Mathematics Interdisciplinary Research **6** (2021) 185 – 194

Fixed Point of Multi-valued Mizoguchi-Takahashi's Type Mappings and Answer to the Rouhani-Moradi's Open Problem

Sirous Moradi ⋆ *and Zahra Fathi*

Abstract

The fixed point theorem of Nadler (1966) was extended by Mizoguchi and Takahashi in 1989 and then for multi-valued contraction mappings, the existence of fixed point was demonstrated by Daffer and Kaneko (1995). Their results generalized the Nadler's theorem. In 2009 Kamran generalized Mizoguchi-Takahashi's theorem. His theorem improve Klim and Wadowski results (2007), and extended Hicks and Rhoades (1979) fixed point theorem. Recently Rouhani and Moradi (2010) generalized Daffer and Kaneko's results for two mappings. The results of the current work, extend the previous results given by Kamram (2009), as well as by Rouhani and Moradi (2010), Nadler (1969), Daffer and Kaneko (1995), and Mizoguchi and Takahashi (1986) for tow multi-valued mappings. We also give a positive answer to the Rouhani-Moradi's open problem.

Keywords: fixed point, Mizoguchi-Takahashi fixed point theorem, multi-valued mapping, weak contraction.

2010 Mathematics Subject Classification: 47H10, 54H25, 55M20.

How to cite this article
S. Moradi and Z. Fathi, Fixed point of multi-valued Mizoguchi-Takahashi's Type mappings and answer to the Rouhani-Moradi's open problem, *Math. Interdisc. Res.* **6** (2021) 185 – 194.

⋆Corresponding author (E-mail: moradi.s@lu.ac.ir)
Academic Editor: Ali Farajzadeh
Received 22 July 2020, Accepted 20 September 2021
DOI: 10.22052/MIR.2021.240213.1227

1. Introduction

Let (Y, d) be a metric space. Given a nonempty set $\mathcal{A} \subseteq Y$ and $y \in Y$, the distance between a point y and set $\mathcal{A}$ is displayed with $d(y, \mathcal{A})$. Let

$$\mathcal{CL}(Y) := \{\mathcal{A} \subseteq Y : \mathcal{A} \quad \text{is nonempty and closed}\},$$

$$\mathcal{CB}(Y) := \{\mathcal{A} \subseteq Y : \mathcal{A} \ \text{ is nonempty bounded and closed}\},$$

and

$$\mathcal{K}(Y) := \{\mathcal{A} \subseteq Y : \mathcal{A} \ \text{ is nonempty and compact}\}.$$

For $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{CL}(Y)$, we define $\mathcal{H}(\mathcal{A}_1, \mathcal{A}_2)$ (generalized Hausdorff metric induced by d) by

$$\mathcal{H}(\mathcal{A}_1, \mathcal{A}_2) = \begin{cases} \max\left\{\sup_{x\in\mathcal{A}_1} d(x, \mathcal{A}_2), \sup_{y\in\mathcal{A}_2} d(y, \mathcal{A}_1)\right\} & ; if \ \max < \infty \\ +\infty & ; o.w. \end{cases}$$

Notice that $\mathcal{H}$ is a metric on $\mathcal{CB}(Y)$, and if (Y, d) is complete, then $(\mathcal{CB}(Y), \mathcal{H})$ and $(\mathcal{K}(Y), \mathcal{H})$ are complete too. An element $u \in Y$ is a fixed point of $T : Y \longrightarrow \mathcal{CL}(Y)$ if $u \in Tu$. Throughout this paper, the set of all fixed points of T denotes by $\mathsf{Fix}(T)$. For $u_0 \in Y$, the set $O(T, u_0) = \{u_0, u_1, u_2, ...\}$ is called an orbit of T if $u_n \in Tu_{n-1}$, for all $n \in \mathbb{N}$. Recall that a map $g : Y \longrightarrow \mathbb{R}$ is called T-orbitally lower semi-continuous $(T - o.l.s.c)$ at u_0 [5] if for every sequence $\{u_n\}$ in $O(T, u_0)$ converging to κ, then $g(\kappa) \leq \liminf\limits_{n\to\infty} g(u_n)$.

