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Structure of the Fixed Point of Condensing Set-Valued Maps

Zeinab Soltani

Abstract

In this paper, we present structure of the fixed point set results for condensing set-valued maps. Also, we prove a generalization of the Krasnosel'skii-Perov connectedness principle to the case of condensing set-valued maps.

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1. Introduction and Preliminaries

The famous Schauder fixed point theorem [16] has been generalized in various directions by using different methods, see [1–6,8,12,13,15,17] and reference therein.

The topological structure of the set of fixed points has important applications in structure of the solution sets of differential equations and inclusions. The topological degree is a fundamental tool for proving the existence of fixed point and topological characterization of the set of fixed point and solutions for differential equations, differential inclusions and dynamical systems.

In 1959 Krasnosel'skii and Perov [9] proved a connectedness principle for single valued compact continuous maps and later a generalization of this theorem was proved by B. D. Gel'man in 1997 [7] for connectedness of the set of fixed points of compact continuous set-valued maps. In this paper, we prove a generalization of the Krasnosel'skii-Perov connectedness principle to the case of condensing set-valued maps.

Corresponding author (Email: z.soltani@kashanu.ac.ir)
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In this paper, we present the structure of the fixed point set results for condensing set-valued maps. Our discussion is based on the theory of topological degree for ultimately compact set valued maps developed by Petryshyn and Fitzpatric in 1974 [14].

Now, we introduce some definitions and facts which will be used in the sequel. All topological spaces are assumed to be metric. The set-valued map $T : X \multimap Y$ is said to be:

(i) upper semicontinuous, if for each closed set $B \subseteq Y$, $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X.

(ii) lower semicontinuous if for each open set $V \subseteq Y$, $T^{-}(V) = \{x \in X : T(x) \cap V \neq \emptyset\}$ is open in X.

(iii) continuous if it is both upper and lower semicontinuous.

Suppose that $T : X \multimap Y$ is a set-valued map, the graph of T is defined by $\Gamma_X(T) \subset X \times Y$:

$$\Gamma_X(T) = \{(x, y) : x \in X, y \in T(x)\}.$$

Definition 1.1. Let $T : X \multimap Y$ be a set-valued map. A continuous map $f : X \to Y$ is called an $\varepsilon-$approximation of T if the graph $\Gamma_X(f)$ of f belongs to the $\varepsilon-$neighbourhood of the graph $\Gamma_X(T)$ of T, that is $\Gamma_X(f) \subset U_\varepsilon(\Gamma_X(T))$.

We shall also need some facts about the Kuratowski and Hausdorff measure of noncompactness. Let X be a Banach space and $B(X)$ denote the family of bounded subset of X. Let $A \in B(X)$ then the Kuratowski measure of non-compactness of A is defined as $\gamma(A) = \inf\{\delta > 0| \ A = \cup_{i=1}^{n} A_i, \mathrm{diam(A_i)} \leq \delta\}$, where $\mathrm{diam(A)} = \sup\{d(x, y)|x, y \in A\}$.

Suppose that $T : X \multimap X$ is a set-valued map. T is said to be condensing if $\gamma(T(A)) < \gamma(A)$ for all bounded subsets A of X that $\gamma(A) > 0$ and $T(A) \subseteq B(X)$.

Definition 1.2. [10] Let X be a Banach space. A compact valued map $T : X \multimap X$ is said to be demicompact if whenever (x_n) is bounded sequence and $(d(x_n, T(x_n)))$ is a convergent sequence, then there exists a convergent subsequence (x_{ni}) of (x_n).

Notice that every condensing map is demicompact.

Definition 1.3. Let (X, d) be a metric space, $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. Then, the Pompeiu-Hausdorff metric on $CB(X)$ is given by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$.

