Mathematics Interdisciplinary Research **7** (2022) 385 – 394

Original Scientific Paper

The Non-Coprime Graph of Finite Groups

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Abstract

The non-coprime graph Π_G of a finite group G is a graph with the vertex set $G\setminus\{e\}$, where two distinct vertices u and ν are adjacent if they have non-coprime orders. In this paper, the main properties of Cartesian and tensor product of the non-coprime graph of two finite groups are investigated. We also describe the non-coprime graph of some groups including the dihedral and semi-dihedral groups. Some open questions are also proposed.

Keywords: Coprime graph, dihedral group, semi-dihedral group, dicyclic group.

2010 Mathematics Subject Classification: 05C15, 20D15, Secondary: 05C75.

How to cite this article
G. Aghababaei-Beni and A. Jafarzadeh, The non-coprime graph of finite groups, *Math. Interdisc. Res.* **7** (2022) 385-394.

1. Introduction

In this paper, all groups are finite and all graphs are undirected simple graphs without loops or multiple edges. For every graph Π, the sets of all vertices and edges of Π are denoted by $V(\Pi)$ and $E(\Pi)$, respectively.

Given a group G, there are different ways to associate a graph to G, including the prime graph [5], commuting graph [3], and Cayley graphs which have a long history and valuable applications.

The non-coprime graph Π_G of a finite group G is a graph with $G \setminus \{e\}$ as the vertex set and two distinct vertices u and ν are adjacent if $(|u|, |\nu|) \neq 1$. This

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Academic Editor: Ali Reza Ashrafi
Received 17 February 2019, Accepted 27 February 2019
DOI: 10.22052/mir.2019.172393.1117

graph was first introduced in [4]. The relative non-coprime graph $\Pi_{(G,H)}$ of G and a subgroup H of G is a spanning subgraph of Π_G where two distinct vertices u and ν are adjacent if at least one of them belongs to H. Clearly $\Pi_{(G,G)} = \Pi_G$. In the next section, we investigate the Cartesian and tensor product of the non-coprime graph of two groups. In the last section, some properties of the non-coprime graph of the groups D_{2n}, U_{2nm}, V_{8n}, T_{4n} and SD_{8n} such as their clique numbers, chromatic numbers, and connectivity are investigated. We show that the non-coprime graph of all these groups are perfect.

We use the notation $u \sim v$ to show that two vertices u and v are adjacent in the background graph. For integers m and n, we use the notations (m,n) and $[m,n]$ for the greatest common divisor and the least common multiple of a and b, respectively. Also if g is an element of a group G, the notation $|g|$ stands for the order of g in G, i.e. the least positive integer n with $g^n = e$. All other notations are standard and can be found for example in [1, 2].

2. Graph Operations on Non-Coprime Graphs

The aim of this section is to study the non-coprime graph under two graph operations, Cartesian and tensor product of graphs.

Definition 2.1. Let G_1 and G_2 be two graphs. The Cartesian product of G_1 and G_2 denoted $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u,v) and (u',v') are adjacent if $u = u'$ and $v \sim v'$ or $v = v'$ and $u \sim u'$. The tensor product of G_1 and G_2 denoted $G_1 \otimes G_2$, is a graph with a vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u,v) and (u',v') are adjacent if $u \sim u'$ and $v \sim v'$. For an arbitrary group G, we use G^* for the set of all non-identity elements of G.

Lemma 2.2. Let G_1 and G_2 be two groups. Then $\Pi_{G_1} \times \Pi_{G_2}$ and $\Pi_{G_1} \otimes \Pi_{G_2}$ both are proper spanning subgraphs of $\Pi_{G_1^* \times G_2^*}$ which is itself an induced subgraph of $\Pi_{G_1 \times G_2}$ on $G_1^* \times G_2^*$.

