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Fundamental Functor Based on Hypergroups and Groups

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Abstract

The purpose of this paper is to compute the fundamental relations of hypergroups. In this regards first we study some basic properties of fundamental relation of hypergroups, then we show that any given group is isomorphic to the fundamental group of a nontrivial hypergroup. Finally we study the connections between categories of hypergroups and groups via the fundamental relation.

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1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [10]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then many researchers have been worked on this new field of modern algebra and developed it.

The applications of mathematics in other disciplines, for example in informatics, play a key role and they represent, in the last decades, one of the purpose of the study of the experts of Hyperstructures Theory all over the world. Then various connections between hypergroups and other subjects of theoretical and applied

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mathematics have been established. The most important applications in geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy sets and rough sets, automata theory can be found in [3].

A short review of the theory of hypergroups appears in [2]. The relation β (resp. β^*) was introduced on hypergroups by Koskas [9] and was studied mainly by Corsini [2] and Vougiouklis [12]. In [4], Freni proved that in hypergroups the relation β is transitive. Recently, In [5], Freni introduced the relation γ as a generalization of the relation β and proved that in hypergroups, the relation β is transitive. Davvaz et al. [1] introduced the smallest equivalence relation ν^* on a hypergroup H such that the quotient $\frac{H}{\nu^*}$, the set of all equivalence classes, is a nilpotent group and in this paper the characterization of nilpotent groups via strongly regular relations was investigated and several results on the topic were presented.

Recently, M. Hamidi et al. [8] defined a new equivalence relation τ^* on divisible hypergroups and showed that this relation is the smallest strongly regular relation (the fundamental relation) on commutative divisible hypergroups. They proved that $\tau^* \neq \beta^*, \tau^* \neq \gamma^*$, and defined a divisible hypergroup on every nonempty set. They also proved that the quotient of a finite divisible hypergroup by τ^* is the trivial divisible group.

In this paper, we try to construct a connection between the category of hypergroups and groups, thus we need to apply the concept of fundamental relation on hypergroups. We introduce the concept of fundamental group via the relation β^* and show that any nontrivial group is a fundamental group. By considering the concept of fundamental group, we introduce the functors between these categories and try work up national translations of their composites and trivial functor.

2. Preliminaries

In this section we recall some definitions and results from [12], which we need to development our paper. Suppose G be a nonempty set and $P^*(G)$ be the family of all nonempty subsets of G, every function $*_i : G \times G \longrightarrow P^*(G)$ where $i \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$, is called *hyperoperation.* For all $x, y \in G$, $*_i(x, y)$ is called the *hyperproduct* of x, y. An algebraic system $(G, *_1, *_2, \ldots, *_n)$ is called a *hyperstructure* and binary structure $(G, *)$ endowed with only hyperoperation is called a *hypergroupoid.* For any two nonempty subsets A and B of G and $x \in G$:

$$A * B = \bigcup_{a\in A, b\in B} a * b, \qquad A * x = \bigcup_{a\in A} a*x \qquad \textit{and} \qquad x * B = \bigcup_{b\in B} x * b$$

Recall that a *hypergroupoid* $(G, \odot)$ is called a *semihypergroup* if for any $x, y, z \in G$, $(x \odot y) \odot z = x \odot (y \odot z)$ and semihypergroup $(G, \odot)$ is a *hypergroup* if satisfies in *reproduction axiom*, i.e. for any $x \in G, x \odot G = G \odot x = G$. Let G_1 and G_2 be two hypergroups. The map $f : G_1 \to G_2$ is called an *inclusion* homomorphism if for all $x, y \in G, f(x \odot y) \subseteq f(x) \odot f(y)$, and is called a *strong (good)* homomorphism if for all $x, y \in G,\ f(x \odot y) = f(x) \odot f(y)$.

3. Fundamental Groups

Let $(G, \odot)$ be a hypergroup and ρ be an equivalence relation on G. Letting $\frac{G}{\rho} = \{\rho(g) \mid g \in G\}$, be the set of all equivalence classes of G with respect to ρ. Define a hyperoperation $\otimes$ as follows:

$$\rho(a) \otimes \rho(b) = \{\rho(c) \mid c \in \rho(a) \odot \rho(b)\}$$

In [2] it was proved that $(\frac{G}{\rho}, \otimes)$ is a hypergroup if and only if ρ is regular. Moreover, $(\frac{G}{\rho}, \otimes)$ is a group if and only if ρ is strongly regular ([2]). The smallest equivalence relation, β^*, on G such that $(\frac{G}{\beta^*}, \otimes)$ is a group is called the *fundamental relation.* Let $\mathcal{U}$ denote the set of all finite products of elements of G. Define relation β on G by

$$a\beta b \Longleftrightarrow \exists u \in \mathcal{U} : \{a, b\} \in u$$

In [2] it was proved that β^* is the *transitive closure* of β, and $(\frac{G}{\beta^*}, \otimes)$ is called the *fundamental group* of $(G, \odot)$. Moreover, it was rewritten the definition of β^* on G as follows:

$$a\beta^* b \Longleftrightarrow \exists z_1 = a, z_2, ..., z_n = b \in G, u_1, u_2.., u_n \in \mathcal{U} \text{ s.t.} \{z_i, z_{i+1}\} \in u_i, \forall\, 1 \le i \le n.$$