Let X be a metrizable locally convex topological vector space. If $D \subset X$, we denote by $K(D)$ and $CK(D)$ the family of closed convex, and the family of compact convex subsets of D, respectively. We also use $\overline{D}$ and ∂D to denote the closure and boundary of D, respectively. Let $T : D \multimap X$ be a closed convex valued upper semicontinuous map. Suppose that $K_0 = \overline{co}(T(D))$, let η be an ordinal such that K_β is defined for $\beta < \eta$. If η is of the first kind we set $K_\eta = \overline{co}(T(D \cap K_{\eta-1}))$ and if η is of the second kind we set $K_\eta = \cap_{\beta<\eta} K_\beta$. Then K_α is well defined and such that $K_\alpha \subset K_\beta$ if $\alpha > \beta$. Consequently, there exists an ordinal γ such that $K_\beta = K_\gamma$ if $\beta \geq \gamma$. We define $K = K(T, D) = K_\gamma$ and observe that $\overline{co}(T(K \cap D) = K$. The mapping T is called ultimately compact if either $K \cap D = \emptyset$ or $T(K \cap D)$ is relatively compact.

Definition 1.4. [14] Let $D \subset X$ be open and $T : D \multimap X$ be a closed convex valued upper semicontinuous map. Suppose that T is ultimately compact such that $x \notin T(x)$ when $x \in \partial D$. If $K(T, D) \cap D = \emptyset$, we define $deg(I - T, D, 0) = 0$, and when $K(T, D) \cap D \neq \emptyset$, let ρ be a retraction of X onto $K(T, D)$ and define

$$deg(I - T, D, 0) = deg_c(I - T \circ \rho, \rho^{-1}(D), 0), \tag{1}$$

where the right-hand side of (1) means the topological degree defined in [11] for multivalued compact vector fields.

Theorem 1.5. *[14] Let X, D and T be as in Definition 1.4, then the degree given by (1) satisfies the following conditions:*

(1) if $deg(I - T, D, 0) \neq 0$, then T has a fixed point in D;

(2) if $H : D \times [0, 1] \multimap X$ is closed convex valued upper semicontinuous, $H(\overline{D} \cap K' \times [0, 1])$ is relatively compact where $K' = K(H, \overline{D} \times [0, 1])$, and $x \notin H_t(x)$ for $x \in \partial D$ and $t \in [0, 1]$, then $deg(I - H_0, D, 0) = deg(I - H_1, D, 0)$;

(3) if $D = D_1 \cup D_2$, where D_1 and D_2 are open and $D_1 \cap D_2 = \emptyset$ and $x \notin T(x)$ for $x \in \partial D_1 \cup \partial D_2$, then $deg(I - T, D, 0) = deg(I - T, D_1, 0) + deg(I - T, D_2, 0)$.

2. Structure of Fixed Point Set of Condensing Map

In this section, we present structure of fixed point set results for a condensing map.

Theorem 2.1. *Let D be a bounded open subset of a Banach space X and let $T : \overline{D} \multimap X$ be a upper semicontinuous condensing set-valued mapping with compact convex valued such that $deg(I - T, D, 0) \neq 0$. Suppose that there exists a sequence (T_n) of upper semicontinuous condensing mapping of $\overline{D}$ into X such that*

(a) $\delta_n = \sup\{H(T_n(x), T(x)),\ x \in \overline{D}\} \to 0$ as $n \to \infty$;

(b) the equation $x \in T_n(x) + y$ has at most one solution in D if $\|y\| \leq \delta_n$.

Then, the set $F(T)$ of fixed point of T in D is a continuum.

Proof. Since $deg(I-T,D,0)\neq 0$, then (1) of Theorem 1.5 implies T has a fixed point and so $F(T)\neq\emptyset$. Furthermore, since $F(T)\subset T(F(T))$ and T is a condensing map we have that $F(T)$ is compact. Thus, it remains to show that $F(T)$ is connected. Assume that $F=F(T)$ is not connected. Hence, there exist two nonempty disjoint compact subsets F_1 and F_2 of D such that $F=F_1\cup F_2$ and $d(F_1,F_2)=s$. Let D_1 and D_2 be two disjoint open subsets of D such that $F_1\subset D_1$, $F_2\subset D_2$, $F\subset D_1\cup D_2$, $D_1\cap\overline{D_2}=\emptyset$, $D_2\cap\overline{D_1}=\emptyset$ and $F\subset D_1\cup D_2$. We have $F_i=\{x\in D:\ \ 0\in(I-T)(x)\}\cap D_i$. Since F_i for $i=1,2$, is compact, then there exists U_i such that $\overline{U_i}\subset D_i$. Therefore, for every $x\in\partial D_1\cup\partial D_2$, we have $x\notin T(x)$. By Theorem 1.5,