Proof. It is obvious that $V(\Pi_{G_1} \times \Pi_{G_2}) = V(\Pi_{G_1} \otimes \Pi_{G_2}) = V(\Pi_{G_1^* \times G_2^*}) = G_1^* \times G_2^*$. Let (u,v) and (u',v') be two arbitrary vertices of $\Pi_{G_1} \times \Pi_{G_2}$ such that $(u,v) \sim (u',v')$. We claim that $(u,v) \sim (u',v')$ in $\Pi_{G_1 \times G_2}$. We have $u = u'$ and $v \sim v'$ or $v = v'$ and $u \sim u'$. Without loss of generality, let $u = u'$ and $v \sim v'$. Then $(|v|, |v'|) \neq 1$. Hence there exists a positive integer $d \neq 1$ such that $d||v|$ and $d||v'|$. Therefore $d||(u,v)|$ and $d||(u',v')|$ which implies that they are adjacent in $\Pi_{G_1 \times G_2}$. The proof is similar for $\Pi_{G_1} \otimes \Pi_{G_2}$. □

The next corollary is an obvious result of the previous lemma:

Corollary 2.3. Let G_1 and G_2 be two groups and $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$ be nontrivial subgroups of G_1 and G_2, respectively. Then $\Pi_{G_1,H_1} \times \Pi_{G_2,H_2}$ and $\Pi_{G_1,H_1} \otimes \Pi_{G_2,H_2}$ are proper subgraphs of $\Pi_{G_1^* \times G_2^*, H_1^* \times H_2^*}$.

Remark 1. Let G_1 and G_2 be two groups and $(u, v) \in \Pi_{G_1} \otimes \Pi_{G_2}$. Then by definition, $\deg_{\Pi_{G_1} \otimes \Pi_{G_2}}\big((u, v)\big) = \deg_{\Pi_{G_1}}(u) + \deg_{\Pi_{G_2}}(v)$.

The following theorem is an immediate consequence of the above remark:

Theorem 2.4. Let G_1 and G_2 be two groups of orders p^{k_1} and q^{k_2}, respectively, where p and q are odd primes and k_1 and k_2 are positive integers. Then $\Pi_{G_1} \otimes \Pi_{G_2}$ is a spanning Eulerian subgraph of $\Pi_{G_1^* \times G_2^*}$.

Proof. By Remark 1, it is sufficient to prove that $\Pi_{G_1} \otimes \Pi_{G_2}$ is Eulerian. Since G_1 and G_2 are of prime power orders, their non-coprime graphs are complete. Let (a, b) be an arbitrary vertex of $\Pi_{G_1} \otimes \Pi_{G_2}$. Then

$$\begin{aligned} \deg_{\Pi_{G_1} \otimes \Pi_{G_2}}\big((u, v)\big) &= \deg_{\Pi_{G_1}}(u) + \deg_{\Pi_{G_2}}(v) \\ &= p^{k_1} - 2 \ + \ q^{k_2} - 2 \\ &= p^{k_1} + q^{k_2} - 4. \end{aligned}$$

This means the degree of every vertex of $\Pi_{G_1} \otimes \Pi_{G_2}$ is even. Therefore, the graph is Eulerian. □

3. The Non-Coprime Graph of Some Groups

Suppose $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$ denote the independence, clique, chromatic and covering numbers of a graph G, respectively. In this section, we obtain some results on the non-coprime graph of the dihedral groups, semi-dihedral groups and some other groups. Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b\rangle$ be the dihedral group of order $2n$. We have the following results on its non-coprime graph:

Theorem 3.1. If n is an odd natural number, then

(i) $\omega(\Pi_{D_{2n}}) = \chi(\Pi_{D_{2n}}) = n$,

(ii) $\theta(\Pi_{D_{2n}}) = \theta(\Pi_{Z_n}) + 1 =\mid \pi(Z_n) \mid +1$,

(iii) $\alpha(\Pi_{D_{2n}}) = \theta(\Pi_{D_{2n}})$.

