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Trees with Extreme Values of Second Zagreb Index and Coindex

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Abstract

The second Zagreb index $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices and the second Zagreb coindex $\overline{M_2(G)}$ is equal to the sum of the products of the degrees of pairs of non-adjacent vertices. Kovijanić Vukićević and Popivoda (*Iranian J. Math. Chem.* **5** (2014) 19–29) prove that for any chemical tree of order $n \geq 5$,

$$M_2(T) \leq \begin{cases} 8n-26 & n \equiv 0,1 \pmod 3 \\ 8n-24 & \text{otherwise.} \end{cases}$$

In this paper we present a generalization of the aforementioned bound for all trees in terms of the order and maximum degree. We also give a lower bound on the second Zagreb coindex of trees.

Keywords: Zagreb index, second Zagreb index, second Zagreb coindex, tree.

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1. Introduction

In this paper, G is a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d_v = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by

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$\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of a tree is a vertex of degree 1 and a pendant edge is an edge adjacent to a leaf. Trees with the property $\Delta \leq 4$ are called chemical trees.

The Zagreb indices have been investigated more than forty years ago by Gutman and Trinajstić in [6]. These parameters are important molecular descriptors and have been closely correlated with many chemical properties [6,8]. Hence, they attracted more and more attention from chemists and mathematicians [2–4,11,12].

The first Zagreb index, $M_1 = M_1(G)$, is equal to the sum of squares of the degrees of the vertices. Consult [9] for a good survey on this subject. Also, in [10] we found some lower bound for first Zagreb index of trees.

The second Zagreb index $M_2 = M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices of the graph G, that is,

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v) = \sum_{uv \in E(G)} d_u d_v.$$

Došlić in [5] introduced two new graph invariants, the first and the second Zagreb coindices, defined as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} (d_u + d_v),$$

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_u d_v.$$

Let T be a tree of order n and let n_i be the number of vertices of degree i for each $i = 1, 2, \ldots, \Delta$. Clearly

$$n_1 + n_2 + \cdots + n_\Delta = n \tag{1}$$

and

$$n_1 + 2n_2 + \cdots + \Delta n_\Delta = 2n - 2. \tag{2}$$

By (1) and (2), we have

$$n_2 + 2n_3 + \cdots + (\Delta - 1)n_\Delta = n - 2. \tag{3}$$

Trees with the property $\Delta \leq 4$ are called chemical trees. The following family of trees was introduced in [7]. For $n = (\Delta - 1)k + r$ $(k \geq 2)$, let $\tilde{\mathcal{T}}_n$ be the family of trees T of order n with maximum degree Δ such that:

- If $r = 0$, then T has $k - 1$ vertices of degree Δ and one vertex of degree $\Delta - 2$, and the remaining vertices are pendant.

- If $r = 1$, then T has $k-1$ vertices of degree Δ and one vertex has degree $\Delta - 1$, and the remaining vertices are pendant.
- If $r \geq 2$, then T has k vertices of degree Δ and one vertex has degree $r-1$, and the remaining vertices are pendant.

Theorem A. *[7] If T is a chemical tree of order $n \geq 5$. Then*

$$M_2(T) \leq \begin{cases} 8n - 26, & n \equiv 0, 1 \pmod 3 \\ 8n - 24, & \text{otherwise} \end{cases}$$

with equality if and only if $T \in \tilde{\mathcal{T}}_n$.

In this paper we generalize the aforementioned upper bound and classify all extreme trees.

2. An Upper Bound on the Second Zagreb Index

In this section we present the following upper bound on the second Zagreb index of trees as a generalization of Theorem A.

