Average Degree-Eccentricity Energy of Graphs

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Abstract

The concept of average degree-eccentricity matrix $ADE(G)$ of a connected graph $G$ is introduced. Some coefficients of the characteristic polynomial of $ADE(G)$ are obtained, as well as a bound for the eigenvalues of $ADE(G)$. We also introduce the average degree-eccentricity graph energy and establish bounds for it.

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1. Introduction

Throughout this paper, all graphs are assumed to be simple, finite and connected. Let $G = (V,E)$ be such a graph, with vertex set $V$ and edge set $E$. If $|V| = p$ and $|E| = q$, then $G$ is said to be a $(p,q)$-graph. The degree of a vertex $v$, denoted by $d(v)$, is the number of edges of $G$ incident with $v$. The distance $d(u,v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of a shortest path connecting them. For a vertex $v$ of $G$, the eccentricity of $v$ is $e(v) = \max\{d(v,u), u \in V(G)\}$. For additional graph-theoretical terminologies we refer to [8].

The adjacency matrix of $G$, $A(G) = (a_{ij})$ is a $p \times p$ matrix, such that $a_{ij} = 1$ if $v_i, v_j \in E$ and $a_{ij} = 0$ otherwise. The energy of $G$, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{p} |x_i|$$

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where $x_1, x_2, \ldots, x_p$ are the eigenvalues of $A(G)$. This concept was introduced almost 40 years ago [5] and has been extensively investigated [2, 6, 7, 10]. Eventually, numerous other graph energies have been invented, based on eigenvalues of matrices different from the adjacency matrix; for more details see [1, 6, 7, 9, 11, 13–17] and the references cited therein.

One of these graph energies is the sum-eccentricity energy [15, 17], based on the eigenvalues of the sum-eccentricity matrix $SE$, whose elements are defined as

\[
se_{ij} = \begin{cases} 
  e(v_i) + e(v_j) & \text{if } v_i v_j \in E \\
  0 & \text{otherwise.}
\end{cases}
\]

(2)

Another recently introduced graph energy is the first Zagreb energy [9], based on the eigenvalues of the first Zagreb matrix $ZG$, whose elements are defined as

\[
zg_{ij} = \begin{cases} 
  d(v_i) + d(v_j) & \text{if } v_i v_j \in E \\
  0 & \text{otherwise.}
\end{cases}
\]

(3)

In this article, we introduce the concept of average degree-eccentricity matrix $ADE$.

**Definition 1.1.** Let $G = (V,E)$ be a simple connected graph with $p$ vertices $v_1, v_2, \ldots, v_p$ and let $d_i$ and $e(v_i)$ be, respectively, the degree and eccentricity of $v_i$, $i = 1, 2, \ldots, p$. Then the average degree-eccentricity matrix $ADE = ADE(G)$ of $G$ is the $p \times p$ matrix whose elements are given by

\[
m_{ij} = \begin{cases} 
  \frac{1}{4}[d(v_i) + d(v_j) + e(v_i) + e(v_j)] & \text{if } v_i v_j \in E \\
  0 & \text{otherwise.}
\end{cases}
\]

(4)

Bearing in mind Equations (2) and (3), we see that $ADE$ is conceived as a linear combination of the sum-eccentricity and Zagreb matrices, i.e.,

\[
ADE = \frac{1}{4}[SE + ZG].
\]

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ of $ADE(G)$ form the average degree-eccentricity spectrum or the $ADE$-spectrum of $G$. As usual, the $ADE$-spectrum of $G$ with $n_i$-fold degenerate eigenvalues $\lambda_i$ is written as

\[
S_p(G) = \{(\lambda_1)^{n_1}, (\lambda_2)^{n_2}, \ldots, (\lambda_p)^{n_p}\}.
\]

$ADE$ is a real symmetric matrix. Therefore, its eigenvalues are real numbers, and \( \sum_{i=1}^{p} \lambda_i = 0 \).

The following result will be useful in the proof of our results.
Theorem 1.2. [3] (Gersgorin’s Theorem) Every eigenvalue $\lambda$ of a $p \times p$ matrix $M = (m_{ij})$ satisfies:

$$|\lambda - m_{ii}| \leq \sum_{j=1}^{p} |m_{ij}|.$$  

Corollary 1.3. [4] (Hadamard’s Inequality) If the entries of a $p \times p$ matrix $M$ are bounded by $B$, then $|\text{det}(M)| \leq B^p p^{p/2}$.