Attention, a mapping $T : Y \longrightarrow \mathcal{CB}(Y)$ is weak contraction if

$$\mathcal{H}(Tu, Tv) \leq \beta\mathcal{N}(u, v),$$

for some $0 \leq \beta < 1$ and all $u, v \in Y$ where

$$\mathcal{N}(u, v) := \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv) + d(v, Tu)}{2}\right\}.$$

Fixed points of multi-valued maps have been studied by several authors. We encourage readers to see [1]-[4], [8], [10]-[12] and [15], as well as the references therein, among many more.

The Banach's theorem was extended by Nadler [12] to multi-valued mappings. In 1972 Reich [13] extended the Nadler's theorem. He proved that if $T : Y \longrightarrow \mathcal{K}(Y)$ satisfies the condition

$$\mathcal{H}(Tu, Tv) \leq \beta(d(u, v))d(u, v)$$

for each $u, v \in Y$, where (Y, d) is complete, and where $\beta : \mathbb{R}^+ \longrightarrow [0, 1)$ is a map such that $\limsup\limits_{s\to t^+} \beta(s) < 1$ for all $t \in \mathbb{R}^+$, then T has a fixed point.

After that, Mizoguchi and Takahashi [9] generalized the Nadler and Reich's theorems for multi-valued function $T : Y \longrightarrow \mathcal{CB}(Y)$. In 2009, Kamran extended the Mizoguchi and Takahashi's theorem. His result improved a result by Klim and Wadowski [7], and extended Hicks and Rhoades [5] fixed point theorem for multivaled mappings. After that, Daffer and Kaneko [4] studied the existence of a fixed point for multi-valued weak contraction mapping of a complete metric space Y into $\mathcal{CB}(Y)$. Their result extended the Nadler's theorem. Recently, Rouhani and Moradi [14] extended the Daffer-Kaneko and Nadler's theorems on the common fixed point for two multi-valued generalized weak contraction mappings. Also, they extended the Zhang and Song's theorem [16]. They considered the case that, one of the functions is multi-valued. But for the case where both mappings in Zhang and Song's theorem are multi-valued did not proved. In this paper we prove this open problem under the condition $\limsup_{t\to 0}(1-\frac{\varphi(t)}{t})<1$.

We will recall some definitions and preliminaries in the next section about generalized weak contractions and generalized β-weak contractions mappings. Section 3 contains the main results of this paper and positive answer to the Rouhani-Moradi's open problem. The results extend previous results by Nadler, as well as by Mizoguchi and Takahashi, Daffer and Kaneko, Kamran, Zhang-Song and Rouhani-Moradi.

2. Generalized Contractions

Throughout the paper, (Y,d) is a complete metric space and $\mathcal{H}$ is generalized Hausdorff metric induced by d on $\mathcal{CL}(Y)$.

Two multi-valued functions $T_1, T_2 : Y \longrightarrow \mathcal{CB}(Y)$ are called generalized weak contractions if

$$\mathcal{H}(T_1u, T_2v) \leq \beta\mathcal{M}(u,v) \ \ \forall u,v \in Y,$$

for some $0 \leq \beta < 1$, where

$$\mathcal{M}(u,v) := \max\Big\{d(u,v), d(u,T_1u), d(v,T_2v), \frac{d(u,T_2v)+d(v,T_1u)}{2}\Big\}.$$

The following definition extends the concept of weak contraction for tow multi-valued functions.

Definition 2.1. Two multi-valued functions $T_1, T_2 : Y \longrightarrow \mathcal{CB}(Y)$ are called generalized β-weak contractions if there exists a mapping $\beta : \mathbb{R}^+ \longrightarrow [0,1)$ such that

$$\mathcal{H}(T_1u, T_2v) \leq \beta(d(u,v))\mathcal{M}(u,v),$$

for each $u, v \in Y$.

For $u_0 \in Y$ and sequence $\{u_n\}$ with $u_{2n-1} \in T_1u_{2n-2}$ and $u_{2n} \in T_2u_{2n-1}$ ($\forall n \in \mathbb{N}$), the set $O(T_1,T_2;u_0) = \{u_0,u_1,u_2,...\}$ is called an orbit of $T_1 : Y \longrightarrow$

$\mathcal{CL}(Y)$ and $T_2 : Y \longrightarrow \mathcal{CL}(Y)$. Recall that a map $g : Y \longrightarrow \mathbb{R}$ is called $T_1 - T_2 - o.l.s.c$ at u_0, if for every sequence $\{u_n\}$ in $O(T_1, T_2; u_0)$ converging to κ, then $g(\kappa) \leq \liminf_{n\to\infty} g(u_n)$.

