p$-primitive root of unity, then

$$\beta = \alpha^{\frac{p-1}{q^2}} \text{ and } C^i_{q^2} = \langle \begin{pmatrix} \beta & 0 \\ 0 & \beta^i \end{pmatrix} \rangle,$$

where $2 \leq i \leq (q^2-1)/2$ or $i = kq$ $(k \geq (q+1)/2)$ or $i = q^2-1$. Thus, there are $(q^2+q)/2$ groups with the following presentation:

$$G_i \cong \langle a, b, c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^\beta, a^{-1}ca = c^{\beta^i}, bc = cb \rangle.$$

(d) The cyclic subgroups of order $q^2$ are in the fourth row. If $q = 2$, then two following cases hold:

1) $p = 4k+1$ and so $(p^2-1)/q^2 = 2k(2k+1)$. Hence, there is not a cyclic group of order $q^2$.
2) $p = 4k+3$ and so $p^2-1 = 8(k+1)(2k+1)$. This leads us to verify that $(p^2-1)/q^2 = 2(k+1)(2k+1)$.

In this case, we can construct a cyclic subgroup as follows:

Let $\sigma$ be a generator of multiplicative group $GF(p^2)$, then

$$\sigma^{\frac{p^2-1}{4}} = \alpha + \beta\sqrt{D} : \alpha, \beta, D \in GF(p), \beta \neq 0, \ D \ is \ not \ square$$

and so

$$G \cong \langle a, b, c \mid a^4 = b^p = c^p = 1, a^{-1}ba = b^\alpha c^{\beta D}, a^{-1}ca = b^\beta c^\alpha, bc = cb \rangle.$$

Let $q \neq 2$ and $q^2 \mid p+1$ or $q^2 \mid p-1$. If $q^2 \mid p-1$, then $q \mid p-1$ and so we can not construct a new presentation. Suppose $q^2 \mid p+1$ and $\sigma$ is a multiplicative subgroup of $GF(p^2)$. Then

$$G \cong \langle a, b, c \mid a^{q^2} = b^p = c^p = 1, a^{-1}ba = b^\alpha c^{\beta D}, a^{-1}ca = b^\beta c^\alpha, bc = cb \rangle.$$

4. $G \cong (C_q \times C_q) \ltimes_\varphi (C_p \times C_p)$, first notice that $Aut(C_p \times C_p) \cong GL(2,p)$. We are interested about all automorphism of the form $\varphi : (C_q \times C_q) \hookrightarrow Aut(C_p \times C_p) \cong GL(2,p)$. If $Im\varphi \cong \langle 1 \rangle$, then $G \cong C_q \times C_q \times C_p \times C_p$ is an abelian group of order $p^2q^2$. If $Im\varphi \cong C_q$, then all non-conjugate subgroups of order $q$ in $GL(2,p)$ are as follows:

(a) Subgroups belonging to the first row in Table 1. If $q \mid (p-1)$ and $\alpha$ is a $p$-primitive root of unity, then

$$Im\varphi \cong C_q \cong \langle \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \rangle, \quad \beta = \alpha^{\frac{p-1}{q}}$$

and hence

$$G \cong \langle a, b, c, d \mid a^q = b^q = c^p = d^p = 1, b^{-1}cb = c^{\beta}, b^{-1}db = d^{\beta} \rangle$$

where $[a, c] = [a, d] = [a, b] = [c, d] = 1$.

(b) Subgroups belonging to the second row in Table 1. In this case there is no element of order $q$.

(c) Subgroups belonging to the third row in Table 1. If $q = 2$, then

$$Im\varphi \cong C_q \cong \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

and so

$$G \cong \langle a, b, c, d \mid a^q = b^q = c^p = d^p = 1, b^{-1}cb = c^{-1} \rangle,$$

where $[a, b] = [a, c] = [a, d] = [b, d] = [c, d] = 1$. If $q \neq 2$, $q \mid (p-1)$, then $\alpha$ is a root of unity and $\beta = \alpha^{\frac{p-1}{q}}$. Thus, all roots of the equation $x^q \equiv 1 (mod\ p)$ are

$$x_1 = 1, x_2 = \beta, x_3 = \beta^2, \ldots, x_q = \beta^{q-1}.$$

Hence, there are $(q+1)/2$ non-conjugate cyclic subgroups

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \rangle, \langle \begin{pmatrix} \beta & 0 \\ 0 & \beta^i \end{pmatrix} \rangle,$$

where $i = 2, 3, 4, \ldots, \frac{q-1}{2}$ *or* $i = q - 1$. Similar to our last discussion, we get two presentations for $G$ which are not new.

(d) Subgroup of order $q$ belonging to the fourth row in Table 1. It is not difficult to see that in this case a group of order $q$ exists if $q \neq 2$, $p - 1 \neq kq$ and $q \mid p + 1$. Let $\sigma$ be a root of unity in $GF(p^2)$ and

$$\alpha_0 = \sigma^{\frac{p^2-1}{q}} = \alpha + \beta\sqrt{D} : \alpha, \beta, D \in GF(p), \beta \neq 0,\ D \text{ is not square}.$$

Then

$$\begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0^p \end{pmatrix} \in [\begin{pmatrix} \alpha & \beta D \\ \beta & \alpha \end{pmatrix}]$$

and hence

$$G \cong \langle a, b, c, d \mid a^q = b^q = c^p = d^p = 1, b^{-1}cb = c^{\alpha} d^{\beta D}, b^{-1}db = c^{\beta} d^{\alpha} \rangle,$$

where $[a,b]=[a,c]=[a,d]=[c,d]=1$. Finally, suppose $Im\varphi \cong C_q \times C_q$ and $\alpha$ is $p$-th primitive root of unity, $\beta = \alpha^{\frac{p-1}{q}}$, and $q \mid p-1$. Then

$$C_q \times C_q \cong \langle \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \rangle$$

and so

$$G \cong \langle a,b,c,d \mid a^q = b^q = c^p = d^p = 1, a^{-1}ca = c^{\beta}, b^{-1}db = d^{\beta} \rangle,$$

where $[a,b]=[b,c]=[c,d]=[a,d]=1$.

This completes the proof. □

**Conflicts of Interest.** The authors declare that there is no conflicts of interest regarding the publication of this article.

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