Mathematics Interdisciplinary Research **4** (2019) 11 – 19

An Overview of Mathematical Contributions of Ghiyāth al-Dīn Jamshīd Al-Kāshī [Kāshānī]

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Abstract

In this paper, we study Ghiyāth al-Dīn Jamshīd al-Kāshī's (1380-1429 A.D.) main mathematical achievements. We discuss his *al- Risāla al-muhītīyya* ("The Treatise on the Circumference"), *Risāla al-watar wa'l-jaib* ("The Treatise on the Chord and Sine"), and *Miftāh al-hisāb* ("The Key of Arithmetic"). In particular, we look at al-Kāshī's fundamental theorem, his calculation of π, and his calculation of $\sin 1^\circ$.

Keywords: Ghiyāth al-Dīn Jamshīd al-Kāshī, Jamshīd Kāshānī, *al-Risāla al-muhītīyya, Miftāh al-hisāb, Risāla al-watar wa'l-jaib,* al-Kāshī's fundamental theorem, *Euclid's Elements,* Lambert identity, Ptolemy's theorem, Pythagorean theorem, Pascal triangle, Ruffini-Horner's method.

2010 Mathematics Subject Classification: 01A30, 51-03.

How to cite this article
M. K. Azarian, An Overview of Mathematical Contributions of Ghiyāth al-Dīn Jamshīd Al-Kāshī [Kāshānī], *Math. Interdisc. Res.* **4** (2019) 11-19.

1. Preliminaries

Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (also, known as Jamshīd Kāshānī) was one of the most renowned mathematicians and astronomers in Persian history, and one of the most fascinating medieval mathematicians in the world. Throughout this paper we will refer to him just as Kāshānī. The exact date of his birth is not known, but it was sometime during the last quarter of the fourteenth century (some sources say he was born around 1380) in the Tamerlane (Timur) empire in the city of Kāshān. Kāshānī died on the morning of Wednesday, June 22,

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Academic Editor: Ali Reza Ashrafi
Received 21 October 2017, Accepted 20 January 2018
DOI: 10.22052/mir.2019.167225.1110

1429 (Ramadān 19, 832 A.H.L.) outside of Samarqand at the observatory he had helped to build in Uzbekistān. Kāshānī was known for his extraordinary ability to perform very difficult mental computations. Among the many nicknames given to Kāshānī one could name: "the second Ptolemy", "the pearl of the glory of his age", "the king of engineers", "the reckoner", and "our master of the world". Although Kāshānī's interest was in mathematics and astronomy and he is known as a mathematician and astronomer, he was actually a physician. To support himself financially, like many other scientists of the Middle Ages, he dedicated most of his known scientific treatises to monarchs or other influential rulers of his time in order to get compensation. Although Kāshānī was Persian, in order to increase his readership he wrote most of his mathematical work in Arabic. So, originally, all of Kāshānī's work were written long-hand mostly by himself either in Arabic or Persian. However, now most of his well-known work has been translated or summarized into English, French, German, Russian, and other languages. For a more detailed discussion of Kāshānī's life and his mathematical achievements we refer the reader to the author's paper [6].

2. Al-Risāla al-muhītīyya ("The Treatise on the Circumference")

One of the most significant mathematical achievements of Kāshānī is *al-Risāla al-muhītīyya* ("The Treatise on the Circumference"), which is also known as *Risāla-i muhītīyya,* or simply *Muhītīyya.* This 58-page long treatise, written longhand in Arabic by Kāshānī himself, was completed in Samarqand in July 1424 (Sha'bān 827 A.H.L.). The original manuscript of *al-Risāla al-muhītīyya* was donated by Nāder Shāh (King Nāder) in 1145 A.H.S. to the library of the Āstāne Qudse Razawī, Mashhad, Iran. It consists of an introduction, ten sections, and conclusion.

The calculations of π by Archimedes (c. 287-212 B.C.), Abu'l-Wafā' al-Būzjānī (940-998 A.D.), and Abū Rayhān al-Bīrūnī (973-1048 A.D.) motivated Kāshānī to write this treatise on improving the estimation of π by these three renowned mathematicians. Another motivation for Kāshānī to write this treatise was the fact that there was a real demand for more precise trigonometric tables in connection with advanced research in astronomy. The main tool for the calculation of π is what we will call Kāshānī's fundamental theorem, as follows.