$$deg(I-T,D,0)=deg(I-T,D_1,0)+deg(I-T,D_2,0).$$

We set $\beta=\inf\{d(x,T(x)):\ \ x\in\acute{D}=\overline{D}\setminus(D_1\cup D_2)\}$. Since T is compact valued and demicompact, then $\beta>0$. Indeed, if $\beta=0$, for each $n\in\mathbb{N}$, then there exists $x_n\in\acute{D}$ such that $d(x_n,T(x_n))\leq\frac{1}{n}$. Since T is demicompact, then there exists subsequence (x_{n_k}) of (x_n) such that it converges to $x\in\acute{D}$. Also $T(x)$ is closed then $x\in T(x)$ and that is contradiction. Thus, for each $x\in\acute{D}$,

$$d(x,T(x))\geq\beta. \tag{2}$$

Let $x^*\in F$ be arbitrary. We have $x^*\in T(x^*)$ and for each $n\in\mathbb{N}$, there exists $x_n^*\in T_n(x^*)$ such that $\|x_n^*-x^*\|\leq H(T(x^*),T_n(x^*))$. We define $\tilde{T_n}$ as

$$\tilde{T_n}=T_n(x)+x^*+x_n^*. \tag{3}$$

From (4) and the condition (a), there exists an integer $N_0\geq 1$ such that for each $n\geq N_0,\tilde{T_n}(x)\neq x$ for all $x\in\acute{D}$. Indeed, the condition (a) implies that there is an integer $N_0\geq 1$ such that $\beta-2\delta_n\geq\frac{\beta}{2}$ for $n\geq N_0$. Also, since $\tilde{T_n}(x)$ is compact, then there exists $\tilde{y_n}\in\tilde{T_n}(x)$ such that $d(x,\tilde{T_n}(x))=\|x-\tilde{y_n}\|$ and there exist $y_n\in T_n(x)$ and $z_n\in T(x)$ such that $\tilde{y_n}=y_n+x^*-x_n^*$ and $\|y_n-z_n\|\leq H(T_n(x),T(x))$. Therefore, for $n\geq N_0$

$$\begin{aligned}
d(x,\tilde{T_n}(x)) &= \|x-\tilde{y_n}\|=\|x-y_n+x^*-x_n^*\|\geq\|x-y_n\|-\|x^*-x_n^*\|\\
&\geq \|x-z_n\|-\|z_n-y_n\|-\|x^*-x_n^*\|\|\\
&\geq d(x,T(x))-H(T_n(x),T(x))-H(T(x^*),T_n(x^*))\\
&\geq \beta-2\delta_n\geq\frac{\beta}{2}.
\end{aligned}$$

Thus, $deg(T-\tilde{T_n},D_i,0)$ is well defined for $n\geq N_0, i=1,2$.
Now, for each $n\geq N_0$, we consider the homotopy

$$H_{n\lambda}(x)=H_n(x,\lambda)=\lambda\tilde{T_n}(x)+(1-\lambda)T(x),\quad x\in\overline{D},\ \lambda\in[0,1].$$