Proof. (i) For $1 \leqslant i, j \leqslant n$ we have $|a^i b| = 2$ and $|a^j|$ is odd such that $(|a^i b|, |a^j|) = 1$. Hence $\Pi_{D_{2n}}$ is a graph with two components $\Pi_{<a>}$ and K_n. Therefore $\omega(\Pi_{D_{2n}}) = \chi(\Pi_{D_{2n}}) = n$.
(ii) By proof of (i), the minimum number of cliques that cover D_{2n} is the minimum number of cliques that cover Z_n plus 1.
(iii) It is obvious by (ii). □

Theorem 3.2. If $n = 2^{k_1} p_2^{k_2} \cdots {p_r}^{k_r}$ where $p_1, \ldots, p_r$ are odd primes and $k_1, \ldots, k_r$ are non-negative integers with $k_1 \geq 1$, then $\Pi_{D_{2n}}$ is connected and

(i) $\operatorname{diam}(\Pi_{D_{2n}}) \leq 2$,

(ii) $\theta(\Pi_{D_{2n}}) = \theta(\Pi_{Z_n}) = |\pi(Z_n)|$,

(iii) $\omega(\Pi_{D_{2n}}) = \frac{n(2^{k_1}-1)}{2^{k_1}} + n$.

Proof. (i) D_{2n} has an element of order $2p_2p_3\cdots p_r$ such that it is adjacent to all the other vertices. Hence $\Pi_{D_{2n}}$ is connected and $\text{diam}(\Pi_{D_{2n}}) \leq 2$.
(ii) The minimum number of cliques that cover D_{2n} is the minimum number of cliques that cover Z_n.
(iii) It is obvious by (ii). □

Let V_{8n} be the group defined by $\langle a\ ,\ b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle$. We have the following results on its non-coprime graph:

Theorem 3.3. (i) $\Pi_{V_{8n}}$ is a connected graph.

(ii) $\Pi_{V_{8n}}$ is Eulerian if and only if n is a power of 2.

(iii) $\Pi_{V_{8n}}$ has an Eulerian spanning subgraph.

Proof. (i) Since $\Pi_{<a>}$ is a subgraph of $\Pi_{V_{8n}}$. The order of $\Pi_{V_{8n}-<a>}$ is even, two vertices of order 2 are adjacent and $\Pi_{Z_{2n}}$ is connected, $\Pi_{V_{8n}}$ is connected.
(ii) Let $n = 2^k$. Since $\Pi_{V_{8n}}$ is a complete graph of odd order, the degree of each vertex is even. So, the graph is Eulerian. If $8n = 2^{k_1}p_2^{k_2}\ldots p_r^{k_r}$ with $k_2 \geq 2$, then the number of vertices that are divided by p_i $(2 \leqslant i \leqslant r)$ is even. Hence the degree of p_i is odd and $\Pi_{V_{8n}}$ is not Eulerian.
(iii) Each edge in $\Pi_{V_{8n}}$ is on a triangle, so the graph has an Eulerian spanning subgraph. □

Theorem 3.4. In the graph $\Pi_{V_{8n}}$

(i) If n is a power of 2, then $\omega(\Pi_{V_{8n}}) = 8n - 1$;

(ii) If $n = 2^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$ where $p_2, \ldots, p_r$ are distinct odd prime numbers and $k_1, \ldots, k_r$ are positive integers, then

$$\omega(\Pi_{V_{8n}}) = 8n - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}.$$

Proof. (i) If $n = 2^k$ for some positive integer k, then 2 divides the order of each vertex. Hence $\omega(\Pi_{V_{8n}}) = 8n - 1$.
(ii) If $n = 2^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$, then the number of vertices $v \in V(\Pi_{V_{8n}})$ such that p_i divides the order $|v|$, but 2, p_1, …, p_{i-1}, p_{i+1}, … p_r do not divide $|v|$, is ${p_i}^{k_i} - 1$. Moreover, the number of vertices $v \in V(\Pi_{V_{8n}})$ such that 2 does not divide $|v|$ is $p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$ and so the largest clique has even order. Thus, $\omega(\Pi_{V_{8n}})$ $= 8n - p_2^{k_2}p_3^{k_3}\cdots p_r^{k_r}$. □

Theorem 3.5. $\Pi_{V_{8n}}$ is not a planar graph.