Theorem 2.1. *(Al-Kāshī's [Kāshānī's] Fundamental Theorem) If on a semicircle with the diameter* $\overline{AB} = 2r$ *and the center* E *we consider an arbitrary arc* AG*, and we call the middle of the arc* GB *the point* D*, and we draw the chord* $\overline{AD}$*, then,* $r(2r + \overline{AG}) = \overline{AD}^2$.

Kāshānī's proof of his fundamental theorem is based on four different propositions from *Euclid's Elements* [4]. However, Kāshānī did not burden the reader with mentioning Euclid or the *Elements* in his proof. This is because *Euclid's*

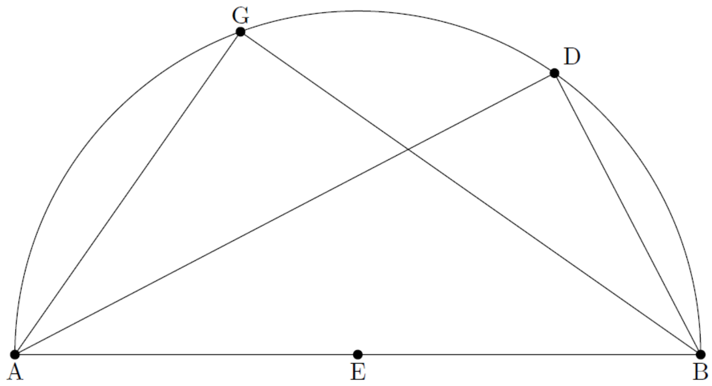

Figure 1: A unit circle.

Elements had been well-known and widely used for geometrical proofs at that time. In an alternative proof of Kāshānī's fundamental theorem we used modern notations, and applied Pythagorean theorem to right triangles [4].

He continued his discussion and he deduced that in a unit circle (Figure 1) if the arc AG is equal to 2θ radians, then, from the right triangle AGB he would get $\overline{AG} = 2\sin\theta$. Similarly, from the right triangle ADB he obtained

$$\overline{AD} = 2\sin\frac{1}{2}(2\theta + \frac{\pi - 2\theta}{2}) = 2\sin(\frac{\pi}{4} + \frac{\theta}{2}).$$

Next, he substituted these values of $\overline{AG}$ and $\overline{AD}$ as well as $r = 1$ in his fundamental theorem to obtain

$$2 + 2\sin\theta = 4\sin^2(\frac{\pi}{4} + \frac{\theta}{2}).$$

That is,

$$\sin(\frac{\pi}{4} + \frac{\theta}{2}) = \sqrt{\frac{1+\sin\theta}{2}}, \ \text{(where } \theta < \frac{\pi}{2}).$$

This is the trigonometric form of Kāshānī's fundamental theorem, which is now known as the Lambert identity. It is intriguing to note that this identity appeared in Johann Heinrich Lambert's work in 1770, while Kāshānī proved it before 1424 (827 A.H.L.), some 346 years earlier than Lambert. It takes a brilliant mind to come up with such a simple concept, and yet with remarkable applications.

Kāshānī's fundamental theorem gave him a distinct advantage over his predecessors by having found the most accurate value of π at that time. Kāshānī applied his fundamental theorem to find the lengths of the sides of inscribed and circumscribed regular polygons each with 3×2^n sides ($n \geq 1$) in a given circle.

Then he determined the number of sides of the inscribed regular polygon in a circle, whose radius is six hundred thousand times the radius of the Earth, in a such a way that the difference between the circumference of the circle and the perimeter of the inscribed regular polygon in this circle, will become less than the width of a horse's hair. Kāshānī used his fundamental theorem to calculate the value of π, correct to 17 (rounded) decimal places, using inscribed and circumscribed polygons, each with $3(2^{28}) = 805,306,368$ sides.