$\tilde{T_n}$ and T are condensing maps and $H_n(0,\lambda)$ is a convex combination of two maps, so it is also a condensing map. Indeed, let $A \subset D$ and $\gamma(A) > 0$ and $\gamma(H_n(A \times [0,1])) \geq \gamma(A)$. We have $H_n(A \times [0,1]) \subset \bar{co}(\tilde{T_n}(A) \cup T(A))$, so

$$\begin{aligned} \gamma(A) &\leq \gamma(H_n(A \times [0,1])) \leq \gamma(\bar{co}(\tilde{T_n}(A) \cup T(A))) \\ &= \gamma(\tilde{T_n}(A) \cup T(A)) = \max\{\gamma(\tilde{T_n}(A)), \gamma(T(A))\}. \end{aligned}$$

Therefore, $\gamma(\tilde{T_n}(A)) \geq \gamma(A)$ or $\gamma(T(A)) \geq \gamma(A)$. But $\tilde{T_n}$ and T are condensing maps which implies that $\gamma(A) = 0$, then $H_n(0,\lambda)$ on $\overline{D} \times [0,1]$ is also a condensing map.

Now, we show that $d(x, H_n(x,\lambda)) > 0$ for each $x \in \acute{D}$ and $n \geq N_0$. First, we take $\varepsilon > 0$ such that $N_\varepsilon(F) \subset D_1 \cup D_2$ and for each $n \geq N_0$, we consider

$$U_n = \{x \in \overline{D} : \quad x \in \lambda\tilde{T_n}(x) + (1-\lambda)T(x) \;\; for\ some\ \lambda \in [0,1]\}.$$

We prove that the set U_n is belong to an $\varepsilon-$neighbourhood of F. We assume that it is not true. Then, for each $m \in \mathbb{N}$, there exists $x_m \in U_n$ such that $x_m \in \frac{1}{m}\tilde{T_n}(x_m) + (1-\frac{1}{m})T(x_m)$ and $x_m \notin N_\varepsilon(F)$. Thus, for any $z \in F$ we have $d(x_m, z) > \varepsilon$. There exist $w_m \in \tilde{T_n}(x_m)$ and $v_n \in T(x_m)$ such that $x_m = \frac{1}{m}w_m + (1-\frac{1}{m})v_m$. Therefore, $\|x_m - v_m\| = \frac{1}{m}\|w_m - v_m\|$ converges to 0. Since T is demicompact, then there exists subsequence (x_{m_k}) of (x_m) such that it converges to $x \in F$, that is a contradiction. Thus for each $n \geq N_0$, $U_n \subset N_\delta(F)$ and for each $x \in \acute{D} \subset (N_\delta(F))^c$ we have $x \notin \lambda\tilde{T_n}(x) + (1-\lambda)T(x)$.

Thus for each $n \geq N_0, i = 1, 2$, we have

$$deg(I - T, D_i, 0) = deg(I - \tilde{T_n}, D_i, 0).$$

Let $x^* \in F_1 \subset D_1$ and $n \geq N_0$, from definition of $\tilde{T_n}$ in (3), we have $x^* \in \tilde{T_n}(x^*)$ and x^* is only fixed point of $\tilde{T_n}$. Indeed, since $x_n^* \in T_n(x^*)$, then $x^* \in \tilde{T_n}(x^*)$, so x^* satisfies the equation $x \in \tilde{T_n}(x) + x^* + x_n^*$ and $H(T(x^*), T_n(x_n^*)) \leq \delta_n$, therefore, condition (b) implies that x^* is only fixed point of $\tilde{T_n}$. Hence, $\tilde{T_n}$ has not fixed point on $\overline{D_2}$, so $deg(I - \tilde{T_n}, D_2, 0) = 0$, i.e., $deg(I - T, D_2, 0) = 0$. Similarly, suppose that $x^* \in F_2 \subset D_2$. By the same argument, we have $deg(I - T, D_2, 0) = 0$. This is contradiction, so $F(T)$ is connected and thus is a continuum. □

Now we prove a generalization of the Krasnosel'skii-Perov connectedness principle to the case of condensing set-valued maps.