Proof. Since there are at least 5 vertices of even order, we have $K_5 \leqslant \Pi_{V_{8n}}$. Hence $\Pi_{V_{8n}}$ is not planar. $\square$

Let U_{2nm} be the group defined by $\langle a\ ,\ b | a^{2n} = b^m = 1, aba^{-1} = b^{-1}\rangle$.

Theorem 3.6. Suppose that $n = 2^{k_1} p_2^{k_2} \dots p_r^{k_r}$ where $p_2, \dots, p_r$ are distinct odd primes and $m = 2^{l_1} q_2^{l_2} \dots q_s^{l_s}$ where $q_2, \dots, q_s$ are distinct odd primes and $k_1, \dots, k_r, l_1, \dots, l_s$ are non-negative integers. Then we have the following statements in U_{2nm}:

(i) The number of elements of even order is $2nm - \dfrac{nm}{2^{k_1+l_1}}$,

(ii) The number of elements of order divided by p_i where $p_i = q_j$ for some j is $2nm - \dfrac{nm}{p_i^{k_i}} - \dfrac{nm}{p_i^{k_i+l_j}}$,

(iii) The number of elements of order divided by p_i if $p_i \nmid m$ is $2nm - \dfrac{2nm}{p_i^{k_i}}$,

(iv) The number of elements of order divided by q_j if $q_j \nmid n$ is $nm - \dfrac{nm}{q_j^{l_j}}$.

Proof. Let i and j be integers with $1 \leq i \leq 2n$ and $1 \leq j \leq m$. It is easily checked that in U_{2nm}, if i is odd, then the order of $a^i b^j$ is $\dfrac{2n}{(i,n)}$, and if i is even, then the order of $a^i b^j$ is $[\dfrac{2n}{(i,2n)}, \dfrac{m}{(j,m)}]$.

(i) To obtain the number of elements of even order, we subtract the number of elements of odd order from $2nm$.

By the above note, if i is odd, then there is no element $a^i b^j$ with even order. The number of all even i's with $\dfrac{2n}{(i,2n)}$ odd is $p_2^{k_2} p_3^{k_3} \dots p_r^{k_r} = \frac{n}{2^{k_1}}$, and the number of j's with $\dfrac{m}{(j,m)}$ odd is $q_2^{l_2} q_3^{l_3} \dots q_s^{l_s} = \frac{m}{2^{l_1}}$. Hence the number of all elements of even order is $2nm - \dfrac{nm}{2^{k_1+l_1}}$.

(ii) Similar to the proof of (i), if i is odd, then the number of elements $a^i b^j$ with $p_i \nmid |a^i b^j|$ is $\dfrac{n}{p_i^{k_i}} m$, and if i is even, then the number of elements $a^i b^j$ with $p_i \nmid |a^i b^j|$ is $\dfrac{n}{p_i^{k_i}} . \dfrac{m}{p_i^{l_t}}$. So the number of elements whose order is divided by p_i is $2nm - \dfrac{nm}{p_i^{k_i}} - \dfrac{nm}{p_i^{k_i+l_t}}$.