The following well-known lemma will be used in the main results of this article. We refer to Kamran [6] for its proof.

Lemma 2.2. *Let (Y, d) be a metric space and $\mathcal{B} \in \mathcal{CL}(Y)$. Then for each $q > 1$ and each $u \in Y$, there exists $b \in \mathcal{B}$ such that*

$$d(u, b) \leq q d(u, \mathcal{B}).$$

3. Main Results

In the following theorem, by the same method in [6] we extends Kamran theorem. Using this theorem we extend Nadler, Mizoguchi and Takahashi, Daffer and Kaneko, and Rouhani and Moradi's theorems.

Theorem 3.1. *Let $T_1, T_2 : Y \longrightarrow \mathcal{CL}(Y)$ be multi-valued mappings such that,*

$$d(v, T_1 v) \leq \beta(d(u, v)) d(u, v) \qquad \forall u \in Y, \forall v \in T_2 u, \tag{1}$$

and

$$d(v, T_2 v) \leq \beta(d(u, v)) d(u, v) \qquad \forall u \in Y, \forall v \in T_1 u, \tag{2}$$

where $\beta : \mathbb{R}^+ \longrightarrow [0, 1)$ is a mapping with $\limsup_{s\to t^+} \beta(s) < 1$ for all $t \in \mathbb{R}^+$. Then,

(i) *For every $u_0 \in Y$, there exists an orbit $\{u_n\}$ of T_1 and T_2 and $\kappa \in Y$ such that $\lim_{n\to\infty} u_n = \kappa$;*

(ii) *$\kappa \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$ if and only if the function $f(u) := d(u, T_1 u)$ or the function $g(u) := d(u, T_2 u)$ is $T_1 - T_2 - o.l.s.c$ at κ.*

Proof. Let $u_0 \in Y$ and $u_1 \in T_1 u_0$. If $u_0 = u_1$, then by using (2), we conclude that $u_0 \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$. Suppose $u_0 \neq u_1$. If $\beta(d(u_0, u_1)) = 0$, then by using (2), $u_1 \in T_2 u_1$ and then from (1), $u_1 \in T_1 u_1$. Hence $u_1 \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$. Suppose $\beta(d(u_0, u_1)) \neq 0$. By taking $q = \sqrt{\frac{1}{\beta(d(u_0,u_1))}}$ and by using Lemma 2.2, there exists $u_2 \in T_2 u_1$ such that

$$d(u_1, u_2) \leq \sqrt{\frac{1}{\beta(d(u_0, u_1))}} d(u_1, T_2 u_1).$$

By the same method if $u_1 = u_2$ $(\beta(d(u_1, u_2)) = 0)$ we conclude that $u_1 \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$ $(u_2 \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2))$. Suppose $u_1 \neq u_2$ and $\beta(d(u_1, u_2)) \neq 0$. Again from Lemma 2.2 there exists $u_3 \in T_1u_2$ such that

$$d(u_2, u_3) \leq \sqrt{\frac{1}{\beta(d(u_1, u_2))}} d(u_2, T_1u_2).$$

By using Lemma 2.2 and using the mathematical induction, there exists a sequence $\{u_n\}$ in Y such that $u_n \neq u_{n-1}$ (for otherwise, $u_{n-1} \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$), $\beta(d(u_{n-1}, u_n)) \neq 0$ (for otherwise, $u_n \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$), $u_{2m} \in T_2u_{2m-1}$ and

$$d(u_{2m-1}, u_{2m}) \leq \sqrt{\frac{1}{\beta(d(u_{2m-2}, u_{2m-1}))}} d(u_{2m-1}, T_2u_{2m-1}),$$

and $u_{2m+1} \in T_1u_{2m}$ and

$$d(u_{2m}, u_{2m+1}) \leq \sqrt{\frac{1}{\beta(d(u_{2m-1}, u_{2m}))}} d(u_{2m}, T_1u_{2m}). \tag{3}$$

