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Calculations of Dihedral Groups Using Circular Indexation

Reza Dianat⋆ *and Mojgan Mogharrab*

Abstract

In this work, a regular polygon with n sides is described by a periodic (circular) sequence with period n. Each element of the sequence represents a vertex of the polygon. Each symmetry of the polygon is the rotation of the polygon around the center-point and/or flipping around a symmetry axis. Here each symmetry is considered as a system that takes an input circular sequence and generates a processed circular output sequence. The system can be described by a permutation function. Permutation functions can be written in a simple form using circular indexation. The operation between the symmetries of the polygon is reduced to the composition of permutation functions, which in turn is easily implemented using periodic sequences. It is also shown that each symmetry is effectively a pure rotation or a pure flip. It is also explained how to synthesize each symmetry using two generating symmetries: time-reversal (flipping around a fixed symmetry axis) and unit-delay (rotation around the center-point by $2\pi/n$ radians clockwise).

Keywords: Dihedral group, permutation, periodic (circular) sequences, composition of functions.

2010 Mathematics Subject Classification: $20B30$, $20E22$.

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1. Introduction

Let P_n be a regular polygon with n sides. Imagine P_n is drawn on a table with the corners indexed sequentially from 0 to $n-1$. Also, imagine that we have another cardboard polygon, the same size as P_n with vertices indexed sequentially from x_0

⋆Corresponding author (E-mail: dianat@pgu.ac.ir)
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to x_{n-1}. The cardboard polygon is put on the table to cover exactly the drawn polygon. Figure 1 demonstrates an example for P_4. The squares are depicted with different sizes so that both squares can be seen.

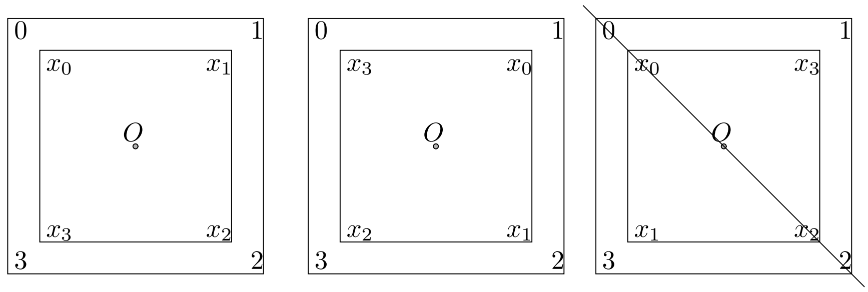

Figure 1: P_4 polygon along with its $2\pi/4$ radians clockwise rotation and its flipped version around the symmetry axis passing through the 0th corner.

A symmetry of P_n is defined as rotating the cardboard polygon around the center O in the table plane and/or its flipping around its symmetry axes, so that the cardboard polygon again covers the drawn polygon on the table. Therefore the in-plane rotations must be an integer multiple of $2\pi/n$ radians. The symmetry axes pass through the center O and a vertex or the midpoint of a side. Note that the flipping happens out of the table plane. The order and the number of rotations and flips are arbitrary.

The group of the symmetries of a polygon is called a dihedral group [2, 8]. Dihedral groups have applications in different fields. Here we are not to mention an inclusive list of applications, but rather we suffice to mention few samples. Dihedral groups can be applied to image processing and pattern recognition [3, 5]. Another application is the design of error-correction codes in telecommunication engineering to combat the channel errors in data transmission [6]. In reference [9], dihedral groups are exploited for remote sensing and feature extraction in the images taken by synthetic aperture radars (SAR images). A frame is a set of over-complete vectors to represent a vector space. It is desirables that the correlation between the members of a frame to be as low as possible. In [10] dihedral groups are deployed to design frames with low-correlation members.

2. System Interpretation of a Symmetry

Now, a permutation function is associated to each symmetry of P_n. If symmetry s takes vertex x_j of the cardboard polygon from corner j to corner i, the associated permutation function is defined as

$$j = f_s(i),\ 0 \leq i, j \leq n-1. \tag{1}$$

The inverse function is itself a permutation and is defined as

$$i = g_s(j),\ 0 \le i, j \le n-1.$$

The group of the associated permutations is called the dihedral group D_n that is of order $2n$. The operation of the group is the composition of functions, so that $f_1 \circ f_2(i) = f_1(f_2(i))$.