The best approximation of π correct to six decimal places obtained before Kāshānī was by the Chinese scientist Tsu Ch'ung-chih, around 480 A.D. Kāshānī's approximation of π seems ordinary to us, as we now know over 22 trillion digits of π (actually, 22,459,157,718,361 decimal digits) as of November 11, 2016, by Peter Trueb [9]. However, Kāshānī's approximation of π, in 1424, was a remarkable achievement for that time, and it far surpassed all approximations of π by all previous mathematicians throughout the world, including Archimedes and Ptolemy. (Archimedes' approximation in 250 B.C. was 3.14, and Ptolemy's result in circa 150 A.D. was 3.14166). It took European scientists 172 years to improve on Kāshānī's approximation of π. François Viète, 155 years later in 1579, obtained a value for π, correct only to 9 decimal places. Adrian Romain, in 1593, calculated the value of π correct to 15 decimal places, and finally, Ludolf van Ceulen obtained the value of π correct to 35 decimal places in 1610. It is interesting to note that Issac Newton used the power series expansion of arctan x to calculate the value of π up to 16 decimal digits of accuracy in 1665 [7]. For an English summary of *Risāla-i muhītīyya,* proof of Kāshānī's fundamental theorem as well as an Eglish translation of the Kāshānī's fundamental theorem, and an English translation of the introduction of *Risāla-i muhītīyya* we refer the reader to the author's papers [2], [3] and [4], respectively.

3. Risāla Al-Watar Wa'l-Jaib "The Treatise on the Chord and Sine"

Risāla al-watar wa'l jaib is one of the three most significant mathematical achievements of Kāshānī. This treatise deals chiefly with the calculation of sine and chord of one-third of an angle with known sine and chord. It is believed that Kāshānī completed this treatise sometime between 1424 and 1427 A.D.. Sadly, the original manuscript is lost. However, since the core of this treatise was about the calculation of $\sin 1°$, several of Kāshānī's colleagues and successors wrote commentaries in Arabic as well as Persian regarding the determination of $\sin 1°$ motivated by Kāshānī's iteration method. Our discussion will be based on *Sharh-i Zaīj-i Ulugh Beg* ("Commentaries on Ulugh Beg's Astronomical Tables"), written in Persian by the Persian astronomer and mathematician Nizām al-Dīn 'Abd al-'Alī ibn Muhammad ibn Husain al-Bīrjandī (also known as 'Abd al-'Alī Bīrjandī; died 1528 A.D.). There are two parts in the calculation of $\sin 1°$. First, Kāshānī applied Ptolemy's theorem to an inscribed quadrilateral to obtain his famous cubic equation, and then he invented an ingenious and quickly converging iteration algorithm to calculate $\sin 1^\circ$ to 16 correct decimal digits (nine correct sexagesimal places) as a root of his cubic equation. It is remarkable that Kāshānī used both geometry and algebra to approximate $\sin 1^\circ$, to any desired accuracy! Not only was this the most fascinating and creative method of approximation, but it was the first approximation method in the history of mathematics, and the most significant achievement in medieval algebra. This was also the most accurate approximation of $\sin 1^\circ$ at that time. The best previous approximations, correct to four sexagesimal places, were obtained in the tenth century by two other Muslim scientists Abu'l-Wafā' al-Būzjānī [Būzjānī] (940-998), and Abu'l-Hasan ibn Yunus (950-1009). Kāshānī's approximation of $\sin 1^\circ$ was not surpassed until the 16th century by Taqī al-Dīn Muhammad al-Asadī (1526–1585).

Kāshānī's calculation of $\sin 1^\circ$ was in sexagesimal system. However, since current readers are more comfortable with the decimal system, we present the calculation of $\sin 1^\circ$ in decimal system.

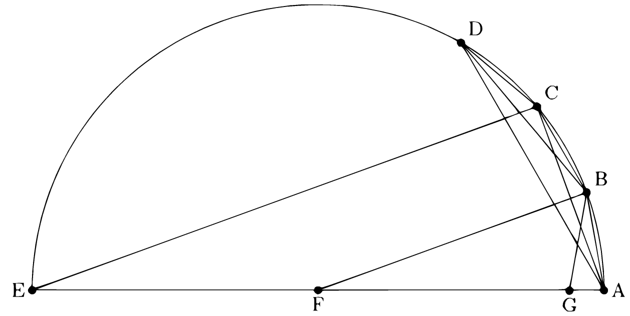

Figure 2: A semicircle.

