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The Non-Coprime Graph of Finite Groups

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Abstract

The non-coprime graph Π_G of a finite group G is a graph with the vertex set $G \setminus \{e\}$, where two distinct vertices u and ν are adjacent if they have non-coprime orders. In this paper, the main properties of the Cartesian and tensor product of the non-coprime graph of two finite groups are investigated. We also describe the non-coprime graph of some special groups including the dihedral and semi-dihedral groups. Some open questions are also proposed.

Keywords: Coprime graph, Dihedral group, Semi-dihedral group, Dicyclic group.

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1. Introduction

In this paper, all groups considered are finite and a graph means an undirected simple graph without loops and multiple edges. For any graph Π, the sets of all vertices and edges of Π are denoted by $V(\Pi)$ and $E(\Pi)$, respectively.

Given a group G, there are different ways to associate a graph to G, including the prime graph [8], commuting graph [4], and Cayley graphs which have a long history and valuable applications.

The non-coprimle graph Π_G of a finite group G is a graph with $G \setminus \{e\}$ as the vertex set and two distinct vertices u and ν are adjacent if $(|u|, |\nu|) \neq 1$. This graph was first introduced in [5]. The relative non-coprime graph $\Pi_{(G,H)}$ of G and a

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subgroup H of G is a spanning subgraph of Π_G where two distinct vertices u and ν are adjacent if at least one of them belongs to H. Clearly $\Pi_{(G,G)} = \Pi_G$. In the next section, we investigate the cartesian and tensor product of the non-coprime graph of two groups. In the last section, some properties of the non-coprime graph of the groups D_{2n}, U_{2nm}, V_{8n}, T_{4n} and SD_{8n} such as their clique numbers, chromatic numbers and connectivity are investigated. We show that the non-coprime graph of all these groups are perfect graphs.

We use the notation $u \sim v$ to show that two vertices u and v are adjacent in the background graph. All other notations are standard and can be found for example in [1].

2. Graph Operations on Non-Coprime Graphs

The aim of this section is to study the non-coprime graph under two graph operations Cartesian and tensor product of graphs.

Definition 2.1. Let G_1 and G_2 be two graphs. The cartesian product of G_1 and G_2 denoted $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u, v) and (u', v') are adjacent if $u = u'$ and $v \sim v'$ or $v = v'$ and $u \sim u'$. The tensor product of G_1 and G_2 denoted $G_1 \otimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u, v) and (u', v') are adjacent if $u \sim u'$ and $v \sim v'$.

For an arbitrary group G, we use G^* for the set of all non-identity elements of G.

Lemma 2.2. Let G_1 and G_2 be two groups. Then $\Pi_{G_1} \times \Pi_{G_2}$ and $\Pi_{G_1} \otimes \Pi_{G_2}$ both are proper spanning subgraphs of $\Pi_{G_1^* \times G_2^*}$ which itself is the induced subgraph of $\Pi_{G_1 \times G_2}$ on $G_1^* \times G_2^*$.

Proof. It is obvious that $V(\Pi_{G_1} \times \Pi_{G_2}) = V(\Pi_{G_1} \otimes \Pi_{G_2}) = V(\Pi_{G_1^* \times G_2^*}) = G_1^* \times G_2^*$. Let (u, v) and (u', v') be two arbitrary vertices of $\Pi_{G_1} \times \Pi_{G_2}$ such that $(u, v) \sim (u', v')$. We claim that $(u, v) \sim (u', v')$ in $\Pi_{G_1 \times G_2}$. We have $u = u'$ and $v \sim v'$ or $v = v'$ and $u \sim u'$. Without loss of generality, let $u = u'$ and $v \sim v'$. Then $(|v|, |v'|) \neq 1$. Hence there exists a positive integer $d \neq 1$ such that $d||v|$ and $d||v'|$. Therefore $d||(u, v)|$ and $d||(u', v')|$ which implies that they are adjacent in $\Pi_{G_1 \times G_2}$. The proof is similar for $\Pi_{G_1} \otimes \Pi_{G_2}$. □

