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Independence Fractals of Graphs as Models in Architecture

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Abstract

Architectural science requires interdisciplinary science interconnection in order to improve this science. Graph theory and geometrical fractal are two examples of branches of mathematics which have applications in architecture and design. In architecture, the vertices are the rooms and the edges are the direct connections between each two rooms. The independence polynomial of a graph G is the polynomial $I(G,x) = \sum i_k x^k$, where i_k denote the number of independent sets of cardinality k in G. The independence fractal of G is the set $I(G) = \lim_{k\to\infty} Roots(I(G^k, x) - 1)$, where $G^k = G[G[\cdots]]$, and $G[H]$ is the lexicographic product for two graphs G and H. In this paper, we consider graphical presentation of a ground plane as a graph G and use the sequences of limit roots of independence polynomials of G^k to present some animated structures for building.

Keywords: Independence fractal, structure, model, architecture.

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1. Introduction

Many architects have been inspired by the structure of the plant structures for building design. As we know the complex structure of the plant follows the mathematical formulas. So mathematics as a basic science can help architectures to design structures (see Figure 1). In mathematics, graph theory is the study of

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graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices, nodes, or points which are connected by edges, arcs, or lines.

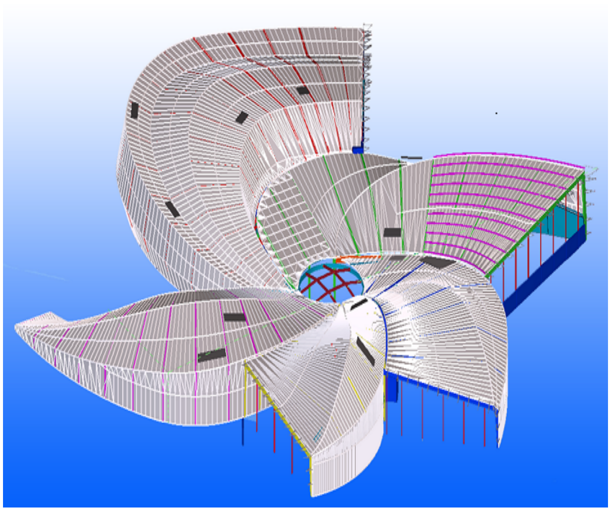

Figure 1: An example of structure of planet in architecture.

One of the most important application of graphs and fractals is to build lighter structures. Structures could be fractal in terms of the overall system which tend to be form finding or/and structures having fractal structural members. They efficiently distribute loads and reduce cost by reducing the quantity of structural materials to the bare minimum. See Figure 2. In the next section, we state some preliminaries in graphs and fractals as models in architecture. In Section 3, we introduce independence fractals which reach us from a graph as ground plane to fractals. Finally, as an example, we consider fan graphs and produce its independence fractals in Section 4.

2. Graphs and Fractals as Models in Architecture

A graph G is a pair $G = (V, E)$, where V and E are the vertex set and the edge set of G, respectively. A graph may be undirected, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be directed from one vertex to another. If u and v are two vertices of G, then an edge of the form $\{u, v\}$ is said to join u and v and that u and v are adjacent. Two or more edges joining the same pair of vertices are called multiple edges, and an edge joining a vertex to itself is called a loop. A graph with no loops or multiple edges is called a simple graph. The degree or valency of a vertex u in a graph G (loopless),

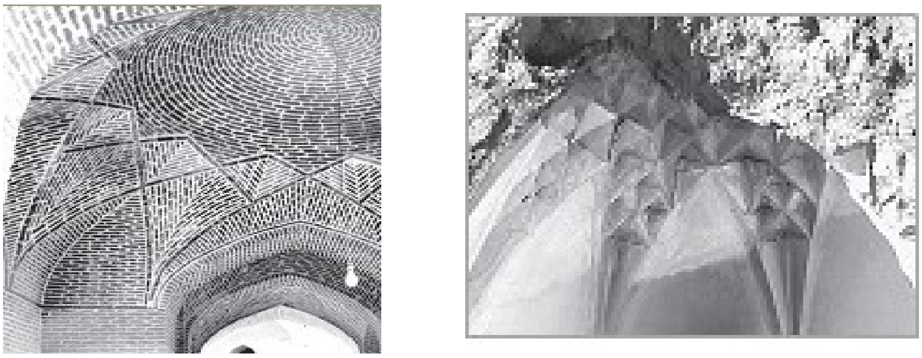

Figure 2: Jame porch in Farah Abad and Ashtari house in Yazd, respectively.