Kāshānī let A, B, C, D be points on a semicircle with center F and radius r (Figure 2) such that $AB = BC = CD$. By Ptolemy's theorem,

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Since $AB = CD = BC$ and $BD = AC$, he obtains

$$AB^2 + BC \cdot AD = AC^2. \tag{1}$$

Then Kāshānī determines the point G on the diameter AE in a such a way that $EC = EG$. He observes that from the similar isosceles triangles ABG and ABF, he has $\dfrac{AB}{AG} = \dfrac{AF}{AB}$. Hence $AG = \dfrac{AB^2}{r}$, and thus

$$EG = 2r - AG = 2r - \frac{AB^2}{r}.$$

From the right triangle AEC, he gets

$$AC^2 = AE^2 - EC^2 = 4r^2 - EG^2, \tag{2}$$

and from (2), he deduces that

$$AC^2 = 4r^2 - (2r - \frac{AB^2}{r})^2 = 4AB^2 - \frac{AB^4}{r^2}. \tag{3}$$

Also, from (1) and (3) he obtains

$$AB^2 + AB \cdot AD = 4AB^2 - \frac{AB^4}{r^2},$$

and consequently he achieves

$$AD = 3AB - \frac{AB^3}{r^2}. \tag{4}$$

If $AB = crd\ 2\alpha$, then clearly $AD = crd\ 6\alpha$. From (4), Kāshānī deduces that

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha. \tag{5}$$

Finally, he let $\alpha = 1°$, $x = \sin 1°$, and uses (5) to obtain Kāshānī's famous cubic equation

$$x = \frac{4}{3}x^3 + \frac{1}{3}\sin 3°. \tag{6}$$

To find an approximation for $\sin 1°$ as a root of (6) Kāshānī proceeds as follows: Since $\sin 1°$ is close to $\frac{1}{3}\sin 3° = 0.0174453\cdots$, he let his initial estimate be $x_0 = 0.01$, and subsequent decimal estimations of the root to be of the form $x = 0.01d_1d_2d_3d_4\ldots$, where $0 \leq d_i \leq 9$. Kāshānī puts this initial estimate as well as the known value of $\frac{1}{3}\sin 3°$ in (6) to get

$$0.01d_1d_2d_3d_4\cdots = \frac{4}{3}(0.01d_1d_2d_3d_4\cdots)^3 + \frac{1}{3}\sin 3°, \tag{7}$$

and then he subtracts 0.01 from both sides to obtain

$$0.00d_1d_2d_3d_4\cdots = \frac{4}{3}(0.01d_1d_2d_3d_4\cdots)^3 + 0.0074453\cdots .$$

Now, the first nonzero digit in the above cubic term is in the sixth decimal place, and since the above equality must hold true digit by digit, he gets $d_1 = 7$, and hence, $x_1 = 0.017$. Next, he substitutes this value of d_1 in (7) and subtracts 0.017 from both sides to get

$$0.000d_2d_3d_4\cdots = \frac{4}{3}(0.017d_2d_3d_4\cdots)^3 + 0.0004453\cdots\ .$$

He applies the same argument as above and gets $d_2 = 4$, and thus $x_2 = 0.0174$. He continues his calculations in a similar fashion to get $d_3 = 5$, $d_4 = 2$, ..., and $d_{20} = 2$, and consequently, Kāshānī achieves the approximation 0.0174524064372835103712 for $\sin 1^\circ$, where the first 16 digits are correct. For an English summary of *Risāla al-watar wa'l-jaib* we refer the reader to the author's paper [1].