The fundamental relation plays an important role in theory of algebraic hyperstructure (for more details see [2, 6, 7, 11]). Let us first survey some simple results on hypergroups such that we will apply in the next sections.

Lemma 3.1. *Let $(G, \odot)$, $(H, \odot')$ be hypergroups and $f : (G, \odot) \to (H, \odot')$ be a homomorphism. Then the following statements are hold:*

- (i) *For any $x, y \in G$, $x\beta^* y$ implies that $f(x)\beta^* f(y)$;*
- (ii) *If f is an injection, then for any $x, y \in G$, $f(x)\beta^* f(y)$ implies that $x\beta^* y$;*
- (iii) *If f is a bijection, then for any $x, y \in G$, $x\beta^* y$ if and only if $f(x)\beta^* f(y)$;*
- (iv) *If f is a bijection. Then for any $x \in G$, $f(\beta^*(x)) = \beta^*(f(x))$.*

Proof. (i) Let $\mathcal{U}$ be the set of all finite products of elements of G, $\mathcal{U}'$ be the set of all finite products of elements of H and $x, y \in G$. Since $x\beta^* y$, then there exists $u \in \mathcal{U}$, such that $\{x, y\} \subseteq u$. Now, for the homomorphism $f : (G, \odot) \to (H, \odot')$ we have $\{f(x), f(y)\} = f\{x, y\} \subseteq f(u) \in \mathcal{U}'$. Therefore, $f(x)\beta^* f(y)$.

(ii) For $x, y \in G$, since $f(x)\beta^* f(y)$, there exists $v \in \mathcal{U}'$, such that $\{f(x), f(y)\} \subseteq v$. Now, for an injection $f : (G, \odot) \to (H, \odot')$, we have

$$\{x, y\} = \{f^{-1}(f(x)), f^{-1}(f(y))\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(v) \in \mathcal{U}.$$

Therefore, $x\beta^*y$.

(iii) By (i) and (ii), the proof is straightforward.

(iv) Let $x \in G$. Then we have

$$f(\beta^*(x)) = \bigcup_{y\in\beta^*(x)} f(y) = \bigcup_{y\beta^*x} f(y).$$

By (iii), for any $x, y \in G$, $x\beta^*y$ if and only if $f(x)\beta^*f(y)$. Therefore,

$$\begin{aligned} f(\beta^*(x)) = \bigcup_{y\in\beta^*(x)} f(y) &= \bigcup_{y\beta^*x} f(y) \\ &= \bigcup_{f(y)\beta^*f(x)} f(y) = \bigcup_{f(y)\in\beta^*(f(x))} f(y) \\ &= \beta^*(f(x)) \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let* $(G_1, \odot_1)$ *and* $(G_2, \odot_2)$ *be isomorphic hypergroups. Then* $(\frac{(G_1,\odot_1)}{\beta^*}, \otimes) \cong (\frac{(G_2,\odot_2)}{\beta^*}, \otimes)$.

Proof. Let $f : (G_1, \odot_1) \longrightarrow (G_2, \odot_2)$ be an isomorphism. Define the mapping $\theta : (\frac{(G_1,\odot_1)}{\beta^*}, \otimes) \longrightarrow (\frac{(G_2,\odot_2)}{\beta^*}, \otimes)$ by $\theta(\beta^*(x)) = \beta^*(f(x))$. By Lemma 3.1, θ is well-defined and one to one. Since f is a homomorphism then θ is an isomorphism. □

Now, briefly introduce the category of hypergroup. The category $\mathcal{H}_g$ consists of the following data:

Objects: $(G, \odot), (H, \odot'), \ldots$ that are hypergroups.
Arrows: $f, g, \ldots$ that are good homomorphisms.