The next corollary is an obvious result of the previous lemma:

Corollary 2.3. Let G_1 and G_2 be two group and $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$ be nontrivial subgroups of G_1 and G_2, respectively. Then

(i) $\Pi_{G_1,H_1} \times \Pi_{G_2,H_2} \subsetneqq \Pi_{G_1^* \times G_2^*, H_1^* \times H_2^*}$; and

(ii) $\Pi_{G_1,H_1} \otimes \Pi_{G_2,H_2} \subsetneqq \Pi_{G_1^* \times G_2^*, H_1^* \times H_2^*}$.

Remark 1. Let G_1 and G_2 be two group and $(u, v) \in \Pi_{G_1} \otimes \Pi_{G_2}$. Then by definition, $\deg_{\Pi_{G_1} \otimes \Pi_{G_2}} \big((u, v)\big) = \deg_{\Pi_{G_1}}(u) + \deg_{\Pi_{G_2}}(v)$.

The following theorem is an immediate consequence of above observation:

Theorem 2.4. Let G_1 and G_2 be two groups of orders p^{k_1} and q^{k_2}, respectively, where p and q are odd primes and k_1 and k_2 are positive integers. Then $\Pi_{G_1} \otimes \Pi_{G_2}$ is a spanning Eulerian subgraph of $\Pi_{G_1^* \times G_2^*}$.

Proof. By Remark **??**, it is sufficient to prove that $\Pi_{G_1} \otimes \Pi_{G_2}$ is Eulerian. Since G_1 and G_2 are of prime power orders, their non-coprime graphs are complete. Let (a, b) be an arbitrary vertex of $\Pi_{G_1} \otimes \Pi_{G_2}$. Then

$$\begin{aligned} \deg_{\Pi_{G_1} \otimes \Pi_{G_2}} \big((u, v)\big) &= \deg_{\Pi_{G_1}}(u) + \deg_{\Pi_{G_2}}(v) \\ &= p^{k_1} - 2 \ + \ q^{k_2} - 2 \\ &= p^{k_1} + q^{k_2} - 4 \end{aligned}$$

and so the degree of every vertex of $\Pi_{G_1} \otimes \Pi_{G_2}$ is even. Therefore, the graph is Eulerian. □

3.The Non-Coprime Graph of Some Groups

Suppose $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$ denote the independence, clique, chromatic and covering numbers of a graph G. In this section, we obtain some results on the non-coprime graph of the Dihedral groups, Semi-dihedral groups and some other groups. Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b\rangle$ be the dihedral group of order $2n$.

Theorem 3.1. If n is an odd natural number, then

(i) $\omega(\Pi_{D_{2n}}) = \chi(\Pi_{D_{2n}}) = n$;

(ii) $\theta(\Pi_{D_{2n}}) = \theta(\Pi_{Z_n}) + 1 =\mid \pi(Z_n) \mid +1$;

(iii) $\alpha(\Pi_{D_{2n}}) = \theta(\Pi_{D_{2n}})$.

Proof. (i) For $1 \leqslant i, j \leqslant n$ we have $|a^i b| = 2$ and $|a^j|$ is odd such that $(|a^i b|, |a^j|) = 1$. Hence $\Pi_{D_{2n}}$ is a graph with two components $\Pi_{<a>}$ and K_n. Therefore $\omega(\Pi_{D_{2n}}) = \chi(\Pi_{D_{2n}}) = n$.

(ii) The minimum number of cliques that cover D_{2n} is by (i) the minimum number of cliques that cover Z_n plus 1.

(iii) It is obvious by (ii). □

Theorem 3.2. If $n = 2^{k_1} p_2^{k_2} \cdots {p_r}^{k_r}$ where $p_1, \ldots, p_r$ are odd primes and $k_1, \ldots, k_r$ are non-negative integers with $k_1 \geq 1$, then $\Pi_{D_{2n}}$ is connected and

(i) $diam(\Pi_{D_{2n}}) \leq 2$;

(ii) $\theta(\Pi_{D_{2n}}) = \theta(\Pi_{Z_n}) = |\pi(Z_n)|$;

(iii) $\omega(\Pi_{D_{2n}}) = \frac{n(2^{k_1}-1)}{2^{k_1}} + n$.