2. Average Degree-Eccentricity Energy

Definition 2.1. The average degree-eccentricity energy $E_{ade}(G)$ of a graph $G$ is

$$E_{ade}(G) = \sum_{i=1}^{p} |\lambda_i|.$$  

Equation (5)

Evidently, the average degree-eccentricity energy is defined in analogy to the ordinary graph energy, Equation (1).

Example 2.2. For a graph $G_1$ in Figure 1,

![Figure 1: G_1.](image)

the average degree-eccentricity matrix of $G_1$ is

$$\text{ADE}(G_1) = \begin{bmatrix} 0 & \frac{9}{4} & 0 & 0 & 0 \\ \frac{9}{4} & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\ \frac{9}{4} & \frac{9}{4} & 0 & \frac{9}{4} & 0 \\ 0 & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\ 0 & 0 & \frac{9}{4} & 0 & 0 \end{bmatrix}.$$  

The characteristic polynomial of $\text{ADE}(G_1)$ is, $P(G_1, \lambda) = |\lambda I_p - \text{ADE}(G_1)| = \lambda^5 - \frac{405}{16} \lambda^3 + \frac{2025}{64} \lambda$, and the average degree-eccentricity eigenvalues of $G_1$ are $\lambda_1 \approx 4.8, \lambda_2 \approx 1.5, \lambda_3 = 0, \lambda_4 \approx -1.5, \lambda_5 \approx -4.8$. Then the average degree-eccentricity energy of $G_1$ is $E_{ade}(G_1) = 4.8 + 1.5 + 1.5 + 4.8 = 12.6$.  

We now calculate the coefficient \( c_i \) of \( \lambda^{p-i} (i = 0, 1, 2, p) \) in the characteristic polynomial of the average degree-eccentricity matrix \( \text{ADE}(G) \). Clearly \( c_0 = 1 \), \( c_1 = \text{trace}(\text{ADE}(G)) = 0 \). Now

\[
c_2 = \sum_{1 \leq i < j \leq p} \begin{vmatrix} 0 & m_{ij} \\ m_{ji} & 0 \end{vmatrix} = \sum_{1 \leq i < j \leq p} -m^2_{ij}.
\]

In view of Equation (4) we get

\[
c_2 = -\sum_{v_i v_j \in E} \left( d(v_i) + e(v_i) + d(v_j) + e(v_j) \right)^2 / 4.
\]

For \( c_3 \) we have

\[
c_3 = (-1)^3 \sum_{1 \leq i < j < r \leq n} \begin{vmatrix} m_{ii} & m_{ij} & m_{ir} \\ m_{ji} & m_{jj} & m_{jr} \\ m_{ri} & m_{rj} & m_{rr} \end{vmatrix}.
\]

The number of non-zero terms in the above sum is equal to the number of triangles in \( G \). Therefore, \( c_3 = 0 \) if \( G \) has no triangle.

Finally, \( c_p = \text{det}(\text{ADE}(G)) \).

**Lemma 2.3.** Let \( G \) be a connected \((p, q)\)-graph and \( uv \in E \). Then

\[
\frac{1}{4} [d(u) + d(v) + e(u) + e(v)] \leq \frac{p}{2}.
\]

Equality in (6) holds for all \( uv \in E \) only if \( G \cong K_p \).

**Proof.** Without loss of generality, we may assume that \( e(u) \leq e(v) \). So, we have

\[
d(u) + d(v) + e(u) + e(v) \leq d(u) + d(v) + 2e(v) \\
\leq d(u) + d(v) + 2[p - (d(u) + d(v)) + 1] \\
= 2p - (d(u) + d(v)) + 2 \leq 2p.
\]

If \( G \cong K_p \), then for any \( uv \in E \) we have \( d(u) = d(v) = p - 1 \) and \( e(u) = e(v) = 1 \), implying that the left-hand side of (6) is equal to \( p/2 \). For all other (connected) graphs, for some \( uv \in E \) the inequality in (6) will be strict. \( \square \)

**Lemma 2.4.** Let \( G \) be a connected \((p, q)\)-graph. Then

\[
\text{trace} \left( \text{ADE}^2(G) \right) \leq \text{trace} \left( \text{ADE}^2(K_p) \right) = \frac{(p - 1)p^3}{4}.
\]