For each arrow $f : (G, \odot) \longrightarrow (H, \odot')$ there are given objects $dom(f) = G$ and $cod(f) = H$ which called *domain* and *codomain*, respectively. Given arrows $f : (G, \odot) \longrightarrow (H, \odot')$ and $g : (H, \odot') \longrightarrow (T, \odot'')$, such that $cod(f) = dom(g)$, there is an arrow $g \circ f : G \longrightarrow T$ called the *composite* of f and g and for any arrows $h : (G, \odot) \longrightarrow (H, \odot'), g : (H, \odot') \longrightarrow (T, \odot''), f : (T, \odot'') \longrightarrow (M, \odot''')$ have $(f \circ g) \circ h = f \circ (g \circ h)$. For object G there is given an arrow $1 : G \longrightarrow G$ which called the *identity arrow* of G and for any arrow $f : G \longrightarrow G$, have $f \circ 1 = 1 \circ f = f$.

Definition 3.3. A group $(G, .)$ is said to be a *fundamental group* if there exists a nontrivial hypergroup, say $(H, \odot)$, such that $(\frac{(H,\odot)}{\beta^*}, \otimes) \cong (G, .)$. In other words, it is equal to the fundamental of nontrivial hypergroup up to isomorphic.

Remark 3.4. We know that on any group $(G, .)$, if define a binary hyperoperation "$\odot$" as $x \odot y = \{x.y\}$ such that is singleton, then $(G, \odot)$ is a *trivial hypergroup*. Therefore, its fundamental group is isomorphic to $(G, .)$. In the following, we define a nontrivial hypergroup such that its fundamental group, be isomorphic to given group $(G, .)$.

Lemma 3.5. *Let $(G,.)$ be a group. Then for any group $(H,.)$, there exists a binary hyperoperation "$\odot$" on group $G\times H$ such that $(G\times H,\odot)$ is a hypergroup.*

Proof. Let $(H,.)$ be a nonzero group. Define a hyperoperation "$\odot$" on $G\times H$, as follows:

$$(g,h)\odot(g',h')=\{(g.g',h),(g.g',h')\}$$

Clearly $\odot$ is associative. We verify reproduction axiom. Let $(g,h)\in(G\times H)$. Since $(g,h)\in(g,h)\odot(1,1)=\{(g.1,s),(g.1,1)\}=\{(g,h),(g,1)\}$, then

$$\begin{aligned}(g,h)\odot(G\times H) &= \bigcup_{(g',h')\in G\times H}(g,h)\odot(g',h')\\ &= \bigcup_{(g',h')\in G\times H}\{(g.g',h),(g.g',h')\}\\ &= G\times H\end{aligned}$$

and similarly, it obtained that $(G\times H)\odot(g,h)=G\times H$. Therefore, $(G\times H,\odot)$ is a hypergroup. □

Example 3.6. Let $G=\{e,a,b,c\}$ be Kline's four-group and $H=G\times\mathbb{Z}_2$. Then define a hyperoperation $\odot$ on H as follows:

$\odot$	e_0	e_1	a_0	a_1	b_0	b_1	c_0	c_1
e_0	$\{e_0\}$	$\{e_0,e_1\}$	$\{a_0\}$	$\{a_0,a_1\}$	$\{b_0\}$	$\{b_0,b_1\}$	$\{c_0\}$	$\{c_0,c_1\}$
e_1	$\{e_0,e_1\}$	$\{e_1\}$	$\{a_0,a_1\}$	$\{a_1\}$	$\{b_0,b_1\}$	$\{b_1\}$	$\{c_0,c_1\}$	$\{c_1\}$
a_0	$\{a_0\}$	$\{a_0,a_1\}$	$\{e_0\}$	$\{e_0,e_1\}$	$\{c_0\}$	$\{c_0,c_1\}$	$\{b_0\}$	$\{b_0,b_1\}$
a_1	$\{a_0,a_1\}$	$\{a_1\}$	$\{e_0,e_1\}$	$\{e_1\}$	$\{c_0,c_1\}$	$\{c_1\}$	$\{b_0,b_1\}$	$\{b_1\}$
b_0	$\{b_0\}$	$\{b_0,b_1\}$	$\{c_0\}$	$\{c_0,c_1\}$	$\{e_0\}$	$\{e_0,e_1\}$	$\{a_0\}$	$\{a_0,a_1\}$
b_1	$\{b_0,b_1\}$	$\{b_1\}$	$\{c_0,c_1\}$	$\{c_0\}$	$\{e_0,e_1\}$	$\{e_1\}$	$\{a_0,a_1\}$	$\{a_1\}$
c_0	$\{c_0\}$	$\{c_0,c_1\}$	$\{b_0\}$	$\{b_0,b_1\}$	$\{a_0\}$	$\{a_0,a_1\}$	$\{e_0\}$	$\{e_0,e_1\}$
c_1	$\{c_0,c_1\}$	$\{e_1\}$	$\{b_0,b_1\}$	$\{b_1\}$	$\{a_0,a_1\}$	$\{a_1\}$	$\{e_0,e_1\}$	$\{e_1\}$

where, for any $w\in G$ and $\overline{x}\in\mathbb{Z}_2$, $w_x=(w,\overline{x})$. A routine calculation shows that $(H,\odot)$ is a hypergroup.