denoted by $deg(u)$, is the number of edges meeting at u. A walk of length k in G is a sequence $v_0, e_1, v_1, e_2, ..., e_k, v_k$ of vertices and edges such that $e_i = \{v_{i-1}, v_i\}$ for all i, and we say that this walk is a walk between v_0 and v_k. Notice that if G is simple, a walk is completely determined by the sequence of vertices. When there is a walk between any given pair of vertices of a graph G (that is G is connected), the distance between two vertices u and v of G, denoted by $d(u, v)$, is defined as the least length of the walks between them. The eccentricity of a vertex u in a graph G, denoted by $ecc(u)$, is the maximum of its distances to other vertices. Observe that the degree of a vertex only depends on the local structure of that vertex, whereas the eccentricity of a vertex depends on the global structure of the graph.

It is easy to see how to use this theory in the study of a ground plan. In architecture, the vertices are the rooms and the edges are the direct connections between each two rooms. Figure 3 shows the ground plan of a small flat of $70m^2$ and a graphical representation of the associated graph, which is a simple graph (see [3]). The vertex assignation is: 1−exterior, 2−hall, 3−livingroom, 4−balcony, 5−corridor, 6−bedroom, 7−bathroom, 8−study, 9−kitchen and 10−dryingplace. The degree and eccentricity sequences are $deg(1) = deg(4) = deg(6) = deg(7) = deg(8) = deg(10) = 1$, $deg(3) = deg(9) = 2$ and $deg(2) = deg(5) = 4$; $ecc(2) = 2, ecc(1) = ecc(3) = ecc(5) = ecc(9) = 3$ and $ecc(4) = ecc(6) = ecc(7) = ecc(8) = ecc(10) = 4$.

The vertices which have lowest degree, correspond to the spaces that require more privacy or because of noises or smells are more isolated. The vertices with medium degree correspond to main spaces and the vertices with highest degree correspond to circulation spaces. The central space, in terms of distances, is the hall. This fact is revealed by the lowest eccentricity of vertex 2. We also observe three levels of eccentricity without gaps: the lowest eccentricity corresponds to the central space, the highest eccentricity corresponds to the perimeter spaces and the medium eccentricity corresponds to the two main spaces, to a circulation space and to the exterior. The absence of cycles, that is the graph is a tree, is the usual way to design this kind of dwellings [3]. One exception to this rule is found when the balcony can be accessible from two or more spaces ([3]).

This easy example shows we can characterize properties of the architectural language, by means of some mathematical invariants associated to graphs.

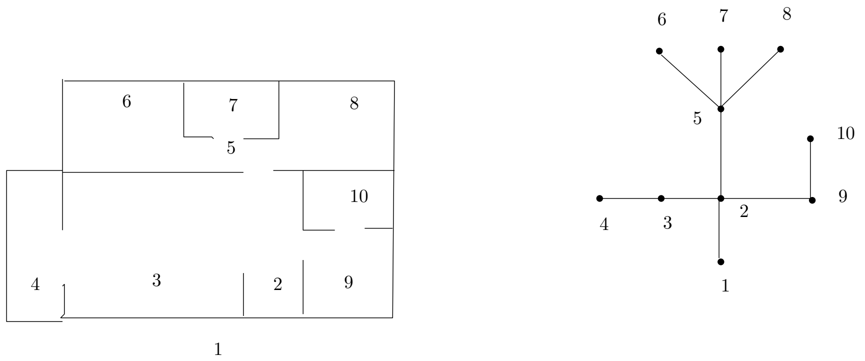

Figure 3: Ground plan and graphical representation of a small flat, respectively.