(iii) If either i is odd or i is even, then the number of elements $a^i b^j$ with $p_i \nmid |a^i b^j|$ is $\dfrac{n}{p_i^{k_i}} .m$. So the number of elements whose order is divided by p_i is

$$2nm - \frac{n}{p_i^{k_i}} m - \frac{n}{p_i^{k_i}} m = 2nm - \frac{2nm}{{p_i}^{k_i}}.$$

(iv) If i is odd, then the number of elements a^ib^j with $q_j \nmid |a^ib^j|$ is nm, and if i is even, then the number of elements a^ib^j with $q_j \nmid |a^ib^j|$ is $\frac{nm}{p_i^{k_i}}$. So the number of elements whose order is divided by q_j is

$$2nm - (nm + \frac{nm}{q_j^{l_j}}) = nm - \frac{nm}{q_j^{l_j}}.$$

□

Theorem 3.7. $\Pi_{U_{2nm}}$ is a planar graph if and only if $n < 2$ and $m < 4$ or $(n, m) = (3, 1)$.

Proof. By previous theorem, k_5 is not subgraph of $\Pi_{U_{2nm}}$ if and only if $m = 1$ and $n = 3$ or $m < 3$ and $n < 2$. □

Theorem 3.8. The graph $\Pi_{U_{2nm}}$

(i) is connected, unless $n = 1$ and m is odd;

(ii) has an end-vertex if it is connected;

(iii) is a line graph when n and m are powers of the same prime p.

Proof. (i) If $n = 1$ and m is odd, the elements of even order are all of order 2, and other elements have odd order. So there is no path between elements of order 2 and elements of even order and hence the graph is not connected. Otherwise, there are elements of order $[m, n]$ that are connected to all vertices of $\Pi_{U_{2nm}}$.
(ii) All elements of order $[m, n]$ are end-vertices.
(iii) If $nm = p^i$, then in $\Pi_{U_{2nm}}$ each vertex is at most in two different cliques and so $\Pi_{U_{2nm}}$ is a line graph. □

Theorem 3.9. In the graph $\Pi_{U_{2nm}}$ we have

$$\omega(\Pi_{U_{2nm}}) \geq \max\{t_1, t_2, t_3, t_4\}$$

in which $t_1{=}nm(2 - \frac{1}{2^{k_1+l_1}})$, $t_2{=}2nm(1 - \frac{1}{p_{r_1}^{k_{r_1}}})$, $t_3 = 2nm - nm(\frac{1}{p_t^{k_t}} + \frac{1}{p_t^{k_t+l_j}})$ and $t_4 = nm - (1 - \frac{1}{q_{s_1}^{l_{s_1}}})$ where $p_t^{k_t}$ is the greatest divisor of n that divides m so that $p_t = q_j$, $p_{r_1}^{k_{r_1}}$ is the greatest divisor of n that does not divide m and $q_{s_1}^{l_{s_1}}$ is the greatest divisor of m that doess not divide n.

Proof. It is enough to obtain the number of elements whose order is a prime power that is divided by m or n and the number of elements whose order is even. So the proof is completed. □

Theorem 3.10. The graph $\Pi_{U_{2nm}}$ has a spaning Eulerian subgraph for all n and m, unless $n = m = 1$ and $(n, m) = (1, 3)$.

Proof. If $\Pi_{U_{2nm}}$ is a disconnected graph, then the number of vertices of even order is more than 2 and there are vertices that are connected to vertices of odd order. So each edge belongs to a triangle. If $\Pi_{U_{2nm}}$ is a connected graph, then it has at least 2 end-vertices and each edge belongs to a triangle. Hence $\Pi_{U_{2nm}}$ is Eulerian. □

Let T_{4n} be the group defined by $\langle a\ ,\ b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$.

Theorem 3.11. The graph $\Pi_{T_{4n}}$

(i) is a connected graph,

(ii) is Eulerian if and only if $n = 2^k$,

(iii) has an Eulerian spanning subgraph,

(iv) $\gamma(\Pi_{T_{4n}}) = 1$,

(v) is not a planar graph if $n > 1$.