A symmetry s of P_n can be thought of a system that takes a sequence x_t, for $0 \le t \le n-1$, as input and produces another sequence y_t, for $0 \le t \le n-1$, as output. The system point of view is well-known in signal processing [7]. Figure 2 demonstrates the system. It means that if the vertices x_t of the cardboard polygon are in the corners t, then, after the symmetry is applied, in the corners t there are vertices y_t. From (1), the associated permutation says that

$$y_t = x_j,\ j = f_s(t).$$

We denote this operation as

$$y_t = x_t s = x_{f_s(t)}. \tag{2}$$

Equation (2) can be interpreted as a binary operation in which the left operand is a sequence, while the right operand is a symmetry. The result of the operation is a sequence.

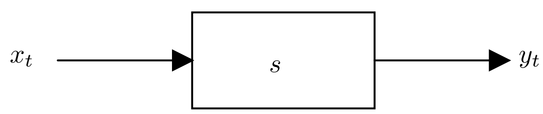

Figure 2: System model of a symmetry.

Suppose that first symmetry s_1 is applied to sequence x_t to produce sequence y_t and then symmetry s_2 is applied to sequence y_t to produce sequence z_t. If the associated permutations are f_1 and f_2 respectively, then we have

$$y_t = x_t s_1 = x_j,\ j = f_1(t) \tag{3}$$

and

$$z_t = y_t s_2 = y_k,\ k = f_2(t). \tag{4}$$

Substituting y_k from (3) into (4), we have

$$z_t = x_k s_1 = x_j,\ j = f_1(k) = f_1(f_2(t)) = f_1 \circ f_2(t). \tag{5}$$

Equation (5) can be interpreted as the series of two systems. The first one takes x_t as input and gives y_t, which is the input to the second system. The output

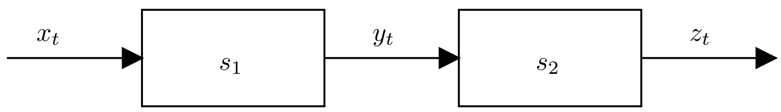

Figure 3: Concatenation of two symmetries.

of the second system is z_t. The effective associated permutation from x_t to z_t is $f_1 \circ f_2$. Figure 3 demonstrates the concatenation of the two systems.

The described procedure can be generalized to more than two systems. If symmetries s_1 to s_k with associated permutations f_1 to f_k are applied in serial to a sequence x_t to produce a sequence y_t, then it is displayed as

$$y_t = x_t s_1 s_2 \ldots s_k = x_j, \; j = f_1 \circ f_2 \ldots \circ f_k(t).$$

3. Two Generator Symmetries: Unit-Delay and Time-Reversal

We want to extend the sequences over the whole $\mathbb{Z}$. To do so, we choose circular indexing to get periodic sequences. Vertices x_0 to x_{n-1} are extended periodically (circularly) to create a periodic sequence x_t, so that $x_{t+n} = x_t, \; \forall t \in \mathbb{Z}$. For squares, for instance, $x_{-1} = x_{-1+4} = x_3$ and so on. Figure 4 depicts the periodic extension. As it is clear from the Figure, $x_{-1} = x_{n-1}$ as a result of circular indexing with period n. Other indices are attributed corresponding vertices in an analogous way.

We introduce two elements of the group of the symmetries of P_n that other elements can be synthesized by them.

3.1. Unit-Delay

This element is a system that produces the sequence $y_t = x_{t-1}$ when it is fed with the sequence x_t as input. We call it unit-delay, as its output in time t is equal to its one-unit delayed input. As it is shown in Figure 4, in time $t = 1$, y_1 is x_0 and so on. Not that $y_0 = x_{-1} = x_{n-1}$ as a natural result of periodic extension of sequences. It is equivalent to $2\pi/n$ radians clockwise rotation of the cardboard polygon. Figure 1 demonstrates an example for $n = 4$. From Figure 4 it is seen that vertices x_{t-1} are in the positions t. The same is true in Figure 1 for the rotated polygon. In general, using the periodic sequences in the interval $0 \leq t \leq n-1$ we can obtain the resultant new positions of the vertices of the polygon.