Using (1) and (3), for all $m \in \mathbb{N}$

$$\begin{aligned} d(u_{2m}, u_{2m+1}) &\leq \sqrt{\frac{1}{\beta(d(u_{2m-1}, u_{2m}))}} d(u_{2m}, T_1u_{2m}) \\ &\leq \sqrt{\beta(d(u_{2m-1}, u_{2m}))} d(u_{2m-1}, u_{2m}) \qquad (4) \\ &< d(u_{2m-1}, u_{2m}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(u_{2m-1}, u_{2m}) &\leq \sqrt{\beta(d(u_{2m-2}, u_{2m-1}))} d(u_{2m-2}, u_{2m-1}) \\ &< d(u_{2m-2}, u_{2m-1}). \end{aligned} \tag{5}$$

Using (4) and (5), for all $n \in \mathbb{N}$

$$\begin{aligned} d(u_n, u_{n+1}) &\leq \sqrt{\beta(d(u_{n-1}, u_n))} d(u_{n-1}, u_n) \\ &< d(u_{n-1}, u_n). \end{aligned} \tag{6}$$

Thus the sequence $\{d(u_{n+1}, u_n)\}$ is decreasing. Therefore

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = a. \tag{7}$$

for some $a \geq 0$. Now we prove that $a = 0$.
For otherwise, by limiting in (7) we get

$$a \leq \sqrt{\limsup_{n\to\infty} \beta(d(u_{n+1}, u_n))} a < a,$$

and this is a contradiction. So $a = 0$.
From (6) we get

$$d(u_n, u_{n+1}) \leq \sqrt{\beta(d(u_{n-1}, u_n))...\beta(d(u_0, u_1))}d(u_0, u_1). \tag{8}$$

Since $\limsup_{r\to 0^+}\beta(r) < 1$, then there exists an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $\beta(t) < \lambda^2$ for all $t \in (0, \varepsilon)$. Also from (7), there exists $N \in \mathbb{N}$ such that $d(u_{n-1}, u_n) < \varepsilon$ for all $n \geq N$. So from (8)

$$\begin{aligned} d(u_n, u_{n+1}) &\leq \lambda^{n-(N-1)}\sqrt{\beta(d(u_{N-2}, u_{N-1}))...\beta(d(u_0, u_1))}d(u_0, u_1) \\ &< \lambda^{n-N+1}d(u_0, u_1), \end{aligned} \tag{9}$$

for all $n \geq N$. Since $\lambda < 1$ and (9) holds, then the sequence $\{u_n\}$ is Cauchy and hence $\lim_{n\to\infty} u_n = \kappa$ for some $\kappa \in Y$. Suppose the function $f(u) = d(u, T_1u)$ is $T_1 - T_2 - o.l.s.c$ at κ. Since $u_{2m+1} \in Tu_{2m}$ for all $m \in \mathbb{N}$

$$d(u_{2m}, T_1u_{2m}) \leq d(u_{2m}, u_{2m+1}). \tag{10}$$

From $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$ and (10) we conclude that $\lim_{m\to\infty} d(u_{2m}, T_1u_{2m}) = 0$. Since f is $T_1 - T_2 - o.l.s.c$ at κ,

$$d(\kappa, T_1\kappa) = f(\kappa) \leq \liminf_{m\to\infty} f(u_{2m}) = \liminf_{m\to\infty} d(u_{2m}, T_1u_{2m}) = 0.$$

Hence $d(\kappa, T_1\kappa) = 0$. Since $T_1\kappa$ is closed, $\kappa \in T_1\kappa$. From (2) and $\kappa \in T_1\kappa$

$$d(\kappa, T_2\kappa) \leq \beta(d(\kappa, \kappa))d(\kappa, \kappa) = 0,$$

and so $\kappa \in T_2\kappa$, since $T_2\kappa$ is closed. Therefore $\kappa \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$.

Conversely, if $\kappa \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$ then $f(\kappa) = 0 \leq \liminf_{n\to\infty} f(u_n)$ and $g(\kappa) = 0 \leq \liminf_{n\to\infty} g(u_n)$ and the proof is completed. □

Remark 1. By taking $T_1 = T_2$ in Theorem 3.1, we conclude the Kamran theorem [6].