Remark 3.7.

(i) The hypergroup $(G\times H,\odot)$ is called the *associated hypergroup* to G via H (or shortly associated hypergroup) and is denoted by G_H.

(ii) The mapping $\varphi:G\longrightarrow G_H$ by $\varphi(g)=(g,1)$ is an embedding.

(iii) G_H is a hypergroup with identity.

(iv) $H=\mathbb{Z}$ and we denote G_H by $\overline{G}$.

(vi) For $H=\mathbb{Z}_2, G_H$ is the smallest associated hypergroup.

Theorem 3.8. *Let $(G_1,.)$ and $(G_2,.)$ be isomorphic groups. Then, for any group $(H,.)$, ${G_1}_H$ and ${G_2}_H$ are isomorphic hypergroups.*

Proof. Let $f : (G_1, .) \longrightarrow (G_2, .)$ be an isomorphism. Define a map $\theta : (G_1 \times H, \odot) \longrightarrow (G_2 \times H, \odot)$ by $\theta(g, h) = (f(g), h)$ where $r \in G_1$. Clearly θ is a bijection. Now, we show that θ is a homomorphism. Let $(g_1, h), (g_2, h') \in G_1 \times H$. Then,

$$\begin{aligned}
\theta((g_1, h) \odot (g_2, h')) &= \theta(\{(g_1.g_2, h), (g_1.g_2, h')\}) \\
&= \{\theta(g_1.g_2, h), \theta(g_1.g_2, h')\} = \{(f(g_1.g_2), h), (f(g_1.g_2), h')\} \\
&= \{(f(g_1).f(g_2), h), (f(g_1).f(g_2), h')\} \\
&= (f(g_1), h) \odot (f(g_2), h')) \\
&= \theta((g_1, h)) \odot \theta((g_2, h'))
\end{aligned}$$

Therefore, θ is an isomorphism and $(G_1 \times H, \odot) \cong (G_2 \times H, \odot)$. □

Theorem 3.9. *Every group is a fundamental group.*

Proof. Let $(G, .)$ be a group. By Lemma 3.5, for any group $(H, .)$, $(G \times H, \odot)$ is a hypergroup. For any $g \in G$ and $(h, h') \in H \times H$, we have $\{(g, h), (g, h')\} = (g, h) \odot (1, h')$, then $(g, h)\beta^*(g, h')$. Thus $\beta^*(g, h) = \{(g, x) \mid x \in H\}$. Define the mapping $\varphi : (\frac{(G \times H, \odot)}{\beta^*}, \otimes) \longrightarrow (G, .)$ by $\varphi(\beta^*(g, h)) = g$. Obviously, φ is well-defined and one to one, since for any $(g, h), (g', h') \in G \times H$, $\beta^*((g, h)) = \beta^*((g', h'))$ if and only if $g = g'$ if and only if $\varphi(\beta^*(g, h)) = \varphi(\beta^*(g', h'))$. Let $(g, h), (g', h') \in G \times H$. Then,

$$\begin{aligned}
\varphi(\beta^*(g, h) \otimes \beta^*(g', h')) = \varphi(\beta^*(g.g', h)) &= \varphi(\beta^*(g.g', h')) \\
&= g.g' = \varphi(\beta^*(g, h)).\varphi(\beta^*(g', h'))
\end{aligned}$$

Thus, φ is a homomorphism. Now, for any $g \in G, \varphi(\beta^*(g, 1)) = g, then$ φ is onto. Therefore, φ is an isomorphism and then $(\frac{(G \times H, \odot)}{\beta^*}, \otimes) \cong (G, .)$. □

Example 3.10. Consider the hypergroup which is defined in Example 3.6. It is easy to see that $\beta^*(a, \overline{0}) = \beta^*(a, \overline{1}) = \{(a, \overline{0}), (a, \overline{1})\}$, $\beta^*(b, \overline{0}) = \beta^*(b, \overline{1}) = \{(b, \overline{0}), (b, \overline{1})\}$, $\beta^*(c, \overline{0}) = \beta^*(c, \overline{1}) = \{(c, \overline{0}), (c, \overline{1})\}$ and $\beta^*(e, \overline{0}) = \beta^*(e, \overline{1}) = \{(e, \overline{0}), (e, \overline{1})\}$. So $(\frac{(G \times \mathbb{Z}_2, \odot)}{\beta^*}, \otimes) = \{\beta^*(a, \overline{0}), \beta^*(b, \overline{0}), \beta^*(c, \overline{0}), \beta^*(e, \overline{0})\} \cong (G, .)$.