Proof. (i) D_{2n} has an element of order $2p_2p_3\cdots p_r$ such that all other vertices are adjacent to which. Hence $\Pi_{D_{2n}}$ is connected and $diam(\Pi_{D_{2n}}) \leq 2$.
(ii) The minimum number of cliques that cover D_{2n} is the minimum number of cliques that cover Z_n.
(iii) It is obvious by (ii). □

Now let V_{8n} be the group presented by $\langle a\ ,\ b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle$, for $n \geq 1$. We have the following results on its non-coprime graph:

Theorem 3.3. (i) $\Pi_{V_{8n}}$ is a connected graph.

(ii) $\Pi_{V_{8n}}$ is Eulerian if and only if n is a power of 2.

(iii) $\Pi_{V_{8n}}$ has an Eulerian spanning subgraph.

(iv) $\gamma(V_{8n}) = 1$

Proof. (i) Since $\Pi_{<a>}$ is a subgraph of $\Pi_{V_{8n}}$, the order of $\Pi_{V_{8n}-<a>}$ is even, two vertices of order 2 are adjacent and $\Pi_{Z_{2n}}$ is connected, $\Pi_{V_{8n}}$ is connected.
(ii) Since $\Pi_{V_{8(2^k)}}$ is a complete graph of odd order, the degree of each vertex is even. So, the graph is Eulerian. If $8n = 2^{k_1}p_2^{k_2}\dots p_r^{k_r}$, then the number of vertices that are divided by p_i $((2 \leqslant i \leqslant r))$ is even. Hence the degree of p_i is odd and $\Pi_{V_{8n}}$ is not Eulerian.
(iii) Each edge in $\Pi_{V_{8n}}$ is on a triangle, so the graph has an Eulerian spanning subgraph.
(iv) Since $\Pi_{V_{8(2^k)}}$ has at least $end - vertices$, $\gamma(\Pi_{V_{8n}}) = 1$. □

Theorem 3.4. In the graph $\Pi_{V_{8n}}$

1. If $n = 2^k$ then $\omega(\Pi_{V_{8n}}) = 8n - 1$;
2. If $n = 2^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$ where $p_2, \dots, p_r$ are distinct odd prime numbers and $k_1, \dots, k_r$ are positive integers, then

$$\omega(\Pi_{V_{8n}}) = 8n - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}.$$

Proof. 1. If $n = 2^k$, then 2 divides the order of each vertex. Hence $\omega(V_{8n}) = 8n - 1$.

2. If $n = 2^{k_1}p_2^{k_2}p_3^{k_3}\dots p_r^{k_r}$, then number of vertices that p_i divides their orders but $2, p_1, \dots, p_{i-1}, p_{i+1}, \dots p_r$ don't divides their orders is ${p_i}^{k_i}-1$. Moreover, the number of vertices that 2 does not divides their orders are $p_2^{k_2}p_3^{k_3}\dots p_r^{k_r}$ and so the largest clique has even order. Thus, $\omega(V_{8n}) = 8n - p_2^{k_2}p_3^{k_3}\dots p_r^{k_r}$.

Hence the result. □

Theorem 3.5. $\Pi_{V_{8n}}$ is not planar graph.