Equality in (7) holds if and only if \( G \cong K_p \).
Proof. Since
\[ \text{ADE}(K_p)_{ij} = \begin{cases} \frac{p}{2} & \text{if } v_i,v_j \in E \\ 0 & \text{otherwise} \end{cases} \]
we get that for \( i \neq j \),
\[ \text{ADE}^2(K_p)_{ij} = (p - 2) \left( \frac{p}{2} \right)^2 \]
whereas for \( i = j \),
\[ \text{ADE}^2(K_p)_{ii} = (p - 1) \left( \frac{p}{2} \right)^2 \]
implying that
\[ \text{trace } \text{ADE}^2(K_p) = p \times (p - 1) \left( \frac{p}{2} \right)^2 = \frac{(p - 1)p^3}{4}. \]

Bearing in mind Lemma 2.3 and formula (4), we immediately see that \( \text{ADE}(G)_{ij} \leq \text{ADE}(K_p)_{ij} \), and that if \( G \not\sim K_p \), then the inequality is strict for at least some of \( ij \). Consequently, inequality (7) holds.

**Theorem 2.5.** For any \((p,q)\)-graph, with average degree-eccentricity eigenvalue \( \lambda_j \),
\[ |\lambda_j| \leq \frac{p(p - 1)}{2}. \]

**Proof.** By Lemma 2.4, the trace of \( \text{ADE}^2(K_p) \) is equal to \( \frac{(p - 1)p^3}{4} \). Then for any \((p,q)\)-graph \( G \) with average degree-eccentricity eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_p \), we have \( \sum \limits_{i=1}^{p} |\lambda_i|^2 \leq \frac{(p - 1)p^3}{4} \). By the Cauchy–Schwarz inequality,
\[ \left( \sum \limits_{i=1}^{p} \lambda_i \right)^2 = (p - 1) \sum \limits_{i=1}^{p} \lambda_i^2. \]

Since \( \sum \limits_{i=1}^{p} \lambda_i^2 = -2c_2 \) and \( \sum \limits_{i=1}^{p} \lambda_i = 0 \), we get
\[ \lambda_j^2 \leq (p - 1) \left[ \frac{(p - 1)p^3}{4} - \lambda_j^2 \right] \]
which implies (8).

**Proposition 2.6.** Let \( G \) be a graph of order \( p \), and average degree-eccentricity eigenvalue \( \lambda_i \). Then
\[ \prod \limits_{i=1}^{p} |\lambda_i| \leq \left( \frac{p}{2} \right)^p p^{p/2}. \]
\textbf{Proof.} By Corollary 1.3 and by the definition of $\text{ADE}$, setting $B = p/2$. \hfill \Box

\textbf{Theorem 2.7.} Let $G$ be a $(p,q)$-graph. Then

$$E_{\text{ade}}(G) \leq \frac{(p - 1)p^2}{2}.$$  

\textbf{Proof.} By Gershgorin’s Theorem and Lemma 2.3, we have

$$E_{\text{ade}}(G) = \sum_{i=1}^{p} |\lambda_i| = \sum_{i=1}^{p} |\lambda_i - 0| \leq \sum_{i=1}^{p} \sum_{j \neq i}^{p} m_{ij}$$

$$\leq \sum_{i=1}^{p} \sum_{j \neq i}^{p} \frac{p}{2} = \frac{(p - 1)p^2}{2}.$$  

\hfill \Box

\textbf{Theorem 2.8.} Let $G$ be a connected $(p,q)$-graph. Then

$$E_{\text{ade}}(G) \geq \sqrt{2(q|\det(\text{ADE})|^{2/p} - c_2)}.$$  

\textbf{Proof.}

$$E_{\text{ade}}(G)^2 = \left(\sum_{i=1}^{p} |\lambda_i| \right)^2 = \sum_{i=1}^{p} \lambda_i^2 + \sum_{i=1 \neq j}^{p} |\lambda_i||\lambda_j| = -2c_2 + \sum_{i=1 \neq j}^{p} |\lambda_i||\lambda_j|.$$  