We display the unit-delay symmetry with d and its corresponding permutation function as

$$f_d(t) = d(t) = t - 1$$

for convenience.

3.2. Time-Reversal

This system when is provided with input x_t, generates output $y_t = x_{-t}$. Again note the indices should be interpreted periodically. In Figure 4 time-reversal is depicted. For example $y_1 = x_{-1} = x_{n-1}$. This system can be understood as the mirror image of sequence x_t relative to the axis $t = 0$ as shown in Figure 4. Therefore the systems exchanges x_1 with x_{-1}, x_2 with x_{-2} and other elements in a similar way. x_0 remains unmoved. If n is an even integer, $y_{n/2} = x_{-n/2} = x_{n-n/2} = x_{n/2}$ and therefore, $x_{n/2}$ does not move as well. Time reversal is equivalent to flipping the cardboard polygon around the axis passing through the corner 0 as shown in Figure 1. Again it is seen that the result of this flipping can be calculated with the aid of the sequence $y_t = x_{-t}$ in Figure 1 with $n = 4$ and $0 \leq t \leq 3$. For convenience, we denote the time-reversal symmetry as r and its permutation as

$$f_r(t) = r(t) = -t.$$

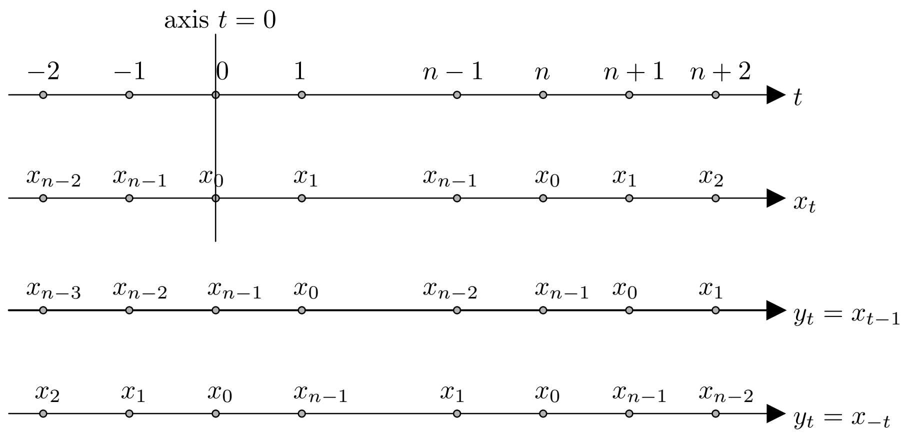

Figure 4: Periodic extension of sequences.

4. General Forms of Symmetries

As it was seen in the previous section, two symmetries were interpreted through periodic sequences and system point of view. In fact, all the symmetries of a polygon can be synthesized by periodic sequences and their calculation can be performed by their interpretation as systems. Each symmetry s of P_n can be attributed to a permutation function of the form $f_s(t) = at+b$, where $a \in \{-1, +1\}$ and b is an integer and $0 \leq b \leq n-1$. When a sequence x_t is processed with the symmetry s, the output sequence is $y_t = x_{at+b}$. In the following through the examples, we explain that how to calculate a and b for each symmetry and then how to synthesize the symmetry by two symmetries: Time-reversal and/or unit-delay.

4.1. Flipping Around an Axis Passing Through a Corner

Here, it is desired to derive the system for flipping the polygon around an axis passing through the corner t_0. As it is seen in Figure 5, it is equivalent to mirror imaging of the input sequence x_t relative to axis t_0. It means that the vertex that is located t units right of the axis t_0 must be exchanged with the vertex in the t units left of the axis t_0. Therefore, if the output of the system is y_t, then

$$y_{t_0+t} = x_{t_0-t}. \tag{6}$$

Replacing t with $t - t_0$ in (6) we get

$$y_t = x_{2t_0-t}. \tag{7}$$

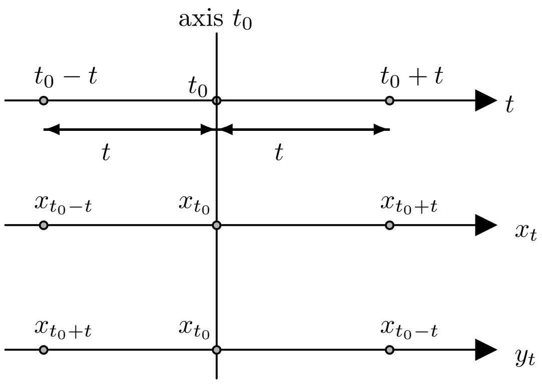

Figure 5: Input and output of the system equivalent to the flipping the polygon around corner t_0.