Theorem 3.11. *Let $(G, \odot)$ be a hypergroup. Then there exist a group H and hyperoperation " $\odot'$ " on $G \times H$ such that $(G, \odot)$ embedded in $(G \times H, \odot')$.*

Proof. Let $(G, \odot)$ be a hypergroup and $H = (\frac{G}{\beta^*}, \otimes)$. Define on $G \times \frac{G}{\beta^*}$, the hyperoperation " $\odot'$ " as follows:

$$(g, \beta^*(h)) \odot' (g', \beta^*(h')) = (g \odot g', \beta^*(h \odot h'))$$

Let $(g_1, \beta^*(h_1)) = (g'_1, \beta^*(h'_1))$ and $(g_2, \beta^*(h_2)) = (g'_2, \beta^*(h'_2))$, then $g_1 = g'_1, g_2 = g'_2, \beta^*(h_1) = \beta^*(h'_1)$ and $\beta^*(h_2) = \beta^*(h'_2)$. Since $\beta^*(h_1) = \beta^*(h'_1)$ and $\beta^*(h_2) = \beta^*(h'_2)$, there exist $u, v \in \mathcal{U}$ such that $\{h_1, h'_1\} \subseteq u$ and $\{h_2, h'_2\} \subseteq v$. Clearly

$$\{h_1 \odot h_2, h'_1 \odot h'_2\} \subseteq \{h_1 \odot h_2, h_1 \odot h'_2, h'_1 \odot h_2, h'_1 \odot h'_2, \} \subseteq u \odot v$$

Then, $\beta^*(h_1 \odot h_2) = \beta^*(h_1' \odot h_2')$. Now, we have $(g_1 \odot g_2, \beta^*(h_1 \odot h_2)) = (g_1' \odot g_2', \beta^*(h_1' \odot h_2'))$. Hence, the hyperoperation " $\odot'$ " is well-defined. Now, we show that $(G \times H, \odot')$ is a hypegroup.

(Associativity): Let $(g, \beta^*(h)), (g', \beta^*(h'))$ and $(g'', \beta^*(h'')) \in G \times H$. Then, $((g, \beta^*(h)) \odot' (g', \beta^*(h'))) \odot' (g'', \beta^*(h'')) = (g \odot g', \beta^*(h \odot h')) \odot' (g'', \beta^*(h'')) = ((g \odot g') \odot g'', (\beta^*(h \odot h') \odot \beta^*(h''))$. Since $(G, \odot)$ is a hypergroup, then $(g \odot g') \odot g'' = g \odot (g' \odot g'')$, $(h \odot h') \odot h'' = h \odot (h' \odot h'')$ and then $((g, \beta^*(h)) \odot' (g', \beta^*(h'))) \odot' (g'', \beta^*(h'')) = (g, \beta^*(h)) \odot' ((g', \beta^*(h')) \odot' (g'', \beta^*(h'')))$.

(Reproduction): Let $(g, \beta^*(h)) \in G \times H$. Since, $g \odot G = G \odot g = G$ and $\frac{G}{\beta^*} = \bigcup_{t \in G} \beta^*(t)$, we have

$$
\begin{aligned}
(g, \beta^*(h)) \odot' (G \times H) &= \bigcup_{(g', \beta^*(h')) \in G \times H} (g, \beta^*(h)) \odot (g', \beta^*(h')) \\
&= \bigcup_{(g', \beta^*(h')) \in G \times H} (g \odot g', \beta^*(h \odot h')) = G \times H.
\end{aligned}
$$

Similarly, we conclud that $(G \times H) \odot' (g, \beta^*(h)) = G \times H$.

Hence, $(G \times H, \odot')$ is a hypergroup. Define the mapping $\varphi : (G, \odot) \to (G \times H, \odot')$, by $\varphi(g) = (g, \beta^*(g))$. Let $g, g' \in G$, then $g = g'$ if and only if $(g, \beta^*(g)) = (g', \beta^*(g'))$ if and only if $\varphi(g) = \varphi(g')$ and so φ is well-defined. Now for any $g, g' \in G$,

$$
\varphi(g \odot g') = (g \odot g', \beta^*(g \odot g')) = (g, \beta^*(g)) \odot' (g', \beta^*(g')) = \varphi(g) \odot' \varphi(g')
$$

Therefore, $(G, \odot)$ embedded in $(G \times H, \odot')$. □

Theorem 3.12. *Let G and H be two sets such that be cocardinal. If $(G, \odot)$ is a hypergroup, then there exists a hyperoperation " $\odot'$ " on H, such that $(G, \odot)$ and $(H, \odot')$ are isomorphic hypergroups.*