The term “fractal" was first used by mathematician Benoit Mandelbrot in 1975. Mandelbrot based it on the Latin fractus meaning “broken" or “fractured", and used it to extend the concept of theoretical fractional dimensions to geometric patterns in nature. A fractal is an abstract object used to describe and simulate naturally occurring objects. Artificially created fractals commonly exhibit similar patterns at increasingly small scales. It is also known as expanding symmetry or evolving symmetry. If the replication is exactly the same at every scale, it is called a self-similar pattern. See Figure 4.

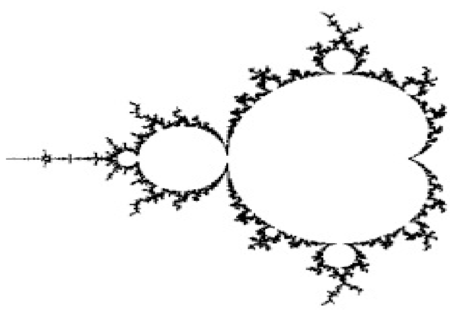

Figure 4: Mandelbort fractal.

3. From Graphs as Ground Plane to Fractals

Architectural forms are handmade and used Euclidean geometry so much, but we can find some fractals components in architecture, too. We need some preliminaries. All graphs in this paper are finite and simple. Actually, our approach for our scope, started from the structure of a plant. Since the structure of a plant are so complicated and have mathematical formula and is near to structure of fractals, therefore we understood that we shall consider fractals. An interesting method for producing fractal is study independence fractal of a graph which has considered recently. Using this attitude to produce structures in architecture. An *independent set* of a graph G is a set of vertices where no two vertices are adjacent. The *independence number* is the size of a maximum independent set in the graph. For a graph G with independence number β, let i_k denote the number of independent sets of cardinality k in G ($k = 0, 1, \ldots, \beta$). The *independence polynomial* of G,

$$I(G, x) = \sum_{k=0}^{\beta} i_k x^k,$$

is the generating polynomial for the independent sequence $(i_0, i_1, i_2, \ldots, i_\beta)$. The path P_4 on 4 vertices, for example, has one independent set of cardinality 0 (the empty set), four independent sets of cardinality 1, and three independent sets of cardinality 2; its independence polynomial is then $I(P_4, x) = 1 + 4x + 3x^2$ ([1]).

For a point $z_0 \in \mathcal{C}$, its *forward orbit* with respect to f is the set $O^+(z_0) = \{f^{ok}(z_0)\}_{k=0}^{\infty}$ where f^{ok} is the map $fofo\ldots of$ and $f^{o(0)}$ as the identity map ([2]).

For a polynomial f, its *filled Julia set* $\mathcal{K}(f)$ is the set of all points z whose forward orbit $O^+(z)$ is bounded in $(\mathcal{C}, |.|)$. Its *Julia set* $\mathcal{J}(f)$ is the boundary $\partial\mathcal{K}(f)$, and its *Fatou set* $\mathcal{F}(f)$ is the complement of $\mathcal{J}(f)$ in $\mathcal{C}$ ([2]).

The *independence fractal* or *independence attractor* of a graph G is the set

$$\mathcal{I}(G) = \lim_{k\to\infty} Roots(I(G^k, x) - 1),$$

where $G^k = G[G[\cdots]]$. (Note that for two graphs G and H, $G[H]$ is the lexicographic product or composition of G and H). For more details about independence fractals the reader is referred to [1, 5, 6]. In Figure 5 some steps of producing an independence fractal of a graph with each of these fractal can be as some animated facades of structures, especially that looks onto a street or open space.