Proof. (i) Since $\Pi_{\mathbb{Z}_{2n}}$ is a subgraph of $\Pi_{T_{4n}}$ and vertices of $\Pi_{T_{4n}-\mathbb{Z}_{2n}}$ are joined to vertices of $\Pi_{\mathbb{Z}_{2n}}$ and since $\Pi_{\mathbb{Z}_{2n}}$ is connected, hence $\Pi_{T_{4n}}$ is connected.
(ii) If $n = 2^k$, then $\Pi_{T_{4n}}$ is a complete graph of order $4n - 1$ and the degree of each vertex is $4n - 2$ which is even. So $\Pi_{T_{4n}}$ is Eulerian. If p is an odd divisor of n, then the degree of the vertex whose order is p is equal to

$$\frac{2n({p_i}^{k_i} - 1)}{{p_i}^{k_i}} - 1,$$

that is odd and hence $\Pi_{T_{4n}}$ is not Eulerian.
(iii) $\Pi_{\mathbb{Z}_{2n}}$ has a spanning Eulerian subgraph and each edge of this subgraph is in a triangle. It is enouph to prove that if $e = v_1v_2$ is an edge, then e be in triangle. If $n = 1$, then $\Pi_{T_{4n}}$ is a complete graph of order 3. If $n > 1$, then there are at least 6 vertices of order 4 in $\Pi_{T_{4n}}$. Now if $| v_1 | = 4$ and $| v_2 | \neq 4$, then the order of v_2 is even, hencen there is an edge between v_2 and a vertex of order 4 such as v_3, so $v_3 \sim v_1 \sim v_2 \sim v_3$ and e is in a triangle. If $| v_1 | = | v_2 | = 4$ proof is similar.
(iv) $\Pi_{T_{4n}}$ has a vertex of order $2n$ that joins to all other vertices, hence $\gamma(\Pi_{T_{4n}}) = 1$.
(v) Obviously Π_{T_4} is of order less than 5 and so is planar. If $n > 1$, then there are at least 6 vertices of order 4 that pairwise join. So $k_5 \leqslant \Pi_{T_{4n}}$ is not planar. □

Theorem 3.12. Let $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ where $p_1, \ldots, p_r$ are distinct prime number and $k_1, \ldots, k_r$ are positive integers.

(i) If n is odd, then $\omega(\Pi_{T_{4n}}) = 3n$.

(ii) If n is even, letting $p_1 = 2$, then $\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}$.

Proof. Since $\Pi_{\mathbb{Z}_{2n}}$ is a subgraph of $\Pi_{T_{4n}}$ and the order of all elements of $\Pi_{T_{4n}-\mathbb{Z}_{2n}}$ is 4 that join to vertices of even order, the clique of even vertices is the largest clique of $\Pi_{T_{4n}}$, so it is sufficient to obtain the the number of vertices of even order.
(i) The number of elements of even order in $\mathbb{Z}_{2n}$ is $\dfrac{2n(2-1)}{2} = n$, hence the number of elements of even order in $\Pi_{T_{4n}}$ is $2n + n = 3n$.
ii) The number of elements of even order in $\mathbb{Z}_{2n}$ is $\dfrac{2n(2^{k_1+1}-1)}{2^{k_1+1}}$, hence the number of elements of even order in $\Pi_{T_{4n}}$ is

$$2n + \frac{2n(2^{k_1+1}-1)}{2^{k_1+1}} = 2n(2 - \frac{1}{2^{k_1+1}}) = 4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}.$$

□

Theorem 3.13. $\chi(\Pi_{T_{4n}}) = \omega(\Pi_{T_{4n}})$.

Proof. Let n be odd. Then the largest clique contains $\omega(\Pi_{T_{4n}}) = 3n$ vertices and remaining $n-1$ vertices have odd order. We can color them with $3n$ colors. If n is even, then the largest clique contains $\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$ vertices where $n = 2^{k_1} p_2^{k_2} \dots p_r^{k_r}$ and $p_2^{k_2} p_3^{k_3} \dots p_r^{k_r} < n$, so $4n - p_2^{k_2} p_3^{k_3} \dots p_r^{k_r} > 3n$, and remaining $n-2$ vertices have odd order. We can color the graph with $\omega(\Pi_{T_{4n}})$ colors and hence $\chi(\Pi_{T_{4n}}) = \omega(\Pi_{T_{4n}})$.