Theorem 2.2. *Let D be a bounded open subset of a Banach space X and $T : \overline{D} \multimap X$ be an upper semicontinuous condensing set-valued map with compact convex valued such that $deg(I - T, D, 0) \neq 0$. Assume that for any $\varepsilon > 0$ and any point $x_1 \in F(T)$ there exists a single-valued condensing map $f_{\varepsilon,x_1} : \overline{D} \to X$ such that:*

(i) the set $X_{\varepsilon,x_1} = \{x \in \overline{D} : x \in \lambda T(x) + (1-\lambda)f_{\varepsilon,x_1}(x)\ for\ some\ \lambda \in [0,1]\}$ belongs to the $\varepsilon-$neighbourhood of $F(T)$;

(ii) *the map* f_{ε,x_1} *has a connected set* F_{ε,x_1} *of fixed points;*

(iii) *there exists a point* $y \in F_{\varepsilon,x_1}$ *such that* $\|y - x_1\| \leq \varepsilon$,

where F_{ε,x_1} *is a set of fixed point of the function* f_{ε,x_1}. *Then the set* $F(T)$ *is a continuum.*

Proof. Since $deg(I - T, D, 0) \neq 0$, then (1) of Theorem 1.5 implies T has a fixed point, so $F(T) \neq \emptyset$. Furthermore, since $F(T) \subset T(F(T))$ and T is a condensing map, then $F(T)$ is compact. Thus, it remains to show that $F(T)$ is connected. Assume that F = F(T) is not connected. Hence, there exist two nonempty disjoint closed subsets F_1 and F_2 of D such that $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. Let $\varepsilon > 0$ be such that $N_\varepsilon(F_1) \cap N_\varepsilon(F_2) = \emptyset$ and $\overline{N_\varepsilon(F_1)} \cup \overline{N_\varepsilon(F_2)} \subset D$. By (3) of Theorem 1.5,

$$deg(I - T, \overline{D}, 0) = deg(I - T, N_\varepsilon(F_1), 0) + deg(I - T, N_\varepsilon(F_2), 0).$$

Consequently, one of the numbers $d_1 = deg(I - T, N_\varepsilon(F_1), 0)$ and $d_2 = deg(I - T, N_\varepsilon(F_2), 0)$ is non-zero. Suppose that $d_1 \neq 0$. We consider an arbitrary point $x_1 \in F_2$ and a map f_{ε,x_1} satisfying the condition of the theorem. The set-valued compact vector fields $I - T$ and $I - f_{\varepsilon,x_1}$ are linearly homotopic on the $N_\varepsilon(F_1)$, indeed, the set X_{ε,x_1} lies in $N_\varepsilon(F) = N_\varepsilon(F_1) \cup N_\varepsilon(F_2)$, then

$$deg(I - T, \overline{N_\varepsilon(F_1)}, 0) = deg(I - f_{\varepsilon,x_1}, \overline{N_\varepsilon(F_2)}, 0) \neq 0,$$

then f_{ε,x_1} has a fixed point in $N_\varepsilon(F_1)$. We deduced that $F_{\varepsilon,x_1} \cap N_\varepsilon(F_1) \neq \emptyset$. By a condition of the theorem, we have $F_{\varepsilon,x_1} \cap N_\varepsilon(F_1) \neq \emptyset$. Since

$$F_{\varepsilon,x_1} \subset X_{\varepsilon,x_1} \subset (N_\varepsilon(F_1) \cup N_\varepsilon(F_2)),$$

the set F_{ε,x_1} is disconnected and that is contradiction. □

Theorem 2.3. *Let* D *be a bounded open subset of a Banach space* X, *and* $T : \overline{D} \multimap X$ *be an upper semicontinuous condensing set-valued mapping with compact convex valued such that* $deg(I - T, D, 0) \neq 0$. *Assume that for any* $\delta > 0$ *and any point* $x_1 \in F(T)$ *there exists a single-valued condensing map* $f_{\delta,x_1} : \overline{D} \to X$ *such that:*

(i) f_{δ,x_1} *is a* $\delta-$*approximation of* T;

(ii) *the map* f_{δ,x_1} *has a connected set* F_{δ,x_1} *of fixed points;*

(iii) *there exists a point* $y \in F_{\delta,x_1}$ *such that* $\|y - x_1\| \leq \delta$.