Theorem 2.1. Let T be a tree of order n and maximum degree Δ. If $n \equiv r \pmod{\Delta - 1}$, then

$$M_2(T) \leq \begin{cases} 2n\Delta - \Delta^2 - 4\Delta + 6 & r = 0 \\ 2n\Delta - \Delta^2 - 3\Delta + 2 & r = 1 \\ 2n\Delta - \Delta^2 - 2\Delta & r = 2 \\ 2n\Delta - \Delta^2 - r\Delta + 2 + r(r-3) & r \geq 3 \end{cases}$$

with equality if and only if $T \in \tilde{\mathcal{T}}_n$.

We start with some lemmas.

Lemma 2.2. If T is a tree with at least two vertices of degree $2 \leq \beta \leq \Delta - 1$, then its second Zagreb index cannot be maximal.

Proof. Let $x, y \in V(T)$ such that $d(x) = d(y) = \beta$, $2 \leq \beta \leq \Delta - 1$.
Let $N(x) = \{x_1, x_2, \ldots, x_\beta\}$, $N(y) = \{y_1, y_2, \ldots, y_\beta\}$, $e_i = xx_i$, $g_i = yy_i$ and $i = 1, 2, \ldots, \beta$.

We consider two cases.

Case 1. $xy \notin E(T)$, that is, x and y are not adjacent.
Without loss of generality, suppose that

$$d(x_1) + d(x_2) + \cdots + d(x_\beta) \leq d(y_1) + d(y_2) + \cdots + d(y_\beta)$$

and the unique path between x and y goes toward the vertices x_1 and y_1. Let T' be a tree, such that from T obtained by remove edge $e_\beta = xx_\beta$ and adding edge yx_β. i.e. $T' = T - e_\beta + yx_\beta$ (see Figure 1).

We will show that $M_2(T) < M_2(T')$. To this end, let $S = \{e_1, e_2, \ldots, e_\beta, g_1, g_2, \ldots, g_\beta\}$. By definition we have

$$\begin{aligned} M_2(T) &= \sum_{uv \notin S} d(u).d(v) + \beta(d(x_1) + \cdots + d(x_\beta)) + \beta(d(y_1) + \cdots + d(y_\beta)), \\ M_2(T') &= \sum_{uv \notin S} d(u).d(v) + (\beta - 1)(d(x_1) + \cdots + d(x_{\beta-1})) \\ &\quad + (\beta + 1)(d(y_1) + \cdots + d(y_\beta) + d(x_\beta)). \end{aligned}$$

Thus

$$\begin{aligned} M_2(T) - M_2(T') &= (d(x_1) + \ldots + d(x_{\beta-1})) - d(x_\beta) - (d(y_1) + \ldots + d(y_\beta)) \\ &= (d(x_1) + \ldots + d(x_\beta)) - (d(y_1) + \ldots + d(y_\beta)) - 2d(x_\beta) \\ &< 0. \end{aligned}$$

Therefore $M_2(T) < M_2(T')$, as desired.

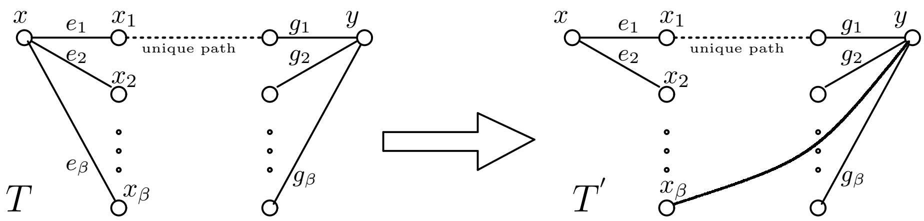

Figure 1: Case 1 - Lemma 2.2.

Case 2. $xy \in E(T)$, that is, x and y are adjacent.