From relation between the arithmetic and geometric means, we get

$$\sum_{i=1 \neq j}^{p} |\lambda_i||\lambda_j| \geq \frac{p(p - 1)}{2} \left(\prod_{i=1 \neq j}^{p} |\lambda_i||\lambda_j|\right)^{1/(p-1)} = \frac{p(p - 1)}{2} \left(\prod_{i=1 \neq j}^{p} |\lambda_i|\right)^{2/p} \geq 2q \left(\prod_{i=1 \neq j}^{p} |\lambda_i|\right)^{2/p} = 2q|\det(\text{ADE})|^{2/p}.$$  

Then

$$E_{\text{ade}}(G)^2 \geq 2q|\det(\text{ADE})|^{2/p} - 2c_2 = 2\left[q|\det(\text{ADE})|^{2/p} - c_2\right]$$  

and finally,

$$E_{\text{ade}}(G) \geq \sqrt{2q|\det(\text{ADE})|^{2/p} - c_2}.$$  

\hfill \Box
Note that Theorem 2.8 and its proof are just a replica of the classical McClelland inequality for ordinary graph energy [12].

Corollary 2.9. Let $G$ be a connected $(p,q)$-graph. Then

$$\sqrt{2(q|\det(ADE)|^{(2/p - c_2/2)}} \leq E_{ade}(G) \leq \frac{(p - 1)p^2}{2}.$$ 

3. Average Degree-Energy of Some Classes of Graphs

In this section, we compute the average degree-eccentricity energies of some well-known graphs.

Example 3.1. Let $G$ be a complete graph $K_p$. Then $S_p(ADE(K_p)) = \{(p/2)^{p - 1}, ((p - 1)(p/2))^1\}$ and $E_{ade}(K_p) = 2(p - 1)(p/2)$.

Proof. Let $G$ be the complete graph $K_p$. Then

$$|\lambda - ADE(K_p)| = \left| \begin{array}{cccc} \lambda & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ -\frac{p}{2} & \lambda & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{p}{2} & -\frac{p}{2} & -\frac{p}{2} & \cdots & \lambda \\ \end{array} \right|$$

$$= \left(\lambda + \frac{p}{2}\right)^{p - 1} \left| \begin{array}{cccc} \lambda & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \end{array} \right|$$

$$= \left(\lambda + \frac{p}{2}\right)^{p - 1} \left[ \lambda - (p - 1)\frac{p}{2} \right].$$

Then the average degree-eccentricity energy of the complete graph is

$$E_{ade}(K_p) = 2(p - 1)\frac{p}{2}.$$ 

Example 3.2. Let $G$ be a complete bipartite graph $K_{m,n}$, $m,n \geq 2$. Then

$$S_p(ADE(K_{m,n})) = \left\{ \left(\frac{p + 4}{4}\sqrt{mn}\right)^1, 0)^{p-2}, \left(-\frac{(p + 4)}{4}\sqrt{mn}\right)^1 \right\}$$

and

$$E_{ade}(K_{m,n}) = \frac{p + 4}{2}\sqrt{mn}.$$
Proof. Let the vertex set of $K_{m,n}$ be $V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$. Then, $p = m + n$, $q = mn$, and

$$|\lambda - \text{ADE}(K_{m,n})| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\ 0 & \lambda & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} & 0 & \lambda & \cdots & 0 \\ -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} & 0 & \cdots & \cdots & \cdots \\ \end{vmatrix}$$

$$= \lambda^p - \left(\frac{p + 4}{4}\right)^2 (mn)\lambda^{p-2} = \lambda^{p-2} \left[\lambda^2 - \left(\frac{p + 4}{4}\right)^2 (mn)\right].$$

Then, $\lambda^{p-2} \left[\lambda^2 - \left(\frac{p + 4}{4}\right)^2 (mn)\right] = 0$ implies $\lambda^{p-2} = 0$, or $\lambda^2 = \left(\frac{p + 4}{4}\right)^2 (mn)$, resulting in (9) and (10).

Example 3.3. For the star graph $K_{1,p-1}$,

$$S_p(\text{ADE}(K_{1,p-1})) = \left\{ \left(\frac{p + 3}{4}\right)^{\sqrt{p-1}}, (0)^{p-2}, \left(-\frac{(p + 3)}{4}\sqrt{p-1}\right)^{1} \right\}$$

and

$$E_{\text{ade}}(K_{1,p-1}) = \frac{p + 3}{2\sqrt{p - 1}}.$$

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References


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