As an example, consider flipping a square P_4 around the axis passing through the corner $t_0 = 1$. From (7), for $0 \leq t \leq 3$, the new positions of the vertices are calculated as $y_0 = x_{2\times1-0} = x_2$, $y_1 = x_{2\times1-1} = x_1$, $y_2 = x_{2\times1-2} = x_0$ and

$y_3 = x_{2\times1-3} = x_{-1} = x_{-1+4} = x_3$. The results can be visualized and verified in Figure 6. Also, from this Figure it is seen that the axis passes through corner $t_0 = 3$ as well. It means that, letting $t_0 = 3$ instead of $t_0 = 1$, we should get the same result. It is easily verified as $y_0 = x_{2\times3-0} = x_6 = x_{6-4} = x_2$, $y_1 = x_{2\times3-1} = x_5 = x_{5-4} = x_1$, $y_2 = x_{2\times3-2} = x_4 = x_{4-4} = x_0$ and $y_3 = x_{2\times3-3} = x_3$. It can also be seen as if we use (7) with $t_0 = 3$ and period $n = 4$, the output sequence is $y_t = x_{2\times3-t} = x_{6-t} = x_{6-t-4} = x_{2-t}$, which is the same output for (7) with $t_0 = 1$.

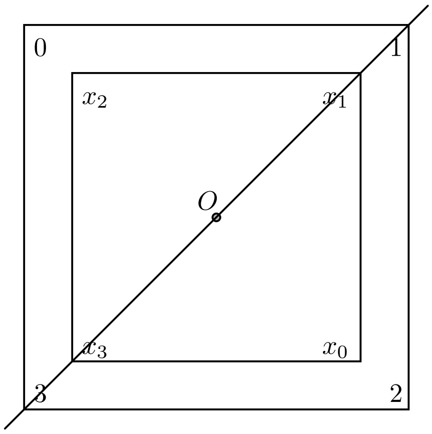

Figure 6: Flipping a square P_4 around the corner 1.

The system for flipping around t_0 can be synthesized with unit-delays and a time-reversal. For example, one way is to apply the time-reversal to the input sequence x_t and then feed the output to a series of $2t_0$ unit-delay systems that effectively are equivalent to a $2t_0$-unit-delay system. To see how this works, note that the output sequence of the time-reversal system is $y_t = x_{-t}$. If this output is the input to a unit-delay system, then the output is $z_t = y_{t-1} = x_{-(t-1)} = x_{1-t}$. If instead of one unit-delay system, we have $2t_0$ unit-delay systems in series, then the output will be $z_t = y_{t-2t_0} = x_{-(t-2t_0)} = x_{2t_0-t}$ as is the corresponding permutation to flipping around the axis passing through the corner t_0 as given in (7).

Another way to synthesize the desired flipping system is that at first the sequence is applied to $2t_0$ inverse-unit-delay systems that are connected serially. The permutation function of the inverse-unit-delay system is $f_i(t) = t + 1$. The effective permutation function of the serial inverses is $t + 2t_0$ so that the output is $y_t = x_{t+2t_0}$. This output is then applied to a time-reversal system to generate the final output $z_t = y_{-t} = x_{-t+2t_0}$, the same result as that of the first solution. These two solutions are demonstrated in Figure 7. The flipping around the corner

t_0 can be formulated as the following:

$$y_t = x_t r d^{2t_0} = x_t d^{-2t_0} r = x_j,\ j = 2t_0 - t\ . \tag{8}$$

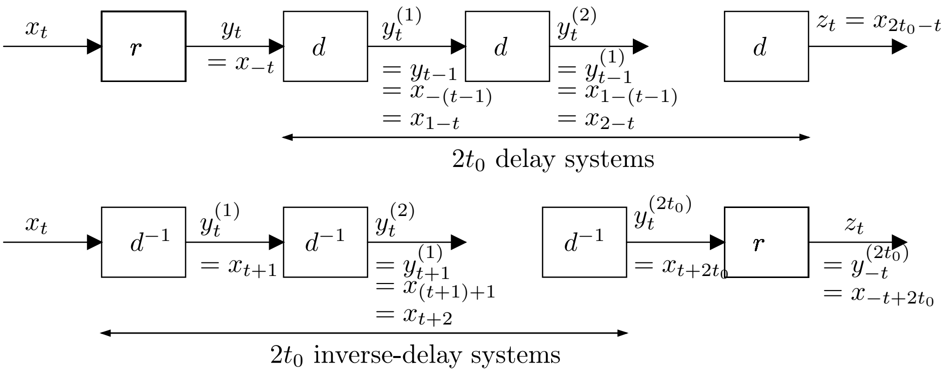

Figure 7: Two syntheses for flipping around corner t_0.

In (8), x_t is the input sequence and y_t is the output sequence and the resultant sequence of the flipping. Symmetry r stands for time-reversal (flipping around the 0th corner) and symmetry d is the unit-delay. Note that each symmetry d is equivalent to $2\pi/n$ radians clockwise rotation of the polygon P_n, while d^{-1} stands for $2\pi/n$ radians counter-clockwise rotation. In terms of the operation of the group of the symmetries of a polygon P_n, equation (8) says that

$$rd^{2t_0} = d^{-2t_0} r,\ \forall t_0 \in \mathbb{Z}.$$

4.2. Flipping Around an Axis Passing Through the Midpoint of a Side

Assume that the polygon P_n is to be flipped around the axis passing through the midpoint of the side connecting corners t_0 and $t_0 + 1$. Again this symmetry is formulated by the aid of sequences. Figure 8 displays the sequence-equivalent of this flipping. As this Figure shows, this operation can be considered as the mirror-imaging of the input sequence relative to the axis passing through the time instant $t_0 + \frac{1}{2}$. It means that the vertex that is located $t + \frac{1}{2}$ units right of the axis should be exchanged with the vertex located $t + \frac{1}{2}$ units left of the axis. Therefore, the following relation holds between the input and output sequences of this flip.

$$\begin{aligned} y_{(t_0+0.5)+t+0.5} &= x_{(t_0+0.5)-(t+0.5)} \\ y_{t_0+t+1} &= x_{t_0-t} \end{aligned} \tag{9}$$

Substituting t with $t - t_0 - 1$ in (9), we get the following:

$$y_t = x_{2t_0+1-t}. \tag{10}$$

As an example, consider a pentagon is flipped around the axis passing through the midpoint of the side from corner $t_0 = 1$ to corner $t_0 + 1 = 2$. Using (10) with $t_0 = 1$ and period $n = 5$, we have $y_0 = x_{2\times1+1-0} = x_3$, $y_1 = x_{2\times1+1-1} = x_2$, $y_2 = x_{2\times1+1-2} = x_1$, $y_3 = x_{2\times1+1-3} = x_0$ and $y_4 = x_{2\times1+1-4} = x_{-1} = x_{-1+5} = x_4$. This example can be visualized in Figure 9. From this Figure it is seen that we can consider the flipping axis passing through the corner 4 instead and get the same answer. To do this, let $t_0 = 4$ and use (7) to get $y_0 = x_{2\times4-0} = x_8 = x_{8-5} = x_3$, $y_1 = x_{2\times4-1} = x_7 = x_{7-5} = x_2$, $y_2 = x_{2\times4-2} = x_6 = x_{6-5} = x_1$, $y_3 = x_{2\times4-3} = x_5 = x_{5-5} = x_0$ and $y_4 = x_{2\times4-4} = x_4$. This is expectable as setting $t_0 = 4$ in (7) gives $y_t = x_{2\times4-t} = x_{8-t} = x_{8-t-5} = x_{3-t}$, which is the same result for (10) with $t_0 = 1$.

