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Golden Ratio: The Mathematics of Beauty

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Abstract

Historically, mathematics and architecture have been associated with one another. Ratios are good example of this interconnection. The origin of ratios can be found in nature, which makes the nature so attractive. As an example, consider the architecture inspired by flowers which seems so harmonic to us. In the same way, the architectural plan of many well-known historical buildings such as mosques and bridges shows a rhythmic balance which according to most experts the reason lies in using the ratios. The golden ratio has been used to analyze the proportions of natural objects as well as building's harmony. In this paper, after recalling the (mathematical) definition of the golden ratio, its ability to describe the harmony in the nature is discussed. When teaching mathematics in the schools, one may refer to this interconnection to encourage students to feel better with mathematics and deepen their understanding of proportion. At the end, the golden ratio decimals as well as its binary digits has been statistically examined to confirm their behavior as a random number generator.

Keywords: Fibonacci sequence, golden ratio, golden rectangular, random number generator.

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1. Introduction

Proportions in geometry, architecture, music and art express the harmonious relationships between the whole and its parts, and within a whole system [27]. The

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selection and use of systems of proportions has always been a vital issue for artists and architects, see [7] and [22]. Since that beautiful harmonious sounds depended on ratios, the architects have been considered the ratios when designing a building, see [4] and [12]. However, architects have fine senses of symmetry of visual forms without considering a precise definition of this concept from the mathematical point of view [18]. Muslim artists have indeed discovered all forms of symmetry that could be represented on a two-dimensional surface. Their striking symmetry suggests balance and serenity, [15] and [32].

One of the coolest facets of architecture is the ability to have buildings be so different so varied in terms of size, shape, and style and yet so similar at their core. One way to achieve this is to keep the proportion between the sizes of elements constant. The so-called golden ratio has been applied for centuries to assure a building's harmony [14]. In the following sections, first the golden ratio is defined mathematically and introduces later from geometrical point of view. Its appearance in nature and architecture are reviewd very briefly. At the end, it is shown that the digits of the golden ration behaves like a random sequence.

2. The History of the Golden Ratio

About 300 B.C., Euclid of Alexandria, the most prominent mathematician of antiquity, gathered and arranged 465 propositions into thirteen books, entitled The Elements [31], denote by $[AB]$ and AB the closed line segment with endpoints A and B and its length, respectively. In the Book VI, he defines that the segment $[AB]$ is divided in extreme and mean ratio by a point $C \in [AB]$, if $AC < CB$ and $\frac{CB}{AC} = \frac{AB}{CB}$. While the proportion $\frac{CB}{AC}$ known as the golden ratio has always existed in mathematics, it is unknown exactly when it was first discovered and applied by mankind. It is reasonable to assume that it has perhaps been discovered and rediscovered throughout history, which explains why it goes under several names, such as golden section, golden mean, golden number, divine proportion, divine section and golden proportion. Its beautiful properties has won the interest of many authors, to mentioned [21] and [28] among others. Many key architects in history, such as, Le Corbusier [3], Pacioli, and Leonardo Da Vinci have used the golden ratio in their works explicitly. Fra Luca Pacioli, an Italian mathematician and painter entered some mathematical basis to painting, published the book De Divina Proporzione and Leonardo da Vinci made the illustrations of his book. In addition, many recent publications discussed the appearance of the golden ratio in designing the historical buildings in Iran, see [12] and [23].

3. The Golden Ratio in Mathematics

In mathematics two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Expressed algebraically, for quantities a and b with $a > b > 0$,

$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi,$$

or

$$\varphi^2 - \varphi - 1 = 0, \tag{1}$$

where, the Greek letter phi (φ or ϕ) represents the golden ratio. where, the Greek letter phi (φ or ϕ) represents the golden ratio, which was invented at the beginning of the twentieth century by the American mathematician Mark Barr and derived from the first three Greek letters in the name of the well-known Greek sculptor Phidias (450 B.C.), who was long supposed to have used the golden ratio in his sculptural works and his creations were considered to be the standard of beauty and harmonious construction based on the human body [10].

It is an irrational number (meaning we cannot write it as a simple fraction), with a value of:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887.$$

Another interesting relationship involving the golden ratio may be obtained directly from:

$$\varphi = \sqrt{1+\varphi},$$

successive substituting the left hand side for φ on the right hand side gives:

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}.$$

Similarly, replacing $x = \frac{1}{\varphi}$ in the equation (1) yields the quadratic equation $x^2 + x - 1 = 0$, which its positive root is $\frac{1}{\varphi}$, i.e.,

$$\frac{1}{\varphi} = \sqrt{1-\frac{1}{\varphi}} = \sqrt{1-\sqrt{1-\sqrt{1-\cdots}}}.$$

Again, another relationship may be obtained, by using $\varphi = 1+\frac{1}{\varphi} = 1+\frac{1}{1+\frac{1}{\varphi}} = \cdots$ repeatedly, we find that golden ratio is related with the following continued fraction

$$\varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}.$$

Another combined nested square root and continued fraction for φ is derived by successive substituting the formula $\varphi = \sqrt{1+\varphi} = \sqrt{2+\frac{1}{\varphi}}$ which yields:

$$\varphi = \sqrt{2 + \frac{1}{\sqrt{2 + \frac{1}{\sqrt{2+\cdots}}}}}.$$

Furthermore, any power of φ is equal to the sum of the two immediately preceding powers:

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2},$$

thus, from computation cost point of view, having the first two values $\varphi^0 = 1$ and $\varphi^1 \approx 1.61803398$, for calculating φ^n approximately, we can repeatedly apply above recursion relation doing only a single subtraction, rather than a slower multiplication by φ, at each step.

We now use the relationship $\varphi^2 = \varphi + 1$ to inspect successive powers of φ by writing them down to their component parts in the following way.

$$\begin{aligned}
\varphi^3 &= \varphi\varphi^2 = \varphi(\varphi+1) = \varphi^2 + \varphi = (\varphi+1) + \varphi = 2\varphi + 1,\\
\varphi^4 &= \varphi^2\varphi^2 = (\varphi+1)(\varphi+1) = \varphi^2 + 2\varphi + 1 = (\varphi+1) + 2\varphi + 1 = 3\varphi + 2,\\
\varphi^5 &= \varphi^3\varphi^2 = (2\varphi+1)(\varphi+1) = 2\varphi^2 + 3\varphi + 1 = 2(\varphi+1) + 3\varphi + 1 = 5\varphi + 3,\\
\varphi^6 &= \varphi^3\varphi^3 = (2\varphi+1)(2\varphi+1) = 4\varphi^2 + 4\varphi + 1 = 4(\varphi+1) + 4\varphi + 1 = 8\varphi + 5,\\
\varphi^7 &= \varphi^4\varphi^3 = (3\varphi+2)(2\varphi+1) = 6\varphi^2 + 7\varphi + 2 = 6(\varphi+1) + 7\varphi + 1 = 13\varphi + 8,
\end{aligned}$$

and so on. By this point one should be able to see a pattern as further powers of φ is taken. Actually, the end result of each power of φ is equal to a multiple of φ plus a constant, in a linear form $\varphi^n = a_n\varphi + a_{n-1}, n = 2, 3, \ldots$, where a_n and a_{n-1} are both special integers, the so-called Fibonacci numbers, [25]. Applying this pattern enables us to write down the further powers of φ simply, for example $\varphi^8 = 21\varphi + 13$ and $\varphi^9 = 34\varphi + 21$.

A similar recursive equation must also hold for the other root of $\varphi^2 = (\varphi + 1)$, namely $\chi = 1 - \varphi = \frac{1-\sqrt{5}}{2}$, that is $\chi^n = a_n\chi + a_{n-1}, n = 2, 3, \ldots$. Solving these two equations for a_n yields:

$$a_n = \frac{\varphi^n - \chi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^n - \big(\frac{1-\sqrt{5}}{2}\big)^n\Big), \quad n = 0, 1, 2, 3, \ldots.$$

This relation is mainly of theoretical interest [19] and referred to as the Binet's formula, after the French mathematician Jacques Phillipe Marie Binet (1786-1856). The formula allows one to find the value of any number in the Fibonacci sequence $\{a_n\}$, see section 3.2.

3.1 The Kepler Triangle

A Kepler triangle is a right triangle with edge lengths in geometric progression. If we make one side of length 1 and the other side of length a, the hypotenuse must be of length a^2 where a is the geometric ratio. The Pythagorean theorem implies $a^2 = \varphi$ which yields $a = \sqrt{\varphi}$, i.e. the ratio of the edges of a Kepler triangle is linked to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Therefore any Kepler triangle must be simlilar to one of edges $1\colon \sqrt{\varphi}\colon \varphi$, or approximately $1\colon 1.272\colon 1.618$. The smaller angle of a Kepler triangle, say α, is $\alpha = \arctan(\frac{1}{\sqrt{\varphi}})$ radians, for which the interesting relation $\tan\alpha = \cos\alpha = \frac{1}{\sqrt{\varphi}}$ holds.

A triangles with such ratios are named after the German mathematician and astronomer Johannes Kepler (1571-1630), who first demonstrated that this triangle is characterized by a ratio between short side and hypotenuse equal to the golden ratio. Kepler triangles combine two key mathematical concept, the Pythagorean theorem and the golden ratio, that fascinated Kepler deeply, as he expressed in this quotation by Kepler [8]:

"Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into mean and extreme ratio. The first we may compare to a mass of gold, the second we may call a precious jewel". Some sources claim that a triangle with dimensions closely approximating a Kepler triangle can be recognized in the Great Pyramid of Giza [9].

3.2 More on the Fibonacci Sequence

The Fibonacci sequence, named after Leonardo Fibonacci an Italian born in 1175 AD, also a plot element in "The Da Vinci Code, provides yet another way to derive φ mathematically. The series is quite simple. Start with 0 and add 1 to get 1. Then repeat the process of adding each two numbers in the series to determine the next one: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and so on. This pattern is also found in the diagonals of Pascals Triangle. The relationship to the golden ratio or φ is found by dividing each number by the one before it. The further you go in the series, the closer the result gets to φ, [29]. For example: $\frac{1}{1} = 1$, $\frac{2}{2} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} = 1.666$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, if we go further into the series, it will find that $\frac{233}{144} = 1.61805$. The golden number is not just the limit of the sequence of

$$\left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \ldots\right),$$

but the sequence of convergent is a sequence of best approximations to φ by rational numbers, see ([11, Chapter 5]).

The relation between the Golden ratio formula and Fibonacci sequence is known [6]. A Fibonacci sequence $\{a_n\}$ is defined by second-order linear difference equation $a_{n+2} - a_{n+1} - a_n = 0$ with $a_0 = 0$, $a_1 = 1$, then we have the following lemma [16].

Lemma 3.1. *The following relations hold:*

$$\begin{aligned} \varphi^n &= a_n\varphi + a_{n-1}, \\ (\varphi-1)^n &= a_{-n}\varphi + a_{-n-1}. \end{aligned} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

The following shows also that the Fibonacci sequence has the analytic form.

Lemma 3.2. *The following relation holds:*

$$a_n = \frac{1}{2\varphi-1}\{\varphi^n - (1-\varphi)^n\}, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

Lemma 3.3. *The following relations hold:*

(i) $\frac{\varphi}{1} = \frac{1+\varphi}{\varphi} = \frac{1+2\varphi}{1+\varphi} = \frac{2+3\varphi}{1+2\varphi} = \frac{3+5\varphi}{2+3\varphi} = \cdots = \frac{a_n+a_{n+1}\varphi}{a_{n-1}+a_n\varphi} \approx 1.61803,$

(ii) $\frac{a_n+a_{n+1}\varphi}{a_{n-1}+a_n\varphi} = \cdots = \frac{2\varphi-3}{5-3\varphi} = \frac{2-\varphi}{2\varphi-3} = \frac{\varphi-1}{2\varphi} = \frac{1}{\varphi-1} = \frac{\varphi}{1},$

(iii) $\frac{0.236}{0.146} \approx \frac{0.382}{0.236} \approx \frac{0.618}{0.382} \approx \frac{1}{0.618} \approx \frac{1.618}{1} \approx \frac{2618}{1618} \approx \frac{4236}{2618} \approx \cdots \approx 1.618.$

Plants, and especially their flowers, have captivated the imaginations of artists and poets over the centuries. Plants show all the features of geometry from the spiral of cactus spines, the fractal pattern of the branches and roots of a tree, or the symmetry of the leaf arrangements on the pea plants. The leaf pattern and number of petals on most plants include Fibonacci numbers. For example, white Calla Lily having one petal, Euphorbia having two petals, Trillium with three petals, Hibiscus having five petals, Bloodroot with eight petals, Black Eyed Susan having thirteen petals, Shasta Daisy having 21 petals and Daisy with 34 petals, [1].

3.3 The Golden Rectangle

The golden section (or proportion) is the basis of the golden rectangle, whose sides are in golden proportion $1\colon \varphi$ to each other. The golden rectangle is considered as one of the shape for representing φ in two dimensions and turned to be the most visually pleasing of all rectangles. The followings successive rectangles approach toward the golden rectangle in the limit. Starting from two 1×1 square one above another, one first adjoins a square of side length 2 to their right. Then one adjoins a square of side length 2 below and so on following a clockwise movement. The ratios of width over height for the successive rectangles run through quotients of the form $\frac{a_{n+1}}{a_n}, n = 0, 1, 2, \ldots$, where $\{a_n\}$ is a Fibonacci sequence. Indeed, the successive rectangles are not exact golden rectangles but they approach toward the golden rectangle.

A distinctive feature of rectangle is that when a square section is removed, the remainder is another golden rectangle; that is, with the same aspect ratio as the first. Square removal can be repeated infinitely, in which case corresponding corners of the squares form an infinite sequence of points on the golden spiral, the unique logarithmic spiral with this property. Applications appeared in all kinds of design, art, architecture, advertising, packaging, and engineering; and can therefore be found readily in everyday objects.

Golden rectangles can be found in the shape of playing cards, windows, book covers, file cards, ancient buildings, and modern skyscrapers. Many artists have incorporated the golden rectangle in to their works because of its aesthetic appeal. It is believed by some researchers that classical Greek sculptures of the human body were proportioned so that the ratio of the total height to the height of the navel was the golden ratio.

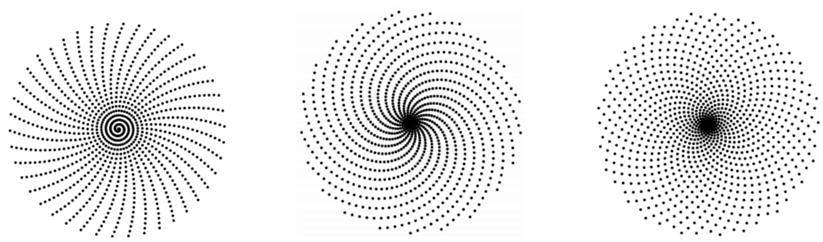
4. The Golden Ratio in the Nature

The golden ratio manifests throughout living things such as plants, animals and throughout the human body and considered as an important part of human beauty [24], [30]. For example, the golden the relationship lies in the balance between a person's height and the height at the navel, in the human face, in the ratio of the length of the arm to section formed by the forearm and hand [5].

Geometric proportions are the fundamental part of the nature which act as the key part of the observed order in the structure of beautiful patterns. Consider, for example, the study of pattern observed in leaf-arrangement, on which the leaves spread all around the stem, so that new leaves don't block sun from older leaves, or so that the maximum amount of rain absorbed down to the roots. This regular arrangement is an important aspect of plant form, known as spiral phyllotaxis and is common in arrangement of seeds, scales on a cone axis, sunflower heads, etc. [17] and [26]. Since nature has many different methods of survival, we do not see this kind of spiral growth in all plants. A spiral is a curve starts from the origin and moves away from this point as it turn around it. A simple way to rearrange circular (x, y) pints having the same distance from the center into a spiral form is to multiply x and y by a factor t which increases for each point and then rotate the point (tx, ty) by angle θ, see Figure 1. To see the difference, three values has been used as rotation parameter for generating the patterns, two of which looks not successful to mimic the sunflowers spirals.

5. The Golden Ratio in Architecture

Looking beautiful is an interesting and challenging problem when designing a good structural system. The relationship of smaller parts to the whole can affect whether such system seems threatening, welcoming or impressive. The interrelation between proportion and good looking has been employed since the beginning



(a) The rotation value each time is $\frac{\pi}{16}(3-\sqrt{5})$.
(b) The rotation value each time is $\frac{\pi}{2}(3-\sqrt{5})$.
(c) The rotation value each time is $\pi(3-\sqrt{5})$.

Figure 1: Simulated spirals in a sunflower with different rotation values, source author own programming.

of history in many works of art and architecture. Geometrical analysis of many Persian historical building has proven that a complete of proportions, in particular the golden ratio, was widely used in Persian architecture and it was the basis of Persian aesthetics. In many Persian building, the plan and elevation were set out in a framework of squares and equilateral triangles, whose intersections gave all the important fixed points, such as the width and height of doors; the width, length and height of galleries. A building was not, therefore a collection of odd components, but a harmonious configuration of proportionally related element, which gave movement to space and satisfied the eye. The golden ratio has been masterly used in the design of the Taj-al-Mulk dome date 1088 A. D., in Jami mosque in Isfahan. The dome has an outer diameter of 11.7 m and height of 20 m from the ground level. Its thickness decreases from the base to the apex [13].

6. Digits of the Golden Ratio as a Random Sequence

Compare for example the following sequences of heads and tails generated by a fair coin, which have the same $(\frac{1}{2})^{20}$ probability of occurring:

TTHHHTHHTTHHHTHTTTHT
TTHHTTHHTTHHTTHHTTHH,

based on the definition of the 'randomness' as being unable to 'predict' future events based on past events, most people would probably agree with the randomness of the second sequence while the first is not.

A sequence of independent random numbers with a specified distribution means that each number was obtained just by chance, have no effect on the observed value of the other sequence numbers, and that each number has a specified probability of falling in any given range of values. A uniform distribution on a finite set of numbers is one in which each possible number is equally probable. In a sequence of (uniform) random digits, each of the ten digits 0 through 9 will occur about $\frac{1}{10}$ of the time.

There are many computer aided algorithm developed to produce random numbers that could be either a binary sequences or an integer sequences. This random number generators are actually deterministic algorithms which produce numbers which we expect to resemble truly random ones in some sense. Since fixing the starting point of these deterministic algorithms make it possible to predict the subsequently generated numbers perfectly, none of these algorithm can produce truly random numbers. This explains why such numbers are called 'pseudo-random numbers'. Random number are used by many practical applications including computer simulations, random sampling, numerical analysis, cryptography and communications industry [20].

Different statistical test are designed to test the null hypothesis (H_0) which states a given pseudo-random number generator produces 'sufficiently' random sequence of numbers for their intended use [2]. These randomness tests are probabilistic and involve two types of errors. If the data is random and (H_0) is rejected, type I error is occurred and if the data is non-random and H_0 is accepted, type II error is occurred.

In the following, the first 100,000 of decimal digits of golden ratio has been examined to answer the question of whether these digits form a sequence of random digits. These digits were obtained using PhiCalculator, an easy-to-download-and-use program that compute the golden ratio up to 1 million decimal digits, developed by Alireza Shafaei, which is available for download through https://alireza-shafaei.software.informer.com.

First of all, Table 1 shows the frequency of the digits 0 through 9 for the first 100,000 decimal digits of the golden ratio, confirming these digits are uniformly distributed.

Table 1: Counts of the first 100,000 decimal digits of golden ratio.

digits	0	1	2	3	4
frequency	9986	9963	9950	10079	10041
digits	5	6	7	8	9
frequency	10016	9975	9988	1008	9994

The result of the chi-square test (χ^2=1.3112, df=9, p-value= 0.9983) applied on the counts obtained in Table 1 do not reject the null hypothesis that each of the ten digits 0 through 9 has been occurred with probability $\frac{1}{10}$.

Then the Durbin-Watson (DW) test, is used for testing the null hypothesis claiming the specific lag k autocorrelation in the sequence of the first 100,000 decimal digits of the golden ratio is zero. Autocorrelation means that the data has correlation with its lagged value. Table 2 shows the results of conducting the Durbin-Watson (DW) tests for the first through fifth-order autocorrelation of this sequence, which shows no significant autocorrelation through fifth order. The same results are obtained for higher orders of k. Therefore, according to the results of both chi-square and DW tests the digits of the first 100,000 decimal digits of the golden ratio are indistinguishable from a random sequence, i.e. each element of the sequence is uncorrelated of each other. We conclude that there is no specific

Table 2: Durbin-Watson test applied on the first 100,000 decimal digits of golden ratio.

lag	Autocorrelation	D-W Statistic	p-value
1	+0.002	1.996	0.476
2	−0.002	2.003	0.578
3	+0.001	1.999	0.692
4	+0.002	1.995	0.408
5	−0.000	2.000	0.776

pattern in the decimal digits of the golden ratio.

In the following it is shown that the binary digits of the golden ratio behaves like a random binary sequence, as well. For this aim, the first 45098 bits excluding the very first of the golden ration are examined. The observed frequencies of the numbers of 0s and 1s in this binary sequence are 22529 and 22569, respectively. While half-and-half frequencies is expected under the randomness hypothesis. The result of the chi-square test (χ^2=0.035, df=1, p-value= 0.8506) confirms that the observed versus expected discrepancy is not significant from statistical point of view. Furthermore, applying the DW tests reveals no autocorrelation of this binary sequence for the first through arbitrary higher order k. The WaldWolfowitz runs test on the first 45098 golden ratio bit sequence (p-value= 0.4513) indicates that the corresponding bit generating process is random, i.e. the rest of the sequence can not be predicted from any previous sequence.

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