4. Independence Fractals of Fan Graphs

We first consider a structure with a ground plane and consider its graphical representation. Then using the structure of independence fractals, present some animated structures which we think this idea is very useful to make building with

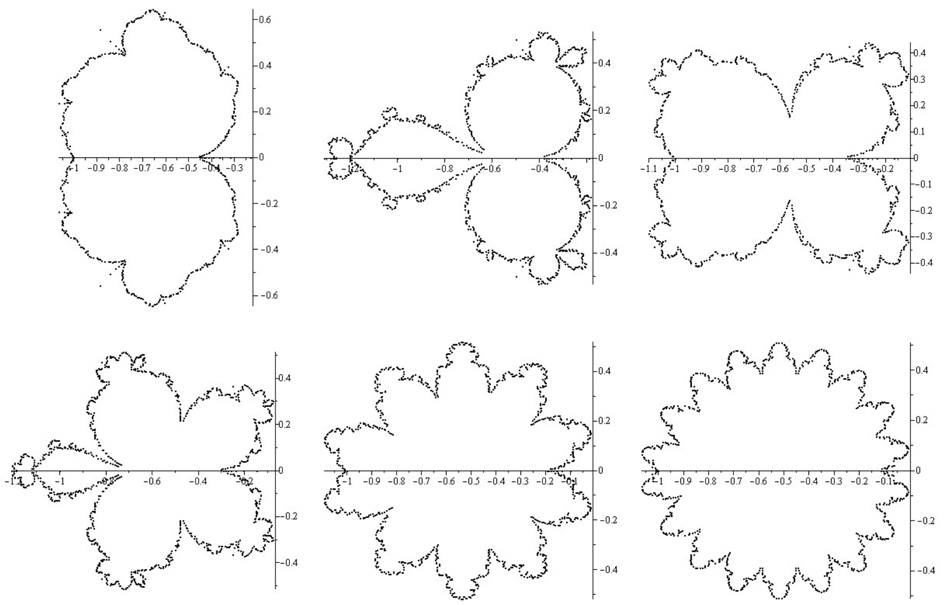

Figure 5: Some steps of producing an independence fractal.

animated structures. In this section we present this idea for a certain graph (certain ground map) and investigate it mathematically, to see how the fractal will be produced from a graph. Let to consider a fan graph as a certain graph.

A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m}+P_n$, where $\overline{K_m}$ is the empty graph on m vertices and P_n is the path graph on n vertices. See the fan $F_{7,6}$ in Figure 6. In this section we obtain the independence fractals of fan graphs. We need the following Theorems:

Theorem 4.1. ([7]) For any vertex v of a graph G,

$$I(G,x) = I(G-v,x) + xI(G-[v],x)$$

where $[v]$ is the closed neighborhood of v, contains of v, together with all vertices incident with v.

Theorem 4.2. ([1]) For any positive integer n,

$$I(P_n,x) = \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(2x+1+2x\cos\frac{2s\pi}{n+2}\right).$$

Theorem 4.3. ([4]) For every two graphs G_1 and G_2,

$$I(G_1+G_2,x) = I(G_1,x) + I(G_2,x) - 1.$$

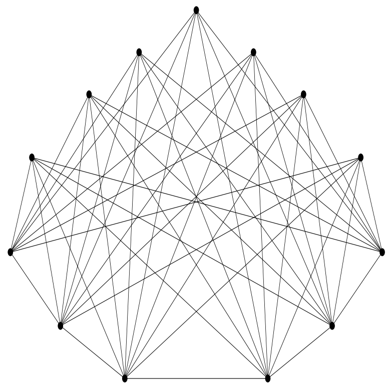

Figure 6: The fan graph $F_{7,6}$.

The following Theorem gives the independence polynomial of $F_{m,n}$:

Theorem 4.4. For every $m, n \in \mathbb{N}$,

$$I(F_{m,n}, x) = (1+x)^m + \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{n+2}\right) - 1.$$

Proof. It is easy to see that G_3^n is join of K_1 and nK_2. Now by Theorem 4.3 we have

$$\begin{aligned} I(F_{m,n}, x) &= I(\overline{K_m}, x) + I(P_n, x) - 1 \\ &= (1+x)^m + \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{n+2}\right) - 1. \end{aligned}$$

□

Here we want to investigate independence fractals (or independence attractors) of some of fan graphs. Hickman in his PhD thesis ([6]) has proved the following result (see also [5]).