Now we extend the Nadler, Mizoguchi and Takahashi, Daffer and Kaneko, and Rouhani-Moradi's theorems.

Corollary 3.2. *Let $T_1, T_2 : Y \longrightarrow \mathcal{CL}(Y)$ be multi-valued mappings such that,*

$$\mathcal{H}(T_1u, T_2v) \leq \beta(d(u, v))\mathcal{M}(u, v) \qquad \forall u, v \in Y,$$

where $\beta : \mathbb{R}^+ \longrightarrow [0, 1)$ is a function such that $\limsup_{s\to t^+}\beta(s) < 1$ for all $t \in \mathbb{R}^+$. Then there exists a point $\kappa \in Y$ such that $\kappa \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$. Moreover, if $T_1\kappa$ or $T_2\kappa$ is a singleton, then the $\mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2) = \{\kappa\}$.

Proof. For every $u \in Y$ and $v \in T_1 u$

$$\begin{aligned} d(v, T_2 v) &\leq \mathcal{H}(T_1 u, T_2 v) \\ &\leq \beta(d(u,v)) \max\Big\{d(u,v), d(u,T_1 u), d(v,T_2 v), \frac{d(u,T_2 v)+d(v,T_1 u)}{2}\Big\}. \end{aligned} \tag{11}$$

Since $v \in T_1 u$, $d(v, T_1 u) = 0$ and $d(v, T_1 u) \leq d(u,v)$. Also $d(u, T_2 v) \leq d(u,v) + d(v, T_2 v)$. Hence from (11)

$$d(v, T_2 v) \leq \beta(d(u,v)) d(u,v).$$

Similarly, for every $u \in Y$ and $v \in T_2 u$

$$d(v, T_1 v) \leq \beta(d(u,v)) d(u,v).$$

Now we prove that the function $g(u) := d(u, T_2 u)$ is $T_1 - T_2 - o.l.s.c.$ Suppose $u_0 \in Y$, $u_{2n-1} \in T_1 u_{2n-2}$ and $u_{2n} \in T_2 u_{2n-1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = y$. We need to show that $g(u) \leq \liminf_{n\to\infty} g(u_n)$. We have

$$\begin{aligned} g(u) &= d(u,T_2 u) \leq d(u,u_{2n}) + d(u_{2n}, T_1 u_{2n}) + \mathcal{H}(T_1 u_{2n}, T_2 u) \\ &\leq d(u,u_{2n}) + d(u_{2n}, T_1 u_{2n}) + \beta(d(u_{2n},u)) \max\{d(u_{2n},u), d(u_{2n}, T_1 u_{2n}), \\ &\quad d(u, T_2 u), \frac{d(u_{2n}, T_2 u) + d(u, T_1 u_{2n})}{2}\} \qquad (12) \\ &\leq d(u,u_{2n}) + d(u_{2n}, u_{2n+1}) + \beta(d(u_{2n},u)) \max\{d(u_{2n},u), d(u_{2n}, u_{2n+1}), \\ &\quad d(u, T_2 u), \frac{d(u_{2n},u) + d(u, T_2 u) + d(u, u_{2n+1})}{2}\}. \end{aligned}$$

From $\lim_{n\to\infty} u_n = u$ and (12),

$$g(u) \leq \limsup_{r\to 0^+} \beta(r) g(u),$$

and this shows that $g(u) = 0$. Hence $g(u) = 0 \leq \liminf_{n\to\infty} g(u_n)$. So g is $T_1 - T_2 - o.l.s.c.$ Therefore, by Theorem 3.1, T_1 and T_2 have a common fixed point.