The vertices x_1 and y_1 from the above construction are the vertices y and x, respectively, and the edges e_1 and g_1 are one and the same edge xy. Similar to the proof of case 1, we suppose that

$$d(x_2) + \cdots + d(x_\beta) \leq d(y_2) + \cdots + d(y_\beta).$$

Let $S = \{e_1 = g_1, e_2, \ldots, e_\beta, g_2, \ldots, g_\beta\}$ (see Figure 2). By definition we have

$$M_2(T) = \sum_{uv \notin S} d(u).d(v) + \beta(d(x_2) + \cdots + d(x_\beta)) + \beta^2 + \beta(d(y_2) + \cdots + d(y_\beta)),$$
$$M_2(T') = \sum_{uv \notin S} d(u).d(v) + (\beta - 1)(d(x_2) + \cdots + d(x_{\beta-1})) + (\beta - 1)(\beta + 1) + (\beta + 1)(d(y_2) + \cdots + d(y_\beta) + d(x_\beta)).$$

Thus

$$\begin{aligned} M_2(T) - M_2(T') &= (d(x_2) + \cdots + d(x_{\beta-1})) - d(x_\beta) + 1 - (d(y_2) + \cdots + d(y_\beta)) \\ &= (d(x_1) + \cdots + d(x_\beta)) - (d(y_1) + \cdots + d(y_\beta)) - 2d(x_\beta) + 1 \\ &< 0. \end{aligned}$$

Since $d(x_\beta) \geq 1$, $-2d(x_\beta) + 1 < 0$. This completes the proof. □

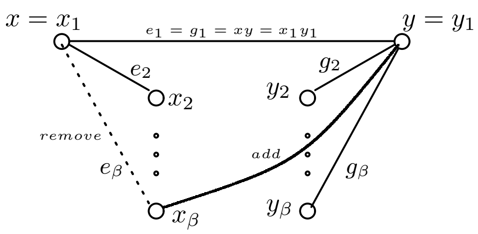

Figure 2: Case 2 - Lemma 2.2.

Lemma 2.3. If T be a tree with at least one vertex of degree α and one vertex of degree β, $2 \leq \alpha < \beta \leq \Delta - 1$, then its second Zagreb index cannot be maximal.

Proof. Let $x, y \in V(T)$ such that $d(x) = \alpha$ and $d(y) = \beta$, $2 \leq \alpha < \beta \leq \Delta - 1$. Let $N(x) = \{x_1, x_2, \ldots, x_\alpha\}$, $N(y) = \{y_1, y_2, \ldots, y_\beta\}$ and $e_i = xx_i$ and $g_j = yy_j$ be the appropriate edges for each $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \beta$.

Without loss of generality, suppose that the unique path between x and y goes toward the vertices x_1 and y_1. (see Figure 3).

Let $S = \{e_1, e_2, \ldots, e_\alpha, g_1, g_2, \ldots, g_\beta\}$. We consider two cases.

Case 1. $xy \notin E(T)$, that is, x and y are not adjacent.

Subcase 1.1 $d(x_1) > d(y_1)$. Let $T' = T - \{e_1, g_1\} + \{yx_1, xy_1\}$. So

$$
\begin{aligned}
M_2(T) &= \sum_{uv\notin S} d(u).d(v) + \alpha(d(x_1)+\cdots+d(x_\alpha)) + \beta(d(y_1)+\cdots+d(y_\beta)),\\
M_2(T') &= \sum_{uv\notin S} d(u).d(v) + \alpha(d(y_1)+d(x_2)+\cdots+d(x_\alpha)) + \beta(d(x_1)\\
&\quad + d(y_2)+\cdots+d(y_\beta).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M_2(T) - M_2(T') &= d(x_1)(\alpha-\beta) + d(y_1)(\beta-\alpha)\\
&= (d(x_1)-d(y_1))(\alpha-\beta)\\
&< 0.
\end{aligned}
$$

Because, by hypothesis, $\alpha < \beta$ and $d(y_1) < d(x_1)$.