□

Theorem 3.14. In graph $\Pi_{T_{4n}}$

(i) $\operatorname{diam}(\Pi_{T_{4n}}) \leqslant 2$.

(ii) $\Pi_{T_{4n}}$ is a line graph if and only if $n = p^k$ or $n = 2^k$ or $n = 2^k p^k$, where p is prime.

(iii) if $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, then the number of end-vertices is

$$\begin{cases} (p_1^{k_1}-1)(p_2^{k_2}-1)\cdots(p_r^{k_r}-1), & n \text{ is odd}, \\ (2^{k_1+1}-1)(p_2^{k_2}-1)\cdots(p_r^{k_r}-1), & n \text{ is even}. \end{cases}$$

Proof. (i) Since $\operatorname{diam}(\Pi_{\mathbb{Z}_{2n}}) \leqslant 2$, $\operatorname{diam}(\Pi_{T_{4n}}) \leqslant 2$.
(ii) The edge set of $\Pi_{T_{4n}}$ can be partitioned into a set of clique with the property that any vertex lies in at most two clique if and only if n be $n = p^k$ or $n = 2^k$ or $n = 2^k p^k$ by [2, Theorem 1.7.2] the proof is complete.
(iii) The number of $\Pi_{T_{4n}}$ end-vertices is equal to the number of $\Pi_{\mathbb{Z}_{2n}}$ end-vertices that is the number of vertices that $p_1 p_2 \cdots p_r$ divides their order and is $(p_1^{k_1} - 1)(p_2^{k_2}-1)\dots(p_r^{k_r}-1)$ when n is odd and is $(2^{k_1+1}-1)(p_2^{k_2}-1)\dots(p_r^{k_r}-1)$ when n is even. □

Theorem 3.15. Let $SD_{8n} = \langle a\ ,\ b \mid a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle$. Then

(i) $\Pi_{SD_{8n}}$ is connected graph.

(ii) $\Pi_{SD_{8n}}$ is Eulerian if and only if $n = 2^k$ $(0 \leqslant k \in \mathbb{Z})$.

(iii) $\Pi_{V_{8n}}$ has an Eulerian spanning subgraph.

Proof. Since $\Pi_{\mathbb{Z}_{4n}}$ is a subgraph of $\Pi_{SD_{8n}}$ and the order of vertices of $\Pi_{SD_{8n}-\mathbb{Z}_{4n}}$ is even, where $2n$ vertices are of order 2 and $2n$ vertices of order 4 that join to vertices of even order, then this graph has specifications of $\Pi_{\mathbb{Z}_{4n}}$. □

Theorem 3.16. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$.

(i) If n is odd, then $\omega(\Pi_{SD_{8n}}) = 7n$.

(ii) If n is even i.e. $p_1 = 2$, then $\omega(\Pi_{SD_{8n}}) = 8n - p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}$.

Proof. The number of elements of odd order in $\Pi_{SD_{8n}}$ is $p_2^{k_2} \cdots p_r^{k_r} - 1$. Then clique number of $\Pi_{SD_{8n}}$ is the number of vertices that order is even that is

$$8n - 1 - p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r} - 1,$$

then

$$\omega(\Pi_{SD_{8n}}) = 8n - p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}.$$

□

Corollary 3.17. $\chi(\Pi_{SD_{8n}}) = \omega(\Pi_{SD_{8n}})$.

Proof. By previous theorem $7n \leq \omega(\Pi_{SD_{8n}})$, then vertices that remaining of the largest clique are coloring with $7n$ color. Thus $\chi(\Pi_{SD_{8n}}) = \omega(\Pi_{SD_{8n}})$. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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