Proof. Since $|G| = |H|$ (are cocardinal), there exists a bijection $\varphi : G \longrightarrow H$. For any $h_1, h_2 \in H$, define the hyperoperation " $\odot'$ " on H as follows:

$$
h_1 \odot' h_2 = \varphi(g_1 \odot g_2)
$$

We first show that " $\odot'$ " is well-defined. Let $(h_1, h_2) = (h_1', h_2')$, where $h_i = \varphi(g_i)$, $h_i' = \varphi(g_i')$ and $g_i, g_i' \in G$ for $1 \leq i \leq 2$. Then $h_i = h_i'$ implies that $\varphi(g_i) = \varphi(g_i')$. Since φ is a bijection then clearly $g_i = g_i'$ and so $g_1 \odot' g_2 = \varphi(g_1 \odot g_2) = \varphi(g_1' \odot g_2') = h_1' \odot' h_2'$. Moreover,

$$
\varphi(g_1 \odot g_2) = \varphi(g_1) \odot' \varphi(g_2) \tag{1}
$$

Now, by some modifications we can show that $(H, \odot')$ is a hypergroup. Let $g_1, g_2 \in G$. Then, by (1), φ is a homomorphism. Therefore, φ is a homomorphism and hence φ is an isomorphism. □

Corollary 3.13. *Let $(G, .)$ be a group of infinite order ($|G| = \infty$). Then there exists a hyperoperation " $\odot$ " on G such that $(G, .)$ is a fundamental group of itself (i.e. $\frac{(G,\odot)}{\beta^*} \cong (G, .)$).*

Proof. For a given group G, consider the smallest associated hypergroup $(G \times \mathbb{Z}_2, \odot)$. By Theorem 3.9, $(\frac{(G\times H,\odot)}{\beta^*}, \otimes) \cong (G, .)$. Since G is infinite set then $|G| = |G \times \mathbb{Z}_2|$ and by Theorem 3.12, there exists a hyperoperation " $\odot'$ " on G, such that $(G, \odot')$ and $(G \times \mathbb{Z}_2, \odot)$ are isomorphic hypergroups. Now, we have

$$(G, .) \cong (\frac{(G \times \mathbb{Z}_2, \odot)}{\beta^*}, \otimes) \cong (\frac{(G, \odot')}{\beta^*}, \otimes)$$

Therefore, $(G, .)$ is a fundamental group of itself. □

Theorem 3.14. *Every finite group is not fundamental group of itself.*

Proof. Let $(G, .)$ be a finite group and $|G| = n$. If "$\odot$" is hyperoperation on G, such that $(G, \odot)$ is a hypergroup. Then there exist $x, y \in G$ such that $|x \odot y| \geq 2$. Hence, for finite set of indices I and the elements $g_i \in G$ there exist $a, b \in (x \odot y) \odot \prod_{i\in I} g_i$ such that $\beta^*(a) = \beta^*(b)$. Since $\frac{G}{\beta^*} = \{\beta^*(t) \mid t \in G\}$, then, $|\frac{G}{\beta^*}| < n$. Therefore, $(G, .) \ncong (\frac{(G,\odot)}{\beta^*}, \otimes)$. □

4. Category of Hypergroups and Groups

In this section we apply the results that obtained in the previous sections and define functors on categories of $\mathcal{H}_g$ and $\mathcal{G}_r$ (category of groups) and try to work up natural transformations between. For two categories $\mathcal{H}_g$ and $\mathcal{G}_r$, define a categorical morphism (**fundamental functor**) as follows:

$$U : \mathcal{H}_g \longrightarrow \mathcal{G}_r \text{ by } U(G) = (\frac{G}{\beta^*}, \otimes) \tag{2}$$

where $(G, \odot)$ is a hypergroup and for any homomorphism $f : (G_1, \odot_1) \longrightarrow (G_2, \odot_2)$, we define

$$U(f) : (\frac{G_1}{\beta^*}, \otimes) \longrightarrow (\frac{G_2}{\beta^*}, \otimes) \text{ by } U(f) = \beta^*(f) \tag{3}$$

By Corollary 3.2, U is well-defined and we have the next result.

Theorem 4.1. *U is a functor of $\mathcal{H}_g$ to $\mathcal{G}_r$.*

Proof. For any object $(G, \odot)$ of $\mathcal{H}_g$, by (2), $U(G) = (\frac{G}{\beta^*}, \otimes)$ is a group and then $U(G)$ is an object in $\mathcal{G}_r$. Now, we show that for any morphism $f : (G_1, \odot) \longrightarrow (G_2, \odot), Uf$ is a morphism in $\mathcal{G}_r$. Let $\beta^*(x), \beta^*(y) \in \frac{G}{\beta^*}$. Then, by (3),

$$\begin{aligned} Uf(\beta^*(x) \otimes \beta^*(y)) &= Uf(\beta^*(x \odot y)) = \beta^*(f(x \odot y)) \\ &= \beta^*(f(x) \odot f(y)) = Uf(\beta^*(x)) \otimes Uf(\beta^*(y)) \end{aligned}$$