4. Miftāh al-hisāb ("The Key of Arithmetic")

Miftāh al-hisāb, which is also known as "The Calculators' Key", is Kāshānī's best known work. It contains some of Kāshānī's original findings, as well as his improvements on earlier Muslim mathematicians' works. Written primarily as a textbook, *Miftāh al-hisāb* was used for more than five centuries, not only as a textbook, but also as an encyclopedia; it served many generations of students, accountants, astronomers, astrologists, architects, engineers, land surveyors, merchants, other professionals, and even fortune tellers. It took Kāshānī more than 7 years to complete *Miftāh al-hisāb*, which he did on March 2, 1427 (Jamadi al-awwal 3, 830 A.H.L.). To support himself financially, he dedicated this unique mathematical masterpiece to Sultān Ulugh Beg, grandson of Timur and the ruler of Samarqand.

Miftāh al-hisāb is divided into five books preceded by an introduction. Each book consisting of several chapters. The titles of these five books, as well as the number of chapters in each book, respectively are: "On the Arithmetic of Integers" (six chapters), "On the Arithmetic of Fractions" (twelve chapters), "On the Computation of Astronomers" (six chapters), "On the Measurement of Plane Figures and Bodies" (nine chapters preceded by an introduction), and "On the Solutions of Problems by Means of Algebra" (four chapters).

In the first book of *Miftāh al-hisāb*, Kāshānī describes in detail a general method of extracting any root of an integer in the decimal system as well as the sexagesimal system. Later in the nineteenth century this method was rediscovered by European mathematicians, and now it is known as Ruffini-Horner's method. Many Persian mathematicians including Abū Rayhān Bīrūnī, Omar Khayyām, and Nasīr al-Dīn Tūsī worked on the extraction of roots of integers with some success. However, Kāshānī claimed that the general method of extracting roots of integers was his own discovery.

Kāshānī gave a general rule identical to that of Newton for the expansion of $(a + b)^n$, and he made use of a triangular table to find the coefficients of this expansion. Nowadays, this triangle is called the Pascal triangle. Qurbani in his book

[8], claimed that the binomial expansion as well as the Pascal triangle both were discovered by the Persian mathematician, astronomer, scientist, and poet Omar Khayyām about six centuries before the era of Newton and Pascal. However, we know that both the binomial expansion and the Pascal triangle were known before him by Abū Bakr Al-Karajī [Karajī] (953-1029 A.D.) in Persia, and by various mathematicians in China (and even possibly in India). Kāshānī acknowledged that the expansion of $(a+b)^n$ and the triangular table that he used are due to his predecessors, by which he probably meant Omar Khayyām and Nasīr al-Dīn Tūsī (1201-1274 A.D.). Also, Kāshānī presented several trigonometric identities including those identities which are known as the laws of sines and cosines. Moreover, Kāshānī described the "casting out 9" method for checking the accuracy of multiplication as well as division and root extraction. For an English summary of *Miftāh al-hisāb* we refer the reader to the author's paper [5].

5. Other Mathematical Contributions of Kāshānī

Other mathematical discoveries of Kāshānī are in conjunction with his many works on astronomy. The best known of Kāshānī's work in astronomy are:

(i) *Sullam al-samā'* ("The Ladder of the Sky" or "The Stairway of Heaven"), completed in Arabic on March 1, 1407 (Ramadān 21, 809 A.H.L.).

(ii) *Mukhtasar dar 'ilm-i hay'at* ("Compendium of Science and Astronomy"), completed in Persian in 1410-1411 (813 A.H.L.). This is also known as *Risāla dar hay'at* ("Treatise on Astronomy").

(iii) *Risāla dar sharh-i ālāt-i rasad* ("Treatise on the Explanation of Observational Instruments"), completed in Persian in January 1416 (Dhū al-Qa'da, 818 A.H.L.).

(iv) *Nuzha al-hadāiq* ("The Garden Excursion"), completed in Arabic on February 10, 1416 (Dhū al-Hajja 10, 818 A.H.L.).

(v) *Zīj-i Khāqānī* was completed in 1413-1414 (816 A.H.L.). This is a revised version of *Zīj-i Ilkhānī* of Nasīr al-Dīn al-Tūsī.

Acknowledgements. This paper is the content of the author's keynote address at the International Conference on Architecture and Mathematics at University of Kāshān, Kāshān, Iran. I would like to thank the organizers for the invitation, warm Kāshān hospitality, and their support of my travel.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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