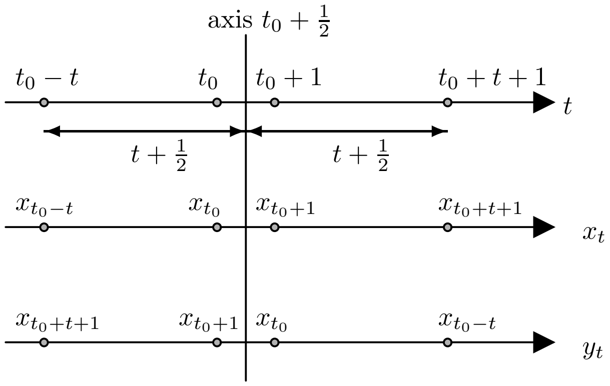

Figure 8: Sequence flipping around the midpoint $t_0 + \frac{1}{2}$.

4.3. General Symmetries

As explained, any symmetry s of the polygon P_n is corresponding to a permutation function $f_s(t) = at + b$, where $a \in \{-1, +1\}$ and $0 \leq b \leq n - 1$. Therefore, the group of the symmetries has totally $2n$ members. If $a = 1$, the symmetry s can be synthesized as $s = d^{-b}$, where d is the unit-delay symmetry and d^{-1} is its inverse. Therefore, d^{-b} is equivalent to the series of b inverse-unit-delay systems. If $a = -1$, then $f(t) = -t+b$. In this case, we can say $s = rd^b$. It means that s can be synthesized by the concatenation of a time-reversal with b unit-delay systems. The procedure is illustrated in Figure 10.

Another way to synthesize the function $f_s(t) = -t+b$ is to apply b inverse-unit-delay systems in series and then a time-reversal so that $s = d^{-b}r$. This synthesis

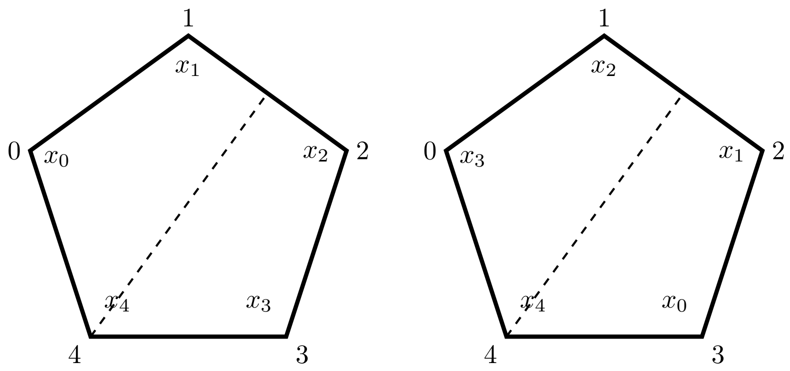

Figure 9: Pentagon P_5 along with its flipped version around the axis passing through the midpoint of the side from corner $t_0 = 1$ to corner $t_0 + 1 = 2$.

is illustrated in Figure 10 too. From these two syntheses the following identity results:

$$rd^m = d^{-m}r,\ m \in \mathbb{Z}$$

Note that the identity holds for all integer m.

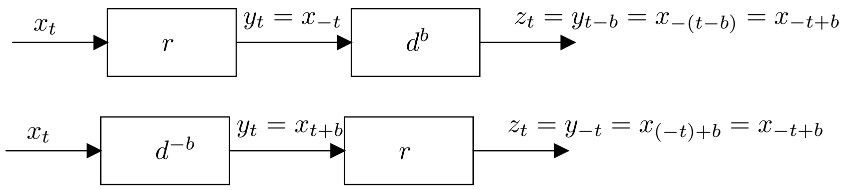

Figure 10: Synthesizing a symmetry.