Proof. Since there is at least 5 vertex of order that 2 divides them order so $K_5 \leqslant \Pi_{V_{8n}}$. Then $\Pi_{V_{8n}}$ is not planar graph. ☐

Theorem 3.6. Let $U_{2nm} =< a\ ,\ b|a^{2n} = b^m = 1, aba^{-1} = b^{-1}$. If $n = 2^{k_1}p_2^{k_2}\dots p_r^{k_r}$, $m = 2^{l_1}q_2^{l_2}\dots q_s^{l_s}$ and for $1\ i \leq r$, p_i are distinct prime and for $1 \leq j \leq s$, q_j are distinct prime and k_i, l_j are in Z. Then in U_{2nm} group:

1. The number of elements that 2 divides them order is $2nm - \dfrac{nm}{2^{k_1+l_1}}$;
2. The number of elements that p_i divides them order and $p_i \mid n$, $p_i \mid m$, $p_i = q_t$ is $2nm - \dfrac{nm}{p_i^{k_i}} - \dfrac{nm}{p_i^{k_i+l_t}}$;
3. The number of elements that p_i divides them order and $p_i \mid n$, $p_i \nmid m$ is $2nm - \dfrac{2nm}{p_i^{k_i}}$;
4. The number of elements that q_j divides them order and $q_j \nmid n$, $q_j \mid m$ is $nm - \dfrac{nm}{q_j^{l_j}}$.

Proof. We know in U_{2nm}

(a) Order of a^ib^j, that $1 \leq i \leq 2n$, i is odd, $1 \leq j \leq m$ is $2\dfrac{n}{(i,n)}$;

(b) Order of a^ib^j, that $1 \leq i \leq 2n$, i is even, $1 \leq j \leq m$ is $[\dfrac{2n}{(i,2n)}, \dfrac{m}{(j,m)}]$;

1. To obtain the number of element of even order we obtain the number of elements that 2 not divides them order in section a and b and sum them and obtain difference $2nm$ of it.
   (a) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is odd, $1 \leq j \leq m$ and $2 \nmid O(a^ib^j)$ is zero.
   (b) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is even, $1 \leq j \leq m$ and $2 \nmid O(a^ib^j)$ is $2nm - (p_2^{k_2}p_3^{k_3}\dots p_r^{k_r})(q_2^{l_2}q_3^{l_3}\dots q_s^{l_s})$;
   That $p_2^{k_2}p_3^{k_3}\dots p_r^{k_r}$ is the number of i that that for them $\dfrac{2n}{(i,2n)}$ is odd and $q_2^{l_2}q_3^{l_3}\dots q_s^{l_s}$ is the number of j that that for them $\dfrac{m}{(j,m)}$ is odd.
   So the number of elements that 2 divides them order is $2nm - \dfrac{nm}{2^{k_1+l_1}}$;
2. To obtain the number of elements that p_i divides them order and $p_i \mid m, p_i \mid n$ obtain the number of element that p_i not divides them order in section a and b and sum them and obtain difference $2nm$ of it.

(a) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is odd, $1 \leq j \leq m$ and $p_i \nmid O(a^ib^j)$ is $\frac{n}{p_i^{k_i}}m$;

(b) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is even, $1 \leq j \leq m$, $p_i \nmid O(a^ib^j)$ and $p_i = q_t$ is $\frac{n}{p_i^{k_i}}.\frac{m}{p_i^{l_t}}$;

So the number of elements that p_i divides them order is $2nm - \frac{nm}{p_i^{k_i}} - \frac{nm}{p_i^{k_i+l_t}}$;

3. To obtain the number of elements that p_i divides them order and $p_i \nmid m, p_i \mid n$ obtain the number of element that p_i not divides them order in section a and b and sum them and obtain difference $2nm$ of it.

(a) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is odd, $1 \leq j \leq m$ and $p_i \nmid O(a^ib^j)$ is $\frac{n}{p_i^{k_i}}m$;

(b) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is even, $1 \leq j \leq m$, $p_i \nmid O(a^ib^j)$ is $\frac{n}{p_i^{k_i}}m$;

So number of elements that p_i divides them order is $2nm - \frac{n}{p_i^{k_i}}m - \frac{n}{p_i^{k_i}}m = 2nm - \frac{2nm}{{p_i}^{k_i}}$;

4. To obtain the number of elements that pq_j divides them order and $q_j \mid m, q_j \nmid n$ obtain the number of element that q_j not divides them order in section a and b and sum them and obtain difference $2nm$ of it.