Hence, if $g : (G_1, \odot) \longrightarrow (G_2, \odot)$ and $f : (G_2, \odot) \longrightarrow (G_3, \odot)$ are morphisms in $\mathcal{H}_g$, then $U(g) : (\frac{G_1}{\beta^*}, \otimes) \longrightarrow (\frac{G_2}{\beta^*}, \otimes)$ by $U(g) = \beta^*(g)$ and $U(f) : (\frac{G_2}{\beta^*}, \otimes) \longrightarrow (\frac{G_3}{\beta^*}, \otimes)$ by $U(f) = \beta^*(f)$ are morphisms in $\mathcal{G}_r$. Now,

$$U(f) \circ U(g) = U(f)(U(g)) = U(f)(\beta^*(g)) = \beta^*(f \circ g) = U(f \circ g)$$

Moreover, for $1 : G \to G$, $U(1) = \beta^*(1)$ and then for any $x \in G$,

$$U(1)(x) = \beta^*(1(x)) = \beta^*(x) = 1_{UG}(x)$$

Therefore, U is a functor of $\mathcal{H}_g$ to $\mathcal{G}_r$. □

Remark 4.2. Let G be a group and

$$\mathcal{B}(G) = \{H \in \mathcal{H}_g \mid \text{ G is isomorphic with to a fundamentl group of H}\}.$$

By Theorem 3.9, $\mathcal{B}(G) \neq \emptyset$.

Now, for $\mathcal{H}_g$ and $\mathcal{G}_r$ and any groups $(G, .)$ and $H = \mathbb{Z}_2$, define a categorical morphism as follows:

$$F : \mathcal{G}_r \longrightarrow \mathcal{H}_g \text{ by } F(G) = G_H \tag{4}$$

and for any group homomorphism $f : (G_1, .) \longrightarrow (G_2, .)$ define

$$F(f) : G_1 \times H \longrightarrow G_2 \times H \text{ by } F(f) = (f, 1) \tag{5}$$

By Theorem 3.8, F is well defined and now, we have the next result.

Theorem 4.3. *F is a functor of $\mathcal{G}_r$ to $\mathcal{H}_g$.*

Proof. For any object $(G, .)$ of $\mathcal{G}_r$, by Theorem 3.9 and (4), $F(G) = G_H$ is a hypergroup and then $F(G)$ is an object in $\mathcal{H}_g$. Now, we show that for any morphism $f : (G_1, \odot) \longrightarrow (G_2, \odot), Ff$ is a morphism in $\mathcal{H}_g$. Let $(g_1, h_1), (g_2, h_2) \in G_1 \times H$. Now, by Theorem 3.9 and (5),

$$\begin{aligned} Ff((g_1, h_1) \odot (g_2, h_2)) &= (f, 1)((g_1, h_1) \odot (g_2, h_2)) \\ &= (f(g_1), h_1) \odot (f(g_2), h_2) = Ff((g_1, h_1)) \odot Ff((g_2, h_2)). \end{aligned}$$

Hence, if $g:(G_1,.)\longrightarrow(G_2,.)$ and $f:(G_2,.)\longrightarrow(G_3,.)$ are morphisms in $\mathcal{G}_r$, then $F(g):(G_1,\odot)\longrightarrow(G_2,\odot)$ by $F(g)=(g,1)$ and $F(f):(G_2,\odot)\longrightarrow(G_3,\odot)$ by $F(f)=(f,1)$ are morphisms in $\mathcal{H}_g$. Now,

$$\begin{aligned}(F(f)\circ F(g))(g,h)=F(f)(F(g))(g,h) &= F(f)(g(r),s)\\ &= (f(g(r)),s)=(f\circ g,1)(g,h)\\ &= F(f\circ g)(g,h)\end{aligned}$$

Moreover, for $1:G\to G$, $F(1)=(1,1)$ and then for any $(g,h)\in G\times H$,

$$F(1)(g,h)=(1,1)(g,h)=(g,h)=1_{UG}(g,h)$$

Therefore, F is a functor of $\mathcal{G}_r$ to $\mathcal{H}_g$. □

Theorem 4.4. *The functor $F:\mathcal{G}_r\longrightarrow\mathcal{H}_g$ is a faithful functor.*

Proof. Let $(G_1,.)$ and $(G_2,.)$ be objects in $\mathcal{G}_r$, $f_1,f_2:G_1\to G_2$ be parallel arrows of $\mathcal{G}_r$ and $F(f_1)=F(f_2)$. Then, for any $(g,h)\in G_1\times H, F(f_1)(g,h)=F(f_2)(g,h)$ implies that $f_1(g)=f_2(g)$ and so $f_1=f_2$. Therefore, F is a faithful functor. □