Then, the set $F(T)$ *is a continuum.*

Proof. By a similar proof as that of Theorem 2.2, $F(T)$ is non-empty and compact. We show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that the map f_{δ,x_1} satisfies the condition of Theorem 2.2.

Let $\varepsilon > 0$ and we set $\beta = \inf\{d(x, T(x)) : \ x \in \overline{D} \setminus N_\varepsilon(F(T))\}$. Since T is compact valued and demicompact, then $\beta > 0$. Indeed, if $\beta = 0$, for each $n \in \mathbb{N}$, then there exists $x_n \in \overline{D} \setminus N_\varepsilon(F(T))$ such that $d(x_n, T(x_n)) \leq \frac{1}{n}$. Since T is demicompact, then there exists subsequence (x_{n_k}) of (x_n) such that it converges to $x \in \acute{D}$. Also $T(x)$ is closed then $x \in T(x)$ and that is a contradiction. Thus, for each $x \in \overline{D} \setminus N_\varepsilon(F(T))$,

$$d(x, T(x)) \geq \beta. \tag{4}$$

We take $\delta > 0$ such that $\delta < \min(\varepsilon, \frac{\beta}{2})$. We prove that the map f_{δ,x_1} satisfies all condition of Theorem 2.2. Clearly, f_{δ,x_1} satisfies the conditions (i) and (ii) of Theorem 2.2. We set

$$X_{\delta,x_1} = \{x \in \overline{D} : x \in \lambda T(x) + (1 - \lambda) f_{\delta,x_1}(x) \ for \ some \ \lambda \in [0, 1]\}.$$

We prove that the set X_{δ,x_1} is belong to a $\delta-$neighbourhood of F. We assume that it is not true. Then, for each $n \in \mathbb{N}$, there exists $x_n \in X_{\delta,x_1}$ such that $x_n \in \frac{1}{n}\tilde{T}(x_n) + (1 - \frac{1}{n}) f_{\delta,x_1}(x_n)$ and $x_n \notin N_\delta(F(T))$. Thus, for any $z \in F(T)$ we have $d(x_n, z) > \delta$. There exists $w_n \in \tilde{T}(x_n)$ such that $x_n = \frac{1}{n} w_n + (1 - \frac{1}{n}) f_{\delta,x_1}(x_n)$. Therefore, $\|x_n - f_{\delta,x_1}(x_n))\| = \frac{1}{n}\|w_n - f_{\delta,x_1}(x_n)\|$ converges to 0. Since f_{δ,x_1} is demicompact, then there exists subsequence (x_{n_k}) of (x_n) such that it is converges to $x = f_{\delta,x_1}$. But f_{δ,x_1} is a $\delta-$approximation of T so $(x, x) \in U_\delta(\Gamma_X(T))$, thus $x \in N_\delta(F(T))$ and that is a contradiction. Therefore, f_{δ,x_1} satisfies the condition of Theorem 2.2 and $F(T)$ is connected. □

Corollary 2.4. *Let D be a bounded open subset of a Banach space X and T : $\overline{D} \multimap X$ be an upper semicontinuous condensing set-valued mapping with compact convex valued such that $deg(I - T, D, 0) \neq 0$. Suppose that T is a non-expansive map. Then $F(T)$ is a continuum.*

Proof. By Lemma 2.9 of [7], for any $\delta > 0$ and any point $x_1 \in F(T)$, there exists a single-valued map f_{δ,x_1} that is $\delta-$approximation of T and satisfies the conditions (ii) and (iii) of Theorem 2.3. Since T is a condensing map, then f_{δ,x_1} is a condensing map. Therefore, by Theorem 2.3, the set $F(T)$ is continuum. □

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Zeinab Soltani
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Kashan,
Kashan, 87317-53153, Iran
E-mail: z.soltani@kashanu.ac.ir