(a) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is odd, $1 \leq j \leq m$ and $q_j \nmid O(a^ib^j)$ is nm;

(b) The number of elements a^ib^j, that $1 \leq i \leq 2n$, i is even, $1 \leq j \leq m$, $q_j \nmid O(a^ib^j)$ is $\frac{nm}{p_i^{k_i}}$;

So number of elements that q_j divides them order is $2nm - (nm + \frac{nm}{q_j^{l_j}}) = nm - \frac{nm}{q_j^{l_j}}$.

□

Theorem 3.7. $\Pi_{U_{2nm}}$ is planar graph if and only if $n < 2$, $m < 4$ or $n = 3$, $m = 1$.

Proof. By before theorem k_5 is not subgraph of $\Pi_{U_{2nm}}$ if and only if $m = 1$, $n = 3$ or $m < 3$, $n < 2$. □

Theorem 3.8. $\Pi_{U_{2nm}}$

1. Is connected other than $n = 1$, m be odd.
2. If be connected has $end - vertex$
3. Is line graph when $nm = p^i$ and p is prime and $i = 0, 1, 2, \cdots$

Proof. 1. If $n = 1$ and m be odd then elements of even are only order of 2 and other elements has odd order so there is not patd betven elements of order 2 and elements of even order and graph is not connected. In otherwise there are elements of order [m,n] that is connect to all vertices of U_{2nm}.

2. All elements of order [m,n] are $end - vertex$.
3. If $nm = p^i$ that p is prime, then in $\Pi_{U_{2nm}}$ each vertex is at most in two clique and $\Pi_{U_{2nm}}$ is line graph.

□

Theorem 3.9. If $n = 2^{k_1}p_2^{k_2}\dots p_r^{k_r}$, $m = 2^{l_1}q_2^{l_2}\dots q_s^{l_s}$ and for $1\ i \leq r$, p_i are distinct prime and for $1 \leq j \leq s$, q_j are distinct prime and k_i, l_j are in Z. Then in $\Pi_{U_{2nm}}$:

$$\omega(\Pi_{2nm}) \geq max\{t_1, t_2, t_3, t_4\}$$

that $t_1{=}nm(2 - \frac{1}{2^{k_1+l_1}})$, $t_2{=}2nm(1 - \frac{1}{p_{r_1}^{k_{r_1}}})$, $t_3 = 2nm - nm(\frac{1}{p_t^{k_t}} + \frac{1}{p_t^{k_t+l_j}})$, $t_4 = nm - (1 - \frac{1}{q_{s_1}^{l_{s_1}}})$, $p_t^{k_t}$ is greatest divisor of n that p_t divides m, $p_t = q_j$ and $p_{r_1}^{k_{r_1}}$ is greatest divisor of$n$ that p_{r_1}, m not divides m and $q_{s_1}^{l_{s_1}}$ is greatest divisor of$m$ that q_{s_1} not divides n.

Proof. Enough in $\Pi_{U_{2nm}}$ obtain the number of elements that divides them order prime number that divides m, n or 2 divides them order and obtain maximum of them that is $max\{t_1, t_2, t_3, t_4\}$. So proof complated. □

Theorem 3.10. Other than $n = m = 1$ and $n = 1$ $m = 3$ in otherwise of n, m $\Pi_{U_{2nm}}$ have spaning Eulerian subgraph.

Proof. If $\Pi_{U_{2nm}}$ be unconnected graph, then the number of vertex of even order are larger than 2 and there are vertices that connect to vertex of odd order, so sach edge belong to triangle. If $\Pi_{U_{2nm}}$ be connected graph, then hase at least 2 evd-vertex and each edge belong to triangle. Then $\Pi_{U_{2nm}}$ is Eulerian. □

Theorem 3.11. Let $T_{4n} =< a\ ,\ b|a^{2n} = 1a^n = b^2, b^{-1}ab = a^{-1} >$. Then $\Pi_{T_{4n}}$

1. Is connected graph.
2. Is Eulerian if $n = 2^k$ and otherwise not Eulerian.

3. Has Eulerian spanning subgraph.
4. $\gamma(\Pi_{T_{4n}}) = 1$
5. If $n \neq 1$ not planar graph.

Proof. 1. Since $\Pi_{Z_{2n}}$ is subgraph of $\Pi_{T_{4n}}$ and order of $\Pi_{T_{4n}-Z_{2n}}$ are 4 that join to vertices of order even and $\Pi_{Z_{2n}}$ is connected, then $\Pi_{T_{4n}}$ is connected.