Theorem 4.5. *On objects of $\mathcal{G}_r$, $U\circ F=1$.*

Proof. For any object $(G,.)$ in $\mathcal{G}_r$ by Theorem 3.9, (2) and (4)

$$(U\circ F)(G,.)=U(G\times H,\odot)=(\frac{((G\times H),\odot)}{\beta^*},\otimes)\cong(G,.) \tag{6}$$

□

Remark 4.6. Consider the hypergroup $(\mathbb{Z}_4,\odot')$ by the following hyperoperation:

$\odot'$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$
$\overline{1}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$
$\overline{2}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$
$\overline{3}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$	$\{\overline{1},\overline{3}\}$	$\{\overline{0},\overline{2}\}$

Clearly,

$$(F\circ U)(\mathbb{Z}_4,\odot')=F(\frac{(\mathbb{Z}_4,\odot')}{\beta^*})\cong F((\mathbb{Z}_2,.))=(\mathbb{Z}_2\times\mathbb{Z}_2,\odot)\ncong(\mathbb{Z}_4,\odot')$$

Therefore, $F\circ U\neq 1$.

Theorem 4.7. *For the functors 1 and $U\circ F:\mathcal{G}_r\longrightarrow\mathcal{G}_r$ there exists a transformation $\tau:1\longrightarrow U\circ F$ such that is natural.*

Proof. For two functors 1 (identity) and $U \circ F$ of $\mathcal{G}_r$ to $\mathcal{G}_r$, define a map $\tau : 1 \longrightarrow U \circ F$ as follows:

$$\tau : 1(G) \longrightarrow (U \circ F)(G) \ \text{ by } \tau(g) = \beta^*(g, 1) \tag{7}$$

Now, for any group homomorphism $f : G \longrightarrow G'$, consider the following diagram:

$$\begin{array}{ccc} 1(G) & \xrightarrow{\tau_G} & (U \circ F)(G) \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle U\circ F(f)} \\ 1(G') & \xrightarrow{\tau_{G'}} & (U \circ F)(G') \end{array}$$

For any $g \in G$ by (5) and (7), we have

$$\begin{aligned} ((U \circ F)(f) \circ \tau)g &= (U \circ F)f(\tau(g)) \\ &= (U \circ F)(f)(\beta^*(g, 1)) \\ &= (\beta^*(f(g)), 1) = \tau_{G'}(f(g)) \\ &= \tau_{G'}(1(f)g) = (\tau_{G'} \circ 1(f))g \end{aligned}$$

Therefore, $\tau : 1 \longrightarrow (U \circ F)$ is a natural transformation. □

Theorem 4.8. *For two functors* 1 *and* $F \circ U : \mathcal{H}_g \longrightarrow \mathcal{H}_g$ *there exists a transformation* $\upsilon : 1 \longrightarrow F \circ U$ *such that is natural.*

Proof. For two functors 1 (identity) and $F \circ U$ of category $\mathcal{H}_g$ to category $\mathcal{H}_g$, define a map $\upsilon : 1 \longrightarrow F \circ U$ as follows:

$$\upsilon : 1(G) \longrightarrow (F \circ U)(G) \ \text{ by } \upsilon(g) = (\beta^*(g), 1) \tag{8}$$

Now, for morphism homomorphism $f : G \longrightarrow G'$, consider the following diagram:

$$\begin{array}{ccc} 1(G) & \xrightarrow{\nu_G} & (F \circ U)(G) \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle F\circ U(f)} \\ 1(G') & \xrightarrow{\nu_{G'}} & (F \circ U)(G') \end{array}$$

For any $g \in G$, by (3) and (8), we have

$$\begin{aligned} ((F \circ U)(f) \circ \upsilon)g &= (F \circ U)f(\upsilon(g)) = (F \circ U)f((\beta^*(g), 1)) \\ &= (\beta^*(f(g)), 1) \\ &= \upsilon_{G'}(f(g)) = \upsilon_{G'}(1(f)g) = (\upsilon_{G'} \circ 1(f))g \end{aligned}$$

Therefore, $\upsilon : 1 \longrightarrow (F \circ U)$ is a natural transformation. □

5. Conclusions

In the present paper, we introduced the notion of fundamental group via the fundamental relation β^* and investigated some of their useful properties. Moreover, we shown that:

(*i*) Any group is a fundamental group.

(*ii*) Any infinite group is a fundamental group of itself, while no finite group is a fundamental group of itself.

(*iii*) By using the concept of fundamental relation, we obtained (faithful) the functors between categories of groups and hypergroups and shown that there exist natural transformations between their combinations and identity functor.

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