Theorem 4.5. Let G be a non-empty graph, and denote $\eta(G)$ be the multiplicity of -1 as a root of $I(G, x)$. If $\eta(G) \leq 1$, then $\mathcal{I}(G) = \mathcal{J}(I(G, x) - 1)$.

Corollary 4.6. *For every* $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{I}(F_{m,m-1}, x) &= \mathcal{J}(I(F_{m,m-1}, x) - 1) \\ &= \mathcal{J}((1+x)^m + \prod_{s=1}^{\lfloor \frac{m}{2} \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{m+1}\right) - 2). \end{aligned}$$

Proof. By Theorem 4.4 obviously -1 is not root of $I(F_{m,m-1},x)$, so $\eta(F_{m,m-1}) \leq 1$ and we have the result by Theorem 4.5. □

The following procedure is procedure that Hickman in his thesis used to construct his plots of independence attractors. We state that procedure exactly here. For more information see [6].

```
Attract := proc (f,z,N)
local A,u,r,xx,yy,OldRts,NewRts,pts,i,j,n,symb:
n := degee(f):
A := table():
pts := NULL:
OldRts := [op ({fsolve(f,x,complex)} minus{z}]:
pts := pts, [Re(z) ,Im(z)] ,op(map([Re,Im] ,OldRts)):
while nops(OldRts) > 0 do
      NewRts := NULL:
      for i from 1 to nops(OldRts) do
           w := [fsolve(f-OldRts [i],x,complex)]:
           for j from 1 to n do
                r := w[j]:
                xx := ceil(N*(Re(r))):
                yy := ceil(N*(Im(r)):
                if A[xx,yy]<> 1 then
                   A[xx,yy] := 1:
                   NewRts := NewRts,r:
                   pts := pts, [Re(r),Im(r)]:
                fi:
            od:
      od:
      OldRts:= [NewRts]:
od:
pts :=[pts]:
symb := cross:
for i from 1 to min(nops(pts),20) do
      if pts[i][2]<> 0 then
           symb := point:
           i := min(nops(pts),20):
      fi:
od:
if symb=point then
     print('Points calculated.  Plotting...'):
else
      print('Attractor is Real.  Plotting...'):
fi:
```

```
plot(pts,style=point,symbol=symb,colour=black,scaling=constrained);
end;
```

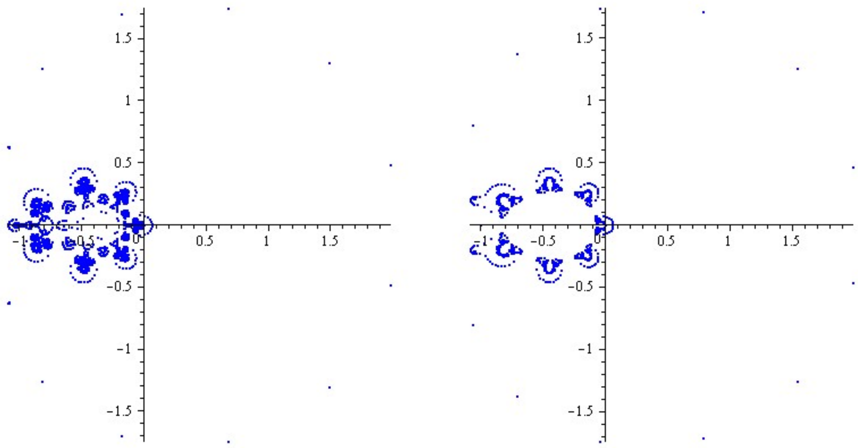

Figure 7: Independence fractal for $F_{21,20}$ and $F_{7,6}$, respectively.

The plots in the right hand of Figure 7, for instance, namely $\mathcal{I}(F_{7,6},x) = \mathcal{J}(I(F_{7,6},x)-1)$ was obtained with the command

> Attract $(I(F_{7,6},x)-1$, -1, 100).

5. Conclusion

Architectural forms are used Euclidean geometry and fractals so much. Since the structure of a plant is so complicated and has mathematical formula and is near to structure of fractals, we focused on fractals. An interesting method for producing fractal is study independence fractal of a graph which has considered recently. This attitude can produced animated structures in architecture.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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