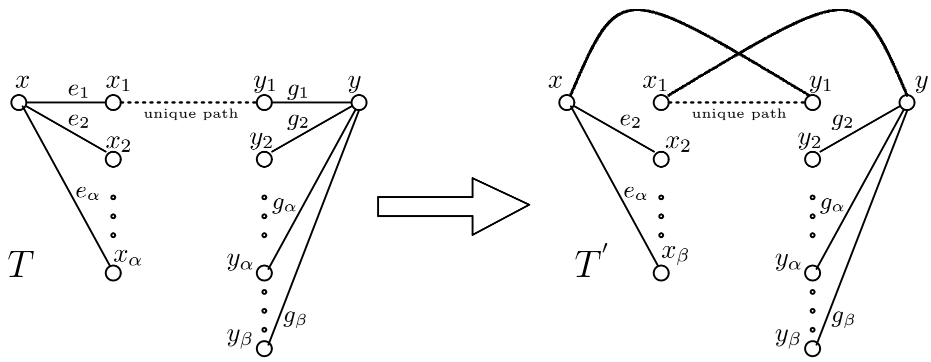

Figure 3: Case 1 - Lemma 2.3.

Subcase 1.2 $d(x_1) \le d(y_1)$ and for some i, j, $d(x_i) > d(y_j)$ $(2 \le i \le \alpha$, $2 \le j \le \beta)$. Let $T' = T - \{e_i, g_j\} + \{yx_i, xy_j\}$. So

$$
\begin{aligned}
M_2(T) &= \sum_{uv\notin S} d(u).d(v) + \alpha(d(x_1)+\ldots+d(x_\alpha)) + \beta(d(y_1)+\ldots+d(y_\beta)),\\
M_2(T') &= \sum_{uv\notin S} d(u).d(v) + \alpha(d(x_1)+\ldots+d(x_\alpha)) + \beta(d(y_1)+\ldots+d(y_\beta))\\
&\quad + (\beta d(x_i) - \alpha d(x_i)) + (\alpha d(y_j) - \beta d(y_j)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M_2(T) - M_2(T') &= d(x_i)(\alpha-\beta) + d(y_j)(\beta-\alpha)\\
&= (d(x_i)-d(y_j))(\alpha-\beta)\\
&< 0.
\end{aligned}
$$

Because, by hypothesis, $\alpha < \beta$ and $d(y_j) < d(x_i)$.

Subcase 1.3 $d(x_1) \leq d(y_1)$ and for all $2 \leq i \leq \alpha$ and $2 \leq j \leq \beta$, we have $d(x_i) \leq d(y_j)$. Let $T' = T - e_\alpha + yx_\alpha$. So

$$\begin{aligned} M_2(T) &= \sum_{uv \notin S} d(u).d(v) + \alpha(d(x_1) + \cdots + d(x_\alpha)) + \beta(d(y_1) + \cdots + d(y_\beta)), \\ M_2(T') &= \sum_{uv \notin S} d(u).d(v) + (\alpha - 1)(d(x_1) + \cdots + d(x_{\alpha-1})) \\ &\quad + (\beta + 1)(d(y_1) + \cdots + d(y_\beta) + d(x_\alpha)). \end{aligned}$$

Therefore

$$\begin{aligned} M_2(T) - M_2(T') &= (d(x_1) + \cdots + d(x_{\alpha-1})) + (\alpha - \beta - 1)d(x_\alpha) \\ &\quad - (d(y_1) + \cdots + d(y_\beta)) \\ &= (d(x_1) + \cdots + d(x_\alpha)) + (\alpha - \beta - 2)d(x_\alpha) \\ &\quad - (d(y_1) + \cdots + d(y_\beta)) \\ &< 0. \end{aligned}$$

Because, by hypothesis and $\alpha < \beta$.

Case 2. $xy \in E(T)$, that is, x and y are adjacent. The vertices x_1 and y_1 from the above construction are the vertices y and x, respectively, and the edges e_1 and g_1 are one and the same edge xy. Let $S = \{e_2, \ldots, e_\alpha, g_2, \ldots, g_\beta\}$. We consider two subcases.