2. If $n = 2^k$, then $\Pi_{T_{4n}}$ is is complete graph of order $4n - 1$ and degree of each vertex is $4n-2$ and is even, so $\Pi_{T_{4n}}$ is Eulerian. If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, degree of vertex that order is p_i equal

$$\frac{2n({p_i}^{k_i} - 1)}{{p_i}^{k_i}} - 1$$

That is odd and $\Pi_{T_{4n}}$ is not Eulerian.

$$\frac{2n({p_i}^{k_i} - 1)}{{p_i}^{k_i}} - 1$$

3. $\Pi_{Z_{2n}}$ has spanning Eulerian subgraph, then each edge of this sub graph is in triangle. Eenouph that prove if $e = v_1v_2$ is edge where $| v_1 |= 4, | v_2 |\neq 4$ or$| v_1 |=| v_2 |= 4$ then e be in triangle. If $n = 1$, then $\Pi_{T_{4n}}$ is complete graph of order 3. If $n \neq 1$ there is at least 6 vertex of order 4 in$\Pi_{T_{4n}}$ and if $| v_1 |= 4, | v_2 |\neq 4$ then order of v_2 is even, then there is edge between v_2 and vertex of order 4 as v_3, so $v_3 \sim v_1 \sim v_2 \sim v_3$ and e is in triangle. If $| v_1 |=| v_2 |= 4$ proof is similarity.

4. $\Pi_{T_{4n}}$ has vertex of order $2n$ that adjoin to all vertices, then $\gamma(\Pi_{T_{4n}}) = 1$

5. Obviously Π_{T_4} is complete graph and planar. If$n \neq 1$ there is at least 6 vertex of order 4 so $k_5 \leqslant \Pi_{T_{4n}}$ is not planar.

□

Theorem 3.12. If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ where p_i are distinct prime number and $k_i \in Z^+$, then :

(i) If n be odd :

$$\omega(\Pi_{T_{4n}}) = 3n$$

(ii) If n be even, i.e $p_1 = 2$, then:

$$\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$$

Proof. Since $\Pi_{Z_{2n}}$ is subgraph of $\Pi_{T_{4n}}$ and order of $\Pi_{T_{4n}-Z_{2n}}$ are 4 that join to vertices of order even, then clique of even vertices is largest clique of $\Pi_{T_{4n}}$, so sufficient to obtainthe the number of even order.

(i) The number of element of even order in Z_{2n} is $\dfrac{2n(2-1)}{2} = n$, then the number of element of even order in $\Pi_{T_{4n}}$ is $2n + n = 3n$

(ii) The number of element of even order in Z_{2n} is $\dfrac{2n(2^{k_1+1}-1)}{2^{k_1+1}}$, then The number of element of even order in $\Pi_{T_{4n}}$ is

$$\omega(\Pi_{T_{4n}}) = 2n + \frac{2n(2^{k_1+1}-1)}{2^{k_1+1}} = 2n(2 - \frac{1}{2^{k_1+1}}) = 4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$$

□

Theorem 3.13.

$$\chi(\Pi_{T_{4n}}) = \omega(\Pi_{T_{4n}})$$

Proof. Let n be odd, then large clique contain $\omega(\Pi_{T_{4n}}) = 3n$ vertices and remaining vertices that are $n-1$ and odd order coloring with $3n$ color. If n be even then large clique contain $\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$ vertices where $n = 2^{k_1} p_2^{k_2} \dots p_r^{k_r}$ and $p_2^{k_2} p_3^{k_3} \dots p_r^{k_r} < n$, so

$$4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r} > 3n$$

and remaining vertices that are at least $n-2$ and coloring with $\omega(\Pi_{T_{4n}})$ color and

$$\chi(\Pi_{T_{4n}}) = \omega(\Pi_{T_{4n}})$$

□

Theorem 3.14. In graph $\Pi_{T_{4n}}$:

1. $diam(\Pi_{T_{4n}}) \leqslant 2$
2. $\Pi_{T_{4n}}$ is line graph if and only if $n = p^k$ or $n = 2^k$ or $n = 2^k p^k$ where p is prime.
3. If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ the number of $\Pi_{T_{4n}}$ end-vertices is:
   (a) If n be odd:
   $$(p_1^{k_1} - 1)(p_2^{k_2} - 1) \dots (p_r^{k_r} - 1)$$
   (b) If n be even i.e. $p_1 = 2$:
   $$(2^{k_1+1} - 1)(p_2^{k_2} - 1) \dots (p_r^{k_r} - 1)$$

Proof. 1. Since $diamm(\Pi_{Z_{2n}}) \leqslant 2$, then by definition of $\Pi_{T_{4n}}$, $diam(\Pi_{T_{4n}}) \leqslant 2$.

2. Edge set of $\Pi_{T_{4n}}$ can be partitioned in to a set of clique with the property that any vertex lies in at most two clique if and only if n be $n = p^k$ or $n = 2^k$ or $n = 2^k p^k$ by theorem 1.7.2 of [3] proof complete.

3. The number of $\Pi_{T_{4n}}$ end-vertices is equal with the number of $\Pi_{Z_{2n}}$ end-vertices that is number of vertices that $p_1p_2\cdots p_r$ divides them order and is $(p_1^{k_1}-1)(p_2^{k_2}-1)\ldots(p_r^{k_r}-1)$ when n is odd and is $(2^{k_1+1}-1)(p_2^{k_2}-1)\ldots(p_r^{k_r}-1)$ when n is even.

□

Theorem 3.15. Let $SD_{8n} =< a\ ,\ b|a^{4n} = b^2 = 1, bab = a^{2n-1}$. Then $\Pi_{SD_{8n}}$

1. *Is connected graph.*
2. *If* $n = 2^k$ *(*$0 \leqslant k \in Z$*)* $\Pi_{SD_{8n}}$ *is Eulerian and otherwise is not Eulerian graph*
3. $\Pi_{V_{8n}}$ *has Eulerian spanning subgraph.*
4. $\gamma(V_{8n}) = 1$

Proof. Since $\Pi_{Z_{4n}}$ is sub graph of $\Pi_{SD_{8n}}$ and order of $\Pi_{SD_{8n}-Z_{4n}}$ vertex are even(2n vertex of order 2 and 2n vertex of order 4) that join to vertices of even order, then this graph has specifications of $\Pi_{Z_{4n}}$ □

Theorem 3.16. Let $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, then

1. If n is odd
$$\omega(\Pi_{SD_{8n}}) = 7n$$
2. If n is even i.e. $p_1 = 2$
$$\omega(\Pi_{SD_{8n}}) = 8n - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$$

Proof. The number of element s of odd order in $\Pi_{SD_{8n}}$ is $p_2^{k_2}\cdots p_r^{k_r}-1$. Then clique number of $\Pi_{SD_{8n}}$ is the number of vertices that order is even that is

$$8n - 1 - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r} - 1$$

then

$$\omega(\Pi_{SD_{8n}}) = 8n - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$$

□

Corollary 3.17.

$$\chi(\Pi_{SD_{8n}}) = \omega(\Pi_{SD_{8n}})$$

Proof. By before theorem $7n \leq \omega(\Pi_{SD_{8n}})$, then vertices that remaining of largest clique are coloring with $7n$ color, then

$$\chi(\Pi_{SD_{8n}}) = \omega(\Pi_{SD_{8n}})$$

□

Confilcts of Interest. The author declares that he has no conflicts of interest.

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