Furthermore, if $\kappa \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$ and $T_1 \kappa = \{\kappa\}$, then the common fixed point is unique. In fact, if $u \in \mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2)$ is arbitrary, then

$$\begin{aligned} d(\kappa, u) &\leq \mathcal{H}(\{\kappa\}, T_2 u) = \mathcal{H}(T_1 \kappa, T_2 u) \leq \beta(d(\kappa,u)) M(\kappa, u) \\ &= \beta(d(\kappa,u)) \max\{d(\kappa,u), d(\kappa, T_1\kappa), d(u, T_2 u), \frac{d(\kappa, T_2 u) + d(u, T_1 \kappa)}{2}\} \\ &\leq \beta(d(\kappa,u)) \max\{d(\kappa,u), 0, 0, \frac{d(\kappa,u) + d(u,\kappa)}{2}\} \\ &= \beta(d(\kappa,u)) d(\kappa,u). \end{aligned}$$

Since $\beta(d(\kappa,u)) < 1$, $d(\kappa,u) = 0$ and so $\kappa = u$. □

Remark 2. By using the last part of the proof of Corollary 3.2, we conclude that, if one of the functions T_1 and T_2 is single valued, then the common fixed point is unique.

The following theorem give a positive answer to the Rouhani-Moradi's open problem under the condition $\limsup_{t\to 0}(1-\frac{\varphi(t)}{t}) < 1$.

Theorem 3.3. *Let $T_1, T_2 : Y \longrightarrow \mathcal{CL}(Y)$ be multi-valued mappings such that,*

$$\mathcal{H}(T_1u, T_2v) \leq \mathcal{M}(u,v) - \varphi(\mathcal{M}(u,v)) \qquad \forall u, v \in Y,$$

where $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is l.s.c with $\varphi(0) = 0$, $\varphi(t) > 0$ for all $t > 0$ and satisfies the condition $\limsup_{t\to 0}(1-\frac{\varphi(t)}{t}) < 1$. Then $\mathsf{Fix}(T_1)\cap\mathsf{Fix}(T_2) \neq \emptyset$. Moreover, for some $\kappa \in \mathsf{Fix}(T_1)\cap\mathsf{Fix}(T_2)$, if $T_1\kappa$ or $T_2\kappa$ is a singleton, then $\mathsf{Fix}(T_1)\cap\mathsf{Fix}(T_2) = \{\kappa\}$ (the common fixed point is unique).

Proof. Let $\beta : \mathbb{R}^+ \longrightarrow [0,1)$ defined by $\beta(t) = 1 - \frac{\varphi(t)}{t}$ for all $t > 0$ and $\beta(0) = 0$. Obviously, $\limsup\limits_{s\to t^+}\beta(s) < 1$ for all $t \in \mathbb{R}^+$. Also for every $u \in X$ and $v \in T_1u$

$$d(v, T_2v) \leq \mathcal{H}(T_1u, T_2v) \leq \mathcal{M}(u,v) - \varphi(\mathcal{M}(u,v)), \tag{13}$$

where

$$\begin{aligned} d(u,v) &\leq \max\Big\{d(u,v), d(u,T_1u), d(v,T_2v), \frac{d(u,T_2v)+d(v,T_1u)}{2}\Big\} \\ &\leq \max\Big\{d(u,v), d(u,v), d(v,T_2v), \frac{d(u,T_2v)+0}{2}\Big\} \\ &\leq \max\Big\{d(u,v), d(v,T_2v), \frac{d(u,v)+d(v,T_2v)}{2}\Big\} \\ &\leq \max\Big\{d(u,v), d(v,T_2v)\Big\}. \end{aligned} \tag{14}$$

Using (13) and (14), we conclude that $d(v, T_2v) \leq d(u, v)$ and from (14), we get $M(u, v) = d(u, v)$. Hence from (13),

$$d(v, T_2v) \leq d(u,v) - \varphi(d(u,v)) = \beta(d(u,v))d(u,v).$$

Similarly, for every $u \in Y$ and $v \in T_2y$,

$$d(v, T_1v) \leq \beta(d(u,v))d(u,v).$$

Therefore, by Theorem 3.1, $\mathsf{Fix}(T_1)\cap\mathsf{Fix}(T_2) \neq \emptyset$.

Alike of Corollary 3.2 we can prove that if $T_1\kappa$ or $T_2\kappa$ is a singleton, then the common fixed point of T_1 and T_2 is unique. □

In the following example, we shows the generality of our results. In particular shows that Theorem 3.1 extends the Kamran's theorem.