4.4. Symmetries in Series

Let two symmetries s_1 and s_2 are connected serially. Further assume that the corresponding permutation functions are $f_1(t) = a_1 t + b_1$ and $f_2(t) = a_2 t + b_2$ respectively. The product $s_1 s_2$ can be calculated by the aid of the corresponding permutation $f_1(f_2(t)) = a_1(a_2 t + b_2) + b_1$. Therefore,

$$f_{s_1 s_2}(t) = a_1 a_2 t + (a_1 b_2 + b_1).$$

The following example is helpful. Let s_1 be the flipping around the vertical axis of a square and s_2 be $2\pi/4$ radians counter-clockwise rotation around the center of the square. For s_1, use (10) with $t_0 = 0$ to get $f_1(t) = 1 - t$. As s_2 is the inverse-unit-delay, we have $f_2(t) = t + 1$. The series product $s_1 s_2$ is equivalent to the permutation $f_1(f_2(t)) = 1 - (t + 1) = -t$. Therefore, the result of this product is a time-reversal. Figure 11 illustrates this example. The reader can easily check that the result is the same as the output of a time-reversal symmetry that is flipping around the axis passing through the 0th corner.

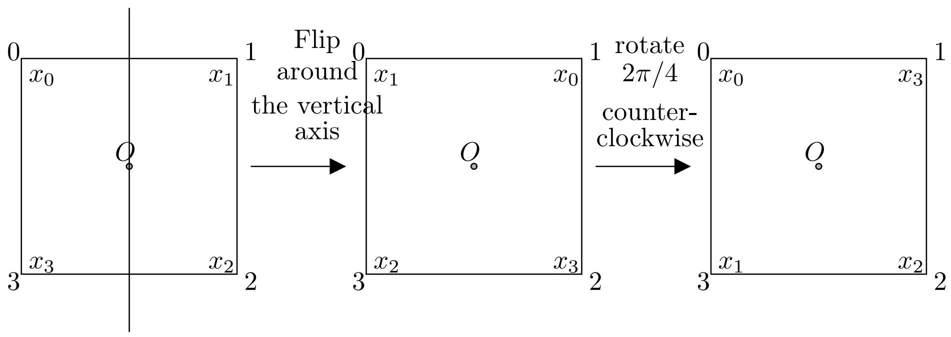

Figure 11: Series of two symmetries.

4.5. Pure-Rotation or Pure-Flip Synthesis of a Symmetry

This fact that each symmetry s can be explained by a corresponding permutation $f_s(t) = at + b$ comes up with this conclusion that any symmetry is effectively a pure rotation or a pure flip. If $a = 1$, s is $2\pi b/n$ radians counter-clockwise rotation of the polygon P_n or equivalently $2\pi(n - b)/n$ radians clockwise rotation. In this case $s = d^{-b} = d^{n-b}$. It is a pure rotation. If $a = -1$, then the symmetry is a flip around the axis passing through the corner $b/2$ for even b using (8). For odd b, from (10), s is a flip around the axis passing through the midpoint of the side connecting corner $t_0 = (b - 1)/2$ to corner $t_0 + 1$. Therefore, for $a = -1$, the symmetry is effectively a pure flip. Table 1 summarizes the results.

5. Proposal for Future Work

Dihedral groups only work on the peripherals of polygons. It seems that generalizations of dihedral groups can be made to include more complex and real problems concerning surfaces, volumes and higher-dimensional spaces. The generalized methods can be options besides the already available methods. As an example, in [1] perturbation method is used to analyze fluid model for blood flow. Another example is the real-time process algebra (RTPA) that is a set of mathematical notations for formally describing system architectures. This model is deployed in [11] for air traffic control. Mathematical models of algebraic topology,

Table 1: Symmetries and their permutations.

Symmetry s	associated permutation $f_s(t) = at + b$
Identity (e)	t
Unit-delay (d)	$t - 1$
Inverse-unit-delay	$t + 1$
b-unit delay	$t - b$
Time-reversal (r)	$-t$
Flip around the axis passing through corner t_0	$-t + 2t_0$
Flip around the axis passing through the midpoint of the side t_0 to $t_0 + 1$	$-t + 2t_0 + 1$

with their applications to computer generation of surfaces and modeling of smart cloud business are proposed in [4].

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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Reza Dianat
Department of Electrical Engineering,
Persian Gulf University,
Bushehr, I. R. Iran
E-mail: dianat@pgu.ac.ir

Mojgan Mogharrab
Department of Pure Mathematics,
Persian Gulf University,
Bushehr, I. R. Iran
E-mail: mmogharab@gmail.com