Subcase 2.1 There exist $2 \leq i \leq \alpha$ and $2 \leq j \leq \beta$, such that $d(x_i) > d(y_j)$. Let $T' = T - \{e_i, g_j\} + \{x_i y, xy_j\}$. So

$$\begin{aligned} M_2(T) &= \sum_{uv \notin S} d(u).d(v) + \alpha(d(x_2) + \cdots + d(x_\alpha)) + \beta(d(y_2) + \cdots + d(y_\beta)), \\ M_2(T') &= \sum_{uv \notin S} d(u).d(v) + \alpha(d(x_2) + \cdots + d(x_\alpha)) + \beta(d(y_2) + \cdots + d(y_\beta)) \\ &\quad - \alpha d(x_i) + \beta d(x_i) - \beta d(y_j) + \alpha d(y_j). \end{aligned}$$

It follows that

$$M_2(T) - M_2(T') = (\alpha - \beta)d(x_i) + (\beta - \alpha)d(y_j) = (\alpha - \beta)(d(x_i) - d(y_j)) < 0.$$

Because, by hypothesis $\alpha - \beta < 0$ and $d(x_i) - d(y_j) > 0$.

Subcase 2.2 For all $2 \leq i \leq \alpha$ and $2 \leq j \leq \beta$, we have $d(x_i) \leq d(y_j)$.

In this case, we suppose that $S = \{e_2, \ldots, e_\alpha, g_2, \ldots, g_\beta, e_1 = g_1 = xy\}$ and $T' = T - e_\alpha + yx_\alpha$. We deduce that

$$\begin{aligned}M_2(T) &= \sum_{uv \notin S} d(u).d(v) + \alpha(d(x_1 = y) + \cdots + d(x_\alpha)) \\ &\quad + \beta(d(y_1 = x) + \cdots + d(y_\beta)), \\ M_2(T') &= \sum_{uv \notin S} d(u).d(v) + (\alpha - 1)(d(x_1 = y) + \cdots + d(x_{\alpha-1})) \\ &\quad + (\beta + 1)(d(y_1 = x) + \cdots + d(y_\beta) + d(x_\alpha)).\end{aligned}$$

Therefore

$$\begin{aligned}M_2(T) - M_2(T') &= (d(y) + d(x_2) + \cdots + d(x_{\alpha-1})) + (\alpha - \beta - 1)d(x_\alpha) \\ &- (d(x) + d(y_2) + \cdots + d(y_\beta)) \\ &= (\alpha - \beta - 1)d(x_\alpha) + (d(y) - d(x)) - (d(y_2) + \cdots + d(y_\beta)) \\ &+ (d(x_2) + \cdots + d(x_{\alpha-1})) \\ &= (\alpha - \beta - 1)d(x_\alpha) + (\beta - \alpha) - (d(y_2) + \cdots + d(y_{\beta-\alpha+2})) \\ &- (d(y_{\beta-\alpha+3}) + \cdots + d(y_\beta)) + (d(x_2) + \cdots + d(x_{\alpha-1})) \\ &< 0.\end{aligned}$$

Because, by hypothesis $(\alpha-\beta-1)d(x_\alpha) < -1$, $(\beta-\alpha)-(d(y_2)+\cdots+d(y_{\beta-\alpha+2})) \leq \beta-\alpha-(\beta-\alpha+1) \leq -1$ and $(d(x_2)+\cdots+d(x_{\alpha-1}))-(d(y_{\beta-\alpha+3})+\cdots+d(y_\beta)) \leq 0$. Consequently, in any cases we have $M_2(T) < M_2(T')$, that is contradiction. □

From the Lemmas 2.2 and 2.3, we make the next conclusion.

Corollary 2.4. If T is tree of order n such that $M_2(T) = \max\{M_2(T') \mid T'$ is a tree of order $n\}$, then T satisfies exactly one of the next two conditions:

(i) all vertices of the graph T have degrees 1 or Δ;

(ii) in $V(T)$ there is exactly one vertex of degree β $(1 < \beta < \Delta)$ and remaining vertices have degrees 1 or Δ.

Proof of Theorem 2.1. By Theorem A, we may assume that $\Delta \geq 5$. Let T be a tree such that

$$M_2(T) = \max\{M_2(T') \mid T' \text{ is a tree of order } n \text{ with maximum degree } \Delta\}.$$

By Corollary 2.4, T has at most one vertex of degree t where $2 \leq t \leq \Delta - 1$. Let A be the set of all pendant edges of T and $B = E(T) \setminus A$. Define the function ω

on $E(T)$ by $w(uv) = d(u)d(v)$. Then

$$M_2(T) = \sum_{e\in A} w(e) + \sum_{e\in B} w(e).$$

There are non-negative integers k, r such that $n = (\Delta-1)k + r$ and $0 \le r \le \Delta - 2$. By (3), we have

$$n_2 + 2n_3 + \cdots + (\Delta-2)n_{\Delta-1} = (\Delta-1)(k - n_\Delta) + r - 2. \tag{4}$$

Case 1. $n_t = 1$.
It follows from (4) that $t + 1 - r = (\Delta-1)(k - n_\Delta)$ and so $n_\Delta = k - \dfrac{t+1-r}{\Delta-1}$. Since $0 \le r \le \Delta-2$ and $2 \le t \le \Delta-1$ and since $\dfrac{t+1-r}{\Delta-1}$ is an integer between 0 and 1, we deduce that one of the following statement holds.

(a) if $r = 0$, then $t = \Delta-2$, $n_\Delta = k-1$, $n_{\Delta-2} = 1$ and $n_1 = n-k$,

(b) if $r = 1$, then $t = \Delta-1$, $n_\Delta = k-1$, $n_{\Delta-1} = 1$ and $n_1 = n-k$,

(c) if $3 \le r \le \Delta-2$, then $t = r-1$, $n_\Delta = k$, $n_{r-1} = 1$ and $n_1 = n-k-1$.

Let V_i be the set consists of all vertices of degree i for each $i = 1, 2, \ldots, \Delta$. Suppose $E_{i,j}$ denotes the set of all edges with one end in V_i and the other end in V_j. Clearly, $E = E_{1,t} \cup E_{1,\Delta} \cup E_{t,\Delta} \cup E_{\Delta,\Delta}$ and $t = |E_{1,t}| + |E_{t,\Delta}|$. Therefore

$$\begin{aligned} M_2(T) &= \sum_{e\in A} w(e) + \sum_{e\in B} w(e) \\ &= (|E_{1,t}|.t + |E_{1,\Delta}|.\Delta) + (|E_{t,\Delta}|.t\Delta + |E_{\Delta,\Delta}|.\Delta^2) \\ &= (|E_{1,t}|.t + (n_1 - |E_{1,t}|)\Delta) + (|E_{t,\Delta}|.t\Delta + (n - n_1 - |E_{t,\Delta}| - 1).\Delta^2) \\ &= (t-\Delta)(|E_{1,t}| + \Delta|E_{t,\Delta}|) + n_1\Delta - n_1\Delta^2 + (n-1)\Delta^2. \qquad (**) \end{aligned}$$

Since $t - \Delta < 0$ and $M_2(T)$ is maximum, we should minimize $|E_{1,t}| + |E_{t,\Delta}|\Delta$. It follows from $t = |E_{1,t}| + |E_{t,\Delta}|$ that $|E_{t,\Delta}| = 1$ and $|E_{1,t}| = t - 1$. Hence,

$$M_2(T) = t^2 - t - 2\Delta^2 + \Delta + n_1\Delta - n_1\Delta^2 + n\Delta^2. \qquad (***)$$

If (a) holds, then $n = (\Delta-1)k$ and by $(***)$ we have

$$\begin{aligned} M_2(T) &= (\Delta-2)^2 - (\Delta-2) - 2\Delta^2 + \Delta + (\Delta-2)k\Delta - (\Delta-2)k\Delta^2 + (\Delta-1)k\Delta^2 \\ &= -\Delta^2 - 4\Delta + 6 - 2k\Delta + 2k\Delta^2 \\ &= 2n\Delta - \Delta^2 - 4\Delta + 6. \end{aligned}$$

If (b) holds, then $n = (\Delta - 1)k + 1$ and by $(***)$ we obtain

$$\begin{aligned} M_2(T) &= (\Delta - 1)^2 - (\Delta - 1) - 2\Delta^2 + \Delta + ((\Delta - 2)k + 1)\Delta - ((\Delta - 2)k + 1)\Delta^2 \\ &+ ((\Delta - 1)k + 1)\Delta^2 \\ &= -\Delta^2 - \Delta + 2 - 2k\Delta + 2k\Delta^2 \\ &= -\Delta^2 - \Delta + 2 + 2(n - 1)\Delta \\ &= 2n\Delta - \Delta^2 - 3\Delta + 2. \end{aligned}$$

If (c) holds, then $n = (\Delta - 1)k + r$ and by $(***)$ we have

$$\begin{aligned} M_2(T) &= t^2 - t - 2\Delta^2 + \Delta + n_1\Delta - n_1\Delta^2 + n\Delta^2 \\ &= r^2 - 3r + 2 - \Delta^2 - 2k\Delta + r\Delta + 2k\Delta^2. \\ &= 2k(\Delta - 1)\Delta - \Delta^2 + r\Delta + 2 + r(r - 3) \\ &= 2n\Delta - \Delta^2 - r\Delta + 2 + r(r - 3). \end{aligned}$$

Case 2. $n_t = 0$.
By (4) we have $(\Delta - 1)(k - n_\Delta) + r - 2 = 0$ that leads to $r = 2$ and $n_\Delta = k$. If follows from $(**)$ that

$$\begin{aligned} M_{2_{max}}(T) &= n_1\Delta - n_1\Delta^2 + (n - 1).\Delta^2 \\ &= ((\Delta - 2)k + 2)\Delta - ((\Delta - 2)k + 2)\Delta^2 + ((\Delta - 1)k + 1)\Delta^2 \\ &= 2\Delta(\Delta - 1)k - \Delta^2 + 2\Delta \\ &= 2\Delta(n - 2) - \Delta^2 + 2\Delta \\ &= 2n\Delta - \Delta^2 - 2\Delta. \end{aligned}$$

This completes the proof. □

3. Lower Bound on the Second Zagreb Coindex among All Trees

In [1], Ashrafi and others proved that for any connected graph G with n vertices and m edges,

$$\overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G).$$

The next corollary is direct consequence this equality and Theorem 2.1.

Corollary 3.1. Let T be a tree of order n and maximum degree Δ. If $n \equiv r \pmod{\Delta - 1}$, then

$$2\overline{M}_2(T) \geq \begin{cases} 4n^2 - 5n(\Delta+2) + 2\Delta^2 + 12(\Delta-1) & r = 0 \\ 4n^2 - 5n(\Delta+2) + 2\Delta^2 + 9\Delta & r = 1 \\ 4n^2 - 5n(\Delta+2) + 2\Delta^2 + 6(\Delta+1) & r = 2 \\ 4n^2 - 5n(\Delta+2) + 2\Delta^2 + (2+3r)\Delta + (7-3r)r & r \geq 3. \end{cases}$$

Proof. From Theorem 2.1, we conclude that $2\overline{M}_2(G) = 4n^2 - 8n + 4 - (2M_2(T) + M_1(T))$. Now by Theorem 2.1 and Corollary 2.1, the proof is straightforward. □

Conflicts of Interests. The authors declare that there are no conflicts of interest regarding the publication of this article.

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