Example 3.4. Let $Y = \{0, \frac{1}{2}, 1\}$ and let d be the Euclidean metric. Let $T_1 : Y \longrightarrow \mathcal{CB}(Y)$ be defined by $T_1 0 = \{0, \frac{1}{2}, 1\}$, $T_1 \frac{1}{2} = \{0, \frac{1}{2}\}$ and $T_1 1 = \{0\}$. Obviously $1 \in T_1 0$ and $d(1, T_1 1) = 1 = d(0, 1)$. So the mapping T_1 does not satisfy the hypothesis of Kamran theorem.

Let $T_2 : Y \longrightarrow \mathcal{CB}(Y)$ be defined by $T_2 x = \{0, \frac{1}{2}\}$ and $\beta : \mathbb{R}^+ \longrightarrow [0, 1)$ be defined by $\beta(x) = \frac{2}{3}$. Obviously,

$$d(v, T_1 v) \leq \frac{2}{3} d(u, v), \quad \forall u \in Y, \forall v \in T_2 u,$$

and

$$d(v, T_2 v) \leq \frac{2}{3} d(u, v), \quad \forall u \in Y, \forall v \in T_1 u.$$

Therefore, all the conditions of Theorem 3.1 hold. Also we have $\mathsf{Fix}(T_1) \cap \mathsf{Fix}(T_2) = \{0\}$.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] M. Abbas and F. Khojasteh, Common f-endpoint for hybrid generalized multi-valued contraction mappings, *RACSAM* **108** (2) (2014) 369 – 375.

[2] S. Benchabaney and S. Djebaliz, Common fixed point for multi-valued (ψ, θ, G)-contraction type maps in metric spaces with a graph structure, *Appl. Math. E-Notes* **19** (2019) 515 – 526.

[3] C. Chifu and G. Petrusel, Existence and data dependence of fixed points and strict fixed points for contractive-type multi-valued operators, *Fixed Point Theory Appl.* **2007** (2007) 34248.

[4] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive multi-valued mappings, *J. Math. Anal. Appl.* **192** (1995) 655 – 666.

[5] T. L. Hicks and B. E. Rhoades, A Banach type fixed point theorem, *Math. Japonica* **24** (1979) 327 – 330.

[6] T. Kamran, Mizoguchi-Takahashi's type fixed point theorem, *Comput. Math. Appl.* **57** (2009) 507 – 511.

[7] D. Klim and D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, *J. Math. Anal. Appl.* **334** (2007) 132 – 139.

[8] Y. Mahendra Singh, G. A. Hirankumar Sharma and M. R. Singh, Common fixed point theorems for (ψ, φ)-weak contractive conditions in metric spaces, *Hacet. J. Math. Stat.* **48** (5) (2019) 1398 – 1408.

[9] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric space, *J. Math. Anal. Appl.* **141** (1989) 177 – 188.

[10] B. Mohammadi, Strict fixed points of Ciric-generalized weak quasicontractive multi-valued mappings of integral type, *Int. J. Nonlinear Anal. Appl.* **9** (2) (2018) 117 – 129.

[11] S. Moradi, Endpoints of multi-valued cyclic contraction mappings, *Int. J. Nonlinear Anal. Appl.* **9** (1) (2018) 203 – 210.

[12] S. B. Nadler, Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969) 475 – 488.

[13] S. Reich, Fixed point of contractive functions, *Boll. Unione Mat. Ital.* **4** (1972) 26 – 42.

[14] B. D. Rouhani and S. Moradi, Common Fixed Point of Multi-valued Generalized ϕ-Weak Contractive Mappings, *Fixed Point Theory Appl.* **2010** (2010) 708984.

[15] N. Shahzad and A. Lone, Fixed points of multimaps which are not necessarily nonexpansive, *Fixed Point Theory Appl.* **2** (2005) 169 – 176.

[16] Q. Zhang and Y. Song, Fixed point theory for generalized φ-weak contractions, *Appl. Math. Lett.* **22** (2009) 75 – 78.

Sirous Moradi
Department of Mathematics,
Faculty of Sciences,
Lorestan University,
Khorramabad 68151-4-4316, Iran
e-mail: moradi.s@lu.ac.ir

Zahra Fathi
Department of Mathematics,
Faculty of Sciences,
Arak University,
Arak 38156-8-8349, Iran
e-mail: zs.